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Abstract

We offer yet another derivation of the perfect graph theorem from the replication lemma. We generalize the pluperfect graph theorem, even its "greedy" version by Chandrasekaran and Tamir [1], in the setting of totally ordered abelian groups.

Key words: perfect graph theorem, totally ordered abelian groups

1 Introduction

I had the good fortune to spend the summer of 1961 in a workshop on combinatorial mathematics conducted at the Rand Corporation in Santa Monica, California. Although I knew most of the other participants from conferences in other venues, this Rand workshop was the first time I met Claude Berge and learned of his conjectures about perfect graphs. The strong conjecture seemed way too difficult for me (and I was right, of course). The weak conjecture (that the complement of a perfect graph is perfect) — now known as the perfect graph theorem — seemed within reach, even though I failed. But, after learning Lovász's proof [8], especially the replication lemma, I thought of the arguments which I describe in Sec. 2 to prove the perfect graph theorem and the pluperfect graph theorem. In Sec. 3, I prove the new result that the pluperfect graph theorem, and its improvement in [1], are valid when the node weights are elements of a totally ordered abelian group.

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2 The perfect graph theorem

Let us first review some definitions. A graph G is *perfect* if G and each of its induced subgraphs have the property that the size of the largest clique is the chromatic number. The perfect graph theorem states that, if G is perfect, so is the complementary graph $\overline{G}$. If G is a graph on n nodes, and $(c_1, \ldots, c_n)$ is a vector of nonnegative integers, let $G(c_1, \ldots, c_n)$ be the graph obtained from G by replacing each node i with a clique of c_i nodes, and edges between these cliques inherited from the edges of G. The replication lemma of Lovász [8] says that G perfect implies $G(c_1, \ldots, c_n)$ perfect.

Now consider an arbitrary perfect graph G, let K denote the (0, 1) matrix in which the rows correspond to all cliques of G, the columns to the nodes of G, with

$$K(i,j) = \begin{cases} 1 & \text{if the } i\text{th clique contains the } j\text{th node,} \\ 0 & \text{otherwise,} \end{cases}$$

and let P denote the polyhedron $\{\, x : \; Kx \leq 1, \, x \geq 0\}$. We use induction on the number of vertices of G to prove, by an argument close to that given in [4], that

$$P \text{ has all its vertices } (0,\, 1) \text{ vectors.} \tag{1}$$

For this purpose, consider an arbitrary vertex, y, of P. If any coordinate of y equals 0, then we are done by the induction hypothesis applied to G with one vertex deleted. Hence we may assume that all coordinates of y are positive, and so y is the unique solution of a system $My = 1$, where M consists of selected rows of K. In particular, all coordinates of y are rational; let us write

$$y = (c_1/d, c_2/d, \ldots, c_n/d)^T$$

with positive integers $c_1, \ldots, c_n, d$. Maximal cliques in the graph $G(c_1, \ldots, c_n)$ are obtained by "weighting" (with weights $c_1, \ldots, c_n$) maximal cliques in G; inequalities $Ky \leq 1$ imply that all maximal cliques in $G(c_1, \ldots, c_n)$ have size at most d; equations $My = 1$ imply that maximal cliques in $G(c_1, \ldots, c_n)$ corresponding to rows in M have size d. By the replication lemma, $G(c_1, \ldots, c_n)$ is perfect, hence its chromatic number is d, and every color class of nodes meets every maximum clique (in one node, of course). But this tells us that there is a (0,1) vector z (corresponding, say, to the red nodes) such that $Mz = 1$. Since y is the unique solution of $My = 1$, we have $y = z$.

To complete the proof of the perfect graph theorem, we prove what Fulkerson [2,3] caled the "pluperfect graph theorem": *(1) implies that for every nonnegative integral vector v, the problem*

$$\textit{minimize} \sum w_i \textit{ such that } w^T K \geq v^T,\ w \geq 0 \tag{2}$$

is solved by an integral vector. This can be done by induction on $\sum v_i$. Certainly the statement is true if $\sum v_i = 0$.

To begin, let OPT denote the optimal value of (2). Duality and (1) guarantee that OPT is an integer. Now consider an arbitrary solution w of (2). If w is an integral vector, then we are done; else let w_i be a non-integral component of w, let K_i^T denote the i-th row of the matrix K, and let $\overline{v}$ denote the positive part of $v - \lceil w_i \rceil K_i$. Since $\sum \overline{v}_i < \sum v_i$, induction hypothesis guarantees the existence of an integral solution $\overline{w}$ of the problem

$$\text{minimize} \sum \overline{w}_i \text{ such that } \overline{w}^T K \geq \overline{v}^T,\ \overline{w} \geq 0. \tag{3}$$

Since w with w_i replaced by zero constitutes a feasible solution of (3), the optimal value of (3) is at most OPT$-w_i$, and so (being integral) it is at most OPT$-\lceil w_i \rceil$. Adding $\lceil w_i \rceil$ to the i-th component of $\overline{w}$, we obtain an integral feasible solution of (2); since the sum of its components is at most OPT, this integral feasible solution of (2) is optimal.

This argument is close to that given in [1], which establishes the stronger statement: if the rows of K are numbered arbitrarily and if OPT denotes the optimal value of (2), then successive maximization of the w_i, subject to the inequalities in (2) and to the requirement that $\sum w_i =$OPT, will yield an optimum w which is integral.

3 Perfect graph theorem with toags

We need some background to establish the themes for this section. Let Γ be a *totally ordered abelian group.* This can be defined in various eqivalent ways: for example, that there is a binary relation "$>$" such that, (a) for every nonzero γ in Γ exactly one of: $\gamma > 0$, $-\gamma > 0$ (alternatively written: $\gamma < 0$), $\gamma = 0$ holds, and (b) $\alpha > 0, \beta > 0$ implies $\alpha + \beta > 0$. We may also say "γ is positive (negative") to mean $\gamma > 0$ (< 0). Familiar examples of totally ordered abelian groups are: the integers, the rationals, the reals; lexicographically ordered vector spaces. A description of all toags (totally ordered abelian groups) is given in a remarkable theorem of Hahn [6].

For any toag Γ, let $Q(\Gamma)$ be its rational extension (there are various ways to define $Q(\Gamma)$, of which the most appealing is to say that the construction of $Q(\Gamma)$ is entirely analogous to the construction of the rational numbers from the integers). Next, consider any polyhedron, in n-dimensional rational space, given by inequalities of the form $Ax \leq b$, where A and b are rational. Let γ be a "vector" in Γ^n. We can consider the primal-dual pair of linear programming problems:

$$\begin{aligned} &\text{maximize } \gamma^T x \text{ such that } Ax \leq b, \\ &\text{minimize } \eta^T b \text{ such that } \eta^T A = \gamma^T,\ \eta \geq 0. \end{aligned} \tag{4}$$

Zimmermann [9] observed that the simplex method applies to these problems (if we extend Γ to $Q(\Gamma)$), hence the duality theorem holds for these problems, if we allow expressions from $Q(\Gamma)$. But perhaps we can find an integral vector x and a vector η with coordinates in Γ itself (eschewing $Q(\Gamma)$) which will verify duality in (4). For the case $\Gamma = \mathbf{Z}$, this issue is a key question in the use of linear programming duality to prove extremal combinatorial theorems. If A and b are integral, suppose it is true that, for every integer vector c such that the problem

$$\text{minimize } y^T b \text{ such that } y^T A = c^T,\ y \geq 0 \tag{5}$$

has an optimum, (5) has an optimum y which is integral. Then (A, b) is called *totally dually integral* (Edmonds and Giles [5]).

Now return to the pluperfect graph theorem. A very interesting proof is given by Chandesekharan and Tamir in [1]. They show that, with the maximal cliques of G indexed in an arbitrary but fixed order, the lexicographically largest optimal solution of (2) is an integral vector. Their proof, and all other proofs known to us, use the ceiling function $\lceil x \rceil$ (or the floor function $\lceil x \rceil$). We will show that, for every vector γ with all coordinates in Γ, the lexicographically largest optimal solution $\overline{\eta}$ — with all coordinates in $Q(\Gamma)$ — of the problem

$$\text{minimize } \sum \eta_i \text{ such that } \eta^T K \geq \gamma^T,\ \ \eta \geq 0 \tag{6}$$

has all coordinates in Γ. (Existence of $\overline{\eta}$ is guaranteed by the simplex method.)

The fact that there is a solution to (6) with all coordinates in Γ follows from a general proposition proved in [7]: if (A, b) is totally dually integral, and γ is any vector all of whose coordinates are in Γ, and such that the optima in (7) exist, then an optimum η exists all of whose coordinates are in Γ. Thus we already know that the pluperfect graph theorem for toags is true, and the issue we are pursuing is whether the Chandrasekaran-Tamir greedy algorithm is correct for toags as well.

We will use induction: assuming that all of $\overline{\eta}_1, \overline{\eta}_2, \ldots, \overline{\eta}_{i-1}$ belong to Γ, we propose to prove that $\overline{\eta}_i$ belongs to Γ. (The induction basis is the case $i = 1$.) For this purpose, note that the vector $\overline{\theta}$ defined by

$$\overline{\theta}_r = \begin{cases} 0 & \text{if } 1 \leq r < i, \\ \overline{\eta}_r & \text{if } i \leq r \end{cases}$$

is the lexicographically largest optimal solution of the problem

$$\text{minimize } \sum \theta_i \text{ such that } \theta^T K \geq \gamma^T - \textstyle\sum_{r=1}^{i-1} \overline{\eta}_r K_r^T, \;\; \theta \geq 0. \tag{7}$$

Let $\overline{x}$ solve the dual problem,

$$\text{maximize } \left(\gamma^T - \textstyle\sum_{r=1}^{i-1} \overline{\eta}_r K_r^T\right) x \quad \text{subject to } x \in P. \tag{8}$$

We may assume that $K_i^T \overline{x} = 1$ (else complementary slackness guarantees that $\overline{\theta}_i = 0$ and we are done since $\overline{\eta}_i = \overline{\theta}_i$ and $0 \in \Gamma$). Next, let $\overline{y}$

$$\text{maximize } \left(\gamma^T - \textstyle\sum_{r=1}^{i-1} \overline{\eta}_r K_r^T\right) y \quad \text{subject to } y \in P, \; K_i^T y = 0$$

and let us write

$$\delta = \left(\gamma^T - \textstyle\sum_{r=1}^{i-1} \overline{\eta}_r K_r^T\right) (\overline{x} - \overline{y});$$

note that $\delta > 0$. Since all the vertices of P are (0,1) vectors, both $\overline{x}$ and $\overline{y}$ are optimal solutions of the problem

$$\text{maximize } \left(\gamma^T - \textstyle\sum_{r=1}^{i-1} \overline{\eta}_r K_r^T - \delta K_i^T\right) z \quad \text{subject to } z \in P; \tag{9}$$

the optimum value of (9) is

$$\left(\gamma^T - \textstyle\sum_{r=1}^{i-1} \overline{\eta}_r K_r^T\right) \overline{y}.$$

Since $\overline{y}$ is an optimal solution of (9), complementary slackness guarantees that the i-th coordinate of every optimal solution of the dual problem,

$$\text{minimize } \sum \xi_i \text{ subject to } \xi^T K \geq \gamma^T - \textstyle\sum_{r=1}^{i-1} \overline{\eta}_r K_r^T - \delta K_i^T, \;\; \xi \geq 0, \tag{10}$$

equals zero. Since $\overline{x}$ is an optimal solution of both (8) and (9), every optimal solution

$$(\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\xi}_{i-1}, 0, \overline{\xi}_{i+1}, \ldots)^T$$

of (10) yields an optimal solution

$$(\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\xi}_{i-1}, \delta, \overline{\xi}_{i+1}, \ldots)^T$$

of (7), and so lexicographic maximality of $\overline{\theta}$ implies $\overline{\xi}_1 = \ldots = \overline{\xi}_{i-1} = 0$ and $\overline{\theta}_i \geq \delta$. Now

$$(\overline{\theta}_1, \overline{\theta}_2, \ldots, \overline{\theta}_{i-1}, \overline{\theta}_i - \delta, \overline{\theta}_{i+1}, \ldots)^T$$

is an optimal solution of (10), and so its i-th coordinate equals zero, which means $\overline{\theta}_i = \delta$; since all vertices of P are (0,1) vectors, δ belongs to Γ.

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