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Line Search Filter Methods for Nonlinear Programming: Local Convergence

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Abstract

A line search method is proposed for nonlinear programming using Fletcher and Leyffer's filter method, which replaces the traditional merit function. Global convergence properties of this method was analyzed in a companion paper. Here a simple modification of the method introducing second order correction steps is presented. It is shown that the proposed method does not suffer from the Maratos effect, so that fast local convergence to strict local solutions is achieved.

Keywords: nonlinear programming – nonconvex constrained optimization – filter method – line search – local convergence – Maratos effect – second order correction

1 Introduction

In this paper we discuss the local convergence properties of the filter algorithm proposed in the companion paper [11]. As mentioned by Fletcher and Leyffer [6], the filter approach can still suffer from the so-called Maratos effect [8], even though it is usually less restrictive in terms of accepting steps than a penalty function approach. The Maratos effect occurs if, even arbitrarily close to a strict local solution of the NLP (1), a full Newton step increases *both* the objective function and the constraint violation, and therefore leads to insufficient progress with respect to the current iterate and is rejected, even though it could be a very good step towards the solution. This can result in poor local convergence behavior. As a remedy, Fletcher and Leyffer propose to improve the search direction, if the full step has been rejected, by means of a second order correction which aims to further reduce infeasibility. In this paper we will show that second order correction steps are indeed able to prevent the Maratos effect.

Recently, Ulbrich has presented a trust region filter method using the Lagrangian function (instead of the objective function) as one of the measures in the filter similar to what we proposed in our companion paper [11], and was able to show fast local convergence without second order correction steps.

Section 2 states the filter line search algorithm with a second order correction step, and the local convergence analysis is presented in Section 3. In Section 4 we briefly discuss how this approach can also be applied to the trust region filter SQP method proposed in [5].

Notation. Norms $\|\cdot\|$ will denote a fixed vector norm and its compatible matrix norm. We will denote by $O(t_k)$ a sequence $\{v_k\}$ satisfying $\|v_k\| \leq \beta\, t_k$ for some constant $\beta > 0$ independent of k, and by $o(t_k)$ a sequence $\{v_k\}$ satisfying $\|v_k\| \leq \beta_k t_k$ for some positive sequence $\{\beta_k\}$ with $\lim_k \beta_k = 0$.

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2 Second Order Correction Steps in a Line Search Filter Method

The presented algorithm is a filter line search algorithm for solving nonlinear optimization problems of the form

$$\min_{x\in\mathbb{R}^n} \quad f(x) \tag{1a}$$

$$\text{s.t.} \quad c(x) = 0 \tag{1b}$$

where the objective function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and the equality constraints $c : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with $m < n$ are sufficiently smooth. The first order optimality (or KKT) conditions of this problem are given by

$$g(x) + A(x)\lambda = 0 \tag{2a}$$

$$c(x) = 0. \tag{2b}$$

with the Lagrangian multipliers λ (see e.g. [10]). The motivation and details of the filter line search can be found in the companion paper [11]. Here, we only restate the formal algorithm, augmented by second order correction steps, which generates the sequence of iterates $\{x_k\}$. We will make use of the following definitions:

$$\theta(x) := \|c(x)\|, \quad g_k := \nabla f(x_k), \quad c_k := c(x_k), \quad A_k := \nabla c(x_k),$$

and H_k will be (an approximation of) the Hessian of the Lagrangian function

$$\mathcal{L}(x,\lambda) := f(x) + c(x)^T\lambda \tag{3}$$

associated with (1) at x_k, assumed to be positive definite in the null space of the constraint Jacobian A_k^T.

Algorithm SOC

Given: Starting point x_0; constants $\theta_{\max} \in (\theta(x_0), \infty]$; $\gamma_\theta, \gamma_f \in (0,1)$; $\delta > 0$; $\gamma_\alpha \in (0,1]$; $s_\theta > 1$; $s_f > 2s_\theta$; $0 < \tau_1 \leq \tau_2 < 1$.

1. *Initialize.* Initialize the filter $\mathcal{F}_0 := \{(\theta, f) \in \mathbb{R}^2 : \theta \geq \theta_{\max}\}$ and the iteration counter $k \leftarrow 0$.

2. *Check convergence.* Stop, if x_k is a local solution (or at least stationary point) of the NLP (1), i.e. if it satisfies the KKT conditions (2) for some $\lambda \in \mathbb{R}^m$.

3. *Compute search direction.* Compute the search direction d_k from the linear system

$$\begin{bmatrix} H_k & A_k \\ A_k^T & 0 \end{bmatrix} \begin{pmatrix} d_k \\ \lambda_k^+ \end{pmatrix} = -\begin{pmatrix} g_k \\ c_k \end{pmatrix}. \tag{4}$$

   If this system is (almost) singular, go to feasibility restoration phase in Step 8.

4. *Backtracking line search.*

   4.1. *Initialize line search.* Set $\alpha_{k,0} = 1$ and $l \leftarrow 0$.

4.2. *Compute new trial point.* If the trial step size becomes too small, i.e. $\alpha_{k,l} < \alpha_k^{\min}$ with

$$\alpha_k^{\min} := \gamma_\alpha \cdot \begin{cases} \min\left\{\gamma_\theta, \frac{\gamma_f \theta(x_k)}{-g_k^T d_k}, \frac{\delta[\theta(x_k)]^{s_\theta}}{[-g_k^T d_k]^{s_f}}\right\} & \text{if } g_k^T d_k < 0 \\ \gamma_\theta & \text{otherwise,} \end{cases} \tag{5}$$

go to the feasibility restoration phase in Step 8. Otherwise, compute the new trial point $x_k(\alpha_{k,l}) := x_k + \alpha_{k,l} d_k$.

4.3. *Check acceptability to the filter.* If $x_k(\alpha_{k,l}) \in \mathcal{F}_k$, reject the trial step size and go to Step 4.5.

4.4. *Check sufficient decrease with respect to current iterate.*

4.4.1. *Case I. The switching condition*

$$g_k^T d_k < 0 \qquad \textit{and} \qquad \alpha_{k,l}\,[-g_k^T d_k]^{s_\varphi} > \delta\,[\theta(x_k)]^{s_\theta} \tag{6}$$

holds: If the Armijo condition for the objective function

$$f(x_k(\alpha_{k,l})) \leq f(x_k) + \eta_f\ \alpha_{k,l}\ g_k^T d_k, \tag{7}$$

holds, accept the trial step $x_{k+1} := x_k(\alpha_{k,l})$ and go to Step 5. Otherwise, go to Step 4.5.

4.4.2. *Case II. The switching condition (6) is not satisfied:* If the sufficient decrease conditions

$$\theta(x_k(\alpha_{k,l})) \leq (1-\gamma_\theta)\theta(x_k) \tag{8a}$$

$$\text{or} \qquad f(x_k(\alpha_{k,l})) \leq f(x_k) - \gamma_f \theta(x_k). \tag{8b}$$

hold, accept the trial step $x_{k+1} := x_k(\alpha_{k,l})$ and go to Step 5. Otherwise, go to Step 4.5.

4.5. *Compute second order correction step.* If $l \neq 0$, go to step 4.8. Otherwise, solve the linear system

$$\begin{bmatrix} H_k^{\text{soc}} & A_k^{\text{soc}} \\ (A_k^{\text{soc}})^T & 0 \end{bmatrix} \begin{pmatrix} d_k^{\text{soc}} \\ \lambda_k^{\text{soc}} \end{pmatrix} = -\begin{pmatrix} g_k^{\text{soc}} \\ c(x_k + d_k) + c_k^{\text{soc}} \end{pmatrix}, \tag{9}$$

(particular admissible choices of $H_k^{\text{soc}}, A_k^{\text{soc}}, g_k^{\text{soc}}, c_k^{\text{soc}}$ are discussed below) to obtain the second order correction step d_k^{soc} and define

$$\bar{x}_{k+1} := x_k + d_k + d_k^{\text{soc}}.$$

4.6. *Check acceptability to the filter.* If $\bar{x}_{k+1} \in \mathcal{F}_k$, reject the second order correction step and go to Step 4.8.

4.7. *Check sufficient decrease with respect to current iterate.*

4.7.1. *Case I. The switching condition (6) holds:* If the Armijo condition for the objective function

$$f(\bar{x}_{k+1}) \leq f(x_k) + \eta_f\ g_k^T d_k \tag{10}$$

holds, accept $x_{k+1} := \bar{x}_{k+1}$ and go to Step 5. Otherwise, go to Step 4.8.

4.7.2. *Case II. The switching condition (6) is not satisfied:* If

$$\begin{aligned} \theta(\bar{x}_{k+1}) &\leq (1-\gamma_\theta)\theta(x_k) && \text{(11a)} \\ \text{or} \quad f(\bar{x}_{k+1}) &\leq f(x_k) - \gamma_f \theta(x_k) && \text{(11b)} \end{aligned}$$

hold, accept $x_{k+1} := \bar{x}_{k+1}$ and go to Step 5. Otherwise, go to Step 4.8.

4.8. *Choose new trial step size.* Choose $\alpha_{k,l+1} \in [\tau_1 \alpha_{k,l}, \tau_2 \alpha_{k,l}]$, set $l \leftarrow l+1$, and go back to Step 4.2.

5. *Accept trial point.* Set $\alpha_k := \alpha_{k,l}$.

6. *Augment filter if necessary.* If one of the conditions (6) or

$$f(x_{k+1}) \leq f(x_k) + \eta_f \alpha_k g_k^T d_k$$

does not hold, augment the filter according to

$$\mathcal{F}_{k+1} := \mathcal{F}_k \cup \left\{(\theta, f) \in \mathbb{R}^2 : \theta \geq (1-\gamma_\theta)\theta(x_k) \quad \text{and} \quad f \geq f(x_k) - \gamma_f \theta(x_k)\right\}; \qquad (12)$$

otherwise leave the filter unchanged, i.e. set $\mathcal{F}_{k+1} := \mathcal{F}_k$.

7. *Continue with next iteration.* Increase the iteration counter $k \leftarrow k+1$ and go back to Step 2.

8. *Feasibility restoration phase.* Compute a new iterate x_{k+1} by decreasing the infeasibility measure θ, so that x_{k+1} satisfies the sufficient decrease conditions (8) and is acceptable to the filter, i.e. $(\theta(x_{k+1}), f(x_{k+1})) \notin \mathcal{F}_k$. Augment the filter according to (12) (for x_k) and continue with the regular iteration in Step 7.

It can be verified easily that this modification of Algorithm I in the companion paper [11] does not affect the global convergence properties proven in [11].

Second order correction steps of the form (9) are discussed by Conn, Gould, and Toint in [3, Section 15.3.2.3]. Here we assume that H_k^{soc} is uniformly positive definite on the null space of $(A_k^{\text{soc}})^T$, and that in a neighborhood of a strict local solution we have

$$g_k^{\text{soc}} = o(\|d_k\|), \qquad A_k - A_k^{\text{soc}} = O(\|d_k\|), \qquad c_k^{\text{soc}} = o(\|d_k\|^2). \qquad (13)$$

In [3], the analysis is made for the particular choices $c_k^{\text{soc}} = 0$, $A_k^{\text{soc}} = A(x_k + p_k)$ for some $p_k = O(\|d_k\|)$, and $H_k = \nabla^2_{xx}\mathcal{L}_\mu(x_k, \lambda_k)$ in (4) for multiplier estimates λ_k. However, the careful reader will be able to verify that the results that we will use from [3] still hold as long as

$$(W_k - H_k)d_k = o(\|d_k\|), \qquad (14)$$

if x_k converges to a strict local solution x_* of the NLP with corresponding multipliers λ_*, where

$$W_k = \nabla^2_{xx}\mathcal{L}(x_k, \lambda_*) \overset{(3)}{=} \nabla^2 f(x_k) + \sum_{i=1}^m (\lambda_*)^{(i)} \nabla^2 c^{(i)}(x_k). \qquad (15)$$

Popular choices for the quantities in the computation of the second order correction step in (9) that satisfy (13) are the following.

(a) $H_k^{\text{soc}} = I$, $g_k^{\text{soc}} = 0$, $c_k^{\text{soc}} = 0$, and $A_k^{\text{soc}} = A_k$ or $A_k^{\text{soc}} = A(x_k + d_k)$, which corresponds to a least-square step for the constraints.

(b) $H_k^{\rm soc} = H_k$, $g_k^{\rm soc} = 0$, $c_k^{\rm soc} = 0$, and $A_k^{\rm soc} = A_k$, which is very inexpensive since this choice allows to reuse the factorization of the linear system (4).

(c) $H_k^{\rm soc}$ being the Hessian approximation corresponding to $x_k + d_k$, $g_k^{\rm soc} = g(x_k + d_k) + A(x_k + d_k)^T \lambda_k^+$, $c_k^{\rm soc} = 0$, and $A_k^{\rm soc} = A(x_k + d_k)$ which corresponds to the step in the next iteration, supposing that $x_k + d_k$ has been accepted. This choice has the flavor of the watchdog technique [2].

(d) If $d_k^{\rm soc}$ is a second order correction step, and $\bar{d}_k^{\rm soc}$ is an additional second order correction step (i.e. with "$c(x_k + d_k)$" replaced by "$c(x_k + d_k + d_k^{\rm soc})$" in (9)), then $d_k^{\rm soc} + \bar{d}_k^{\rm soc}$ can be understood as a single second order correction step for d_k (in that case with $c_k^{\rm soc} \neq 0$). Similarly, several consecutive correction steps can be considered as a single one.

3 Local Convergence Analysis

We start the analysis by stating the necessary assumptions.

Assumptions L. *Assume that $\{x_k\}$ converges to a local solution x_* of the NLP (1) and that the following holds.*

(L1) *The functions f and c are twice continuously differentiable in a neighborhood of x_*.*

(L2) *x_* satisfies the following sufficient second order optimality conditions.*

- *x_* is feasible, i.e. $\theta(x_*) = 0$,*
- *there exists $\lambda_* \in \mathbb{R}^m$ so that the KKT conditions (2) are satisfied for (x_*, λ_*),*
- *the constraint Jacobian $A(x_*)^T$ has full rank, and*
- *the Hessian of the Lagrangian $W_* = \nabla^2_{xx}\mathcal{L}(x_*, \lambda_*)$ is positive definite on the null space of $A(x_*)^T$.*

(L3) *In (4), H_k is uniformly positive definite on the null space of $(A_k)^T$, as well as bounded.*

(L4) *In (9), $H_k^{\rm soc}$ is uniformly positive definite on the null space of $(A_k^{\rm soc})^T$, and (13) holds.*

(L5) *The Hessian approximations H_k in (4) satisfy (14).*

The assumption $x_k \to x_*$ has been discussed in Remark 6 in the companion paper [11] and can be considered to be reasonable. Assumption (L5) is reminiscent of the Dennis-Moré characterization of superlinear convergence [4]. However, this assumption is stronger than necessary for superlinear convergence [1] which requires only that $Z_k^T(W_k - H_k)d_k = o(\|d_k\|)$, where Z_k is a null space matrix for A_k^T.

Note, that the above assumptions imply Assumptions G in the companion paper [11] in a neighborhood of the solution, and therefore the results from [11] remain valid. In particular, from Lemma 1 in [11] we have, that λ_k^+ from (4) is uniformly bounded, and Lemma 4 in [11] implies

$$\theta(x_k) = 0 \quad \Longrightarrow \quad g_k^T d_k < 0 \qquad \text{and} \tag{16}$$

$$\Theta_k := \min\{\theta : (\theta, f) \in \mathcal{F}_k\} > 0 \tag{17}$$

for all k.

First we summarize some preliminary results.

Lemma 1 *Suppose Assumptions L hold. Then there exists a neighborhood U_1 of x_*, so that for all $x_k \in U_1$ we have*

$$
\begin{aligned}
d_k^{\rm soc} &= o(\|d_k\|) &\text{(18a)}\\
c(x_k + d_k + d_k^{\rm soc}) &= o(\|d_k\|^2) &\text{(18b)}
\end{aligned}
$$

Proof. From continuity and full rank of A_*^T, as well as Assumption (L4), we have that the matrix in (9) has a uniformly bounded inverse in the neighborhood of x_*. Hence, since the right hand side is $o(\|d_k\|)$, claim (18a) follows. Furthermore, from

$$
\begin{aligned}
c(x_k + d_k + d_k^{\rm soc}) &= c(x_k + d_k) + A(x_k + d_k)^T d_k^{\rm soc} + O(\|d_k^{\rm soc}\|^2)\\
&\overset{(9)}{=} -c_k^{\rm soc} - (A_k^{\rm soc})^T d_k^{\rm soc} + (A_k + O(\|d_k\|))^T d_k^{\rm soc}\\
&\qquad + O(\|d_k^{\rm soc}\|^2)\\
&\overset{(13)}{=} o(\|d_k\|^2) + O(\|d_k\|\|d_k^{\rm soc}\|) + O(\|d_k^{\rm soc}\|^2)\\
&\overset{(18a)}{=} o(\|d_k\|^2)
\end{aligned}
$$

for x_k close to x_* the claim (18b) follows. □

In order to prove our local convergence result we will make use of two results established in [3] regarding the effect of second order correction steps on the exact penalty function

$$
\phi_\rho(x) = f(x) + \rho\, \theta(x). \tag{19}
$$

Note, that we will employ the exact penalty function only as a technical device, but the algorithm never refers to it. We will also use the following model of the penalty function

$$
q_\rho(x_k, d) = f(x_k) + g_k^T d + \frac{1}{2} d^T H_k d + \rho \left\| A_k^T d + c_k \right\|. \tag{20}
$$

The first result follows from Theorem 15.3.7 in [3].

Lemma 2 *Suppose Assumptions L hold. Let ϕ_ρ be the exact penalty function (19) and q_ρ defined by (20) with $\rho > \|\lambda_*\|_D$, where $\|\cdot\|_D$ is the dual norm to $\|\cdot\|$. Then,*

$$
\lim_{k\to\infty} \frac{\phi_\rho(x_k) - \phi_\rho(x_k + d_k + d_k^{\rm soc})}{q_\rho(x_k, 0) - q_\rho(x_k, d_k)} = 1. \tag{21}
$$

The next result follows from Theorem 15.3.2 in [3].

Lemma 3 *Suppose Assumptions L hold. Let (d_k, λ_k^+) be a solution of the linear system (4), and let $\rho > \|\lambda_k^+\|_D$. Then*

$$
q_\rho(x_k, 0) - q_\rho(x_k, d_k) \geq 0. \tag{22}
$$

The next lemma shows that in a neighborhood of x_* Step 4.7.1 of Algorithm SOC will be successful if the switching condition (6) holds.

Lemma 4 *Suppose Assumptions L hold. Then there exists a neighborhood $U_2 \subseteq U_1$ of x_* so that whenever the switching condition (6) holds, the Armijo condition (10) is satisfied.*

Proof. Choose U_1 to be the neighborhood from Lemma 1. It then follows that for $x_k \in U_1$ satisfying (6) that

$$\theta(x_k) < \delta^{-\frac{1}{s_\theta}}[-g_k^T d_k]^{\frac{s_f}{s_\theta}} = O(\|d_k\|^{\frac{s_f}{s_\theta}}) = o(\|d_k\|^2), \tag{23}$$

since $\frac{s_f}{s_\theta} > 2$ and g_k is uniformly bounded in U_1.

Since $\eta_f < \frac{1}{2}$, Lemma 2 and (22) imply that there exists $K \in \mathbb{N}$ such that for all $k \geq K$ we have for some constant $\rho > 0$ with $\rho > \|\lambda_k^+\|_D$ independent of k that

$$\phi_\rho(x_k) - \phi_\rho(x_k + d_k + d_k^{\mathrm{soc}}) \geq \left(\frac{1}{2} + \eta_f\right)(q_\rho(x_k, 0) - q_\rho(x_k, d_k)). \tag{24}$$

We then have

$$\begin{aligned}
& f(x_k) - f(x_k + d_k + d_k^{\mathrm{soc}}) \\
\overset{(19)}{=} \quad & \phi_\rho(x_k) - \phi_\rho(x_k + d_k + d_k^{\mathrm{soc}}) - \rho\left(\theta(x_k) - \theta(x_k + d_k + d_k^{\mathrm{soc}})\right) \\
\overset{(24),(18b),(23)}{\geq} \quad & \left(\frac{1}{2} + \eta_f\right)(q_\rho(x_k, 0) - q_\rho(x_k, d_k)) + o(\|d_k\|^2) \\
\overset{(20),(23)}{=} \quad & -\left(\frac{1}{2} + \eta_f\right)\left(g_k^T d_k + \frac{1}{2} d_k^T H_k d_k\right) + o(\|d_k\|^2).
\end{aligned}$$

Before continuing, we recall the decomposition from the companion paper [11]

$$\begin{aligned}
d_k &= q_k + p_k, && \text{(25a)} \\
q_k &:= Y_k \bar{q}_k \quad \text{and} \quad p_k := Z_k \bar{p}_k, && \text{(25b)} \\
\bar{q}_k &:= -\left[A_k^T Y_k\right]^{-1} c_k && \text{(25c)} \\
\bar{p}_k &:= -\left[Z_k^T H_k Z_k\right]^{-1} Z_k^T (g_k + H_k q_k) && \text{(25d)}
\end{aligned}$$

where $Z_k \in \mathbb{R}^{n \times (n-m)}$ and $Y_k \in \mathbb{R}^{n \times m}$ are matrices so that the columns of $[Z_k \; Y_k]$ form an orthonormal basis of $\mathbb{R}^n$, and the columns of Z_k are a basis of the null space of A_k^T. Since Assumptions L guarantee that the quantities (25), as well as λ_k^+, are bounded, we can conclude

$$\begin{aligned}
& f(x_k) + \eta_f g_k^T d_k - f(x_k + d_k + d_k^{\mathrm{soc}}) \\
\geq \quad & -\frac{1}{2} g_k^T d_k - \left(\frac{1}{4} + \frac{\eta_f}{2}\right) d_k^T H_k d_k + o(\|d_k\|^2) \\
\overset{(4)}{=} \quad & \frac{1}{2}\left(d_k^T H_k d_k + d_k^T A_k \lambda_k^+\right) - \left(\frac{1}{4} + \frac{\eta_f}{2}\right) d_k^T H_k d_k + o(\|d_k\|^2) \\
\overset{(4)}{=} \quad & \left(\frac{1}{4} - \frac{\eta_f}{2}\right) d_k^T H_k d_k - \frac{1}{2} c(x_k)^T \lambda_k^+ + o(\|d_k\|^2) \\
\overset{(23)}{=} \quad & \left(\frac{1}{4} - \frac{\eta_f}{2}\right) d_k^T H_k d_k + o(\|d_k\|^2) \\
\overset{(25)}{=} \quad & \left(\frac{1}{4} - \frac{\eta_f}{2}\right) \bar{p}_k^T Z_k^T H_k Z_k \bar{p}_k + O(\|q_k\|) + o(\|d_k\|^2).
\end{aligned} \tag{26}$$

Finally, using repeatedly the orthonormality of $[Z_k \; Y_k]$, we have

$$\begin{aligned}
q_k \quad &= \quad O(\bar{q}_k) \overset{(25c)}{=} O(\theta(x_k)) \overset{(23)}{=} o(\|d_k\|^2) \\
&\overset{(25a)}{=} \quad o(p_k^T p_k + q_k^T q_k) \overset{(25b)}{=} o(\|\bar{p}_k\|^2) + o(\|q_k\|^2)
\end{aligned}$$

and therefore $q_k = o(\|\bar{p}_k\|^2)$, as well as

$$d_k \stackrel{(25a)}{=} O(\|q_k\|) + O(\|p_k\|) \stackrel{(25b)}{=} o(\|\bar{p}_k\|) + O(\|\bar{p}_k\|) = O(\|\bar{p}_k\|).$$

Hence, (10) is implied by (26), Assumption (L3) and $\eta_f < \frac{1}{2}$, if x_k is sufficiently close to x_*. □

It remains to show that also the filter and the sufficient reduction criterion (8) do not interfere with the acceptance of full steps close to x_*. The following technical lemmas address this issue and prepare the proof of the main local convergence theorem.

Lemma 5 *Suppose Assumptions L hold. Then there exists a neighborhood $U_3 \subseteq U_2$ (with U_2 from Lemma 4) and constants $\rho_1, \rho_2, \rho_3 > 0$ with*

$$\begin{aligned} \rho_3 &= (1-\gamma_\theta)\rho_2 - \gamma_f && (27a)\\ 2\gamma_\theta\rho_2 &< (1+\gamma_\theta)(\rho_2-\rho_1) - 2\gamma_f && (27b)\\ 2\rho_3 &\geq (1+\gamma_\theta)\rho_1 + (1-\gamma_\theta)\rho_2, && (27c) \end{aligned}$$

so that for all $x_k \in U_3$ we have $\|\lambda_k^+\|_D < \rho_i$ for $i = 1,2,3$. Furthermore, for all $x_k \in U_3$ we have

$$\phi_{\rho_i}(x_k) - \phi_{\rho_i}(x_k + d_k + \bar{d}_k^{\text{soc}}) \geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_i}(x_k,0) - q_{\rho_i}(x_k,d_k)\right) \stackrel{(22)}{\geq} 0 \qquad (28)$$

for $i = 2,3$ and all choices

$$\begin{aligned} \bar{d}_k^{\text{soc}} &= d_k^{\text{soc}}, && (29a)\\ \bar{d}_k^{\text{soc}} &= \sigma_k d_k^{\text{soc}} + d_{k+1} + \sigma_{k+1} d_{k+1}^{\text{soc}}, && (29b)\\ \bar{d}_k^{\text{soc}} &= \sigma_k d_k^{\text{soc}} + d_{k+1} + \sigma_{k+1} d_{k+1}^{\text{soc}} + d_{k+2} + \sigma_{k+2} d_{k+2}^{\text{soc}}, && (29c)\\ \text{or} \quad \bar{d}_k^{\text{soc}} &= \sigma_k d_k^{\text{soc}} + d_{k+1} + \sigma_{k+1} d_{k+1}^{\text{soc}} + d_{k+2} + \sigma_{k+2} d_{k+2}^{\text{soc}} \\ &\quad + d_{k+3} + \sigma_{k+3} d_{k+3}^{\text{soc}}, && (29d) \end{aligned}$$

with $\sigma_k, \sigma_{k+1}, \sigma_{k+2}, \sigma_{k+3} \in \{0,1\}$, as long as $x_{l+1} = x_l + d_l + \sigma_l d_k^{\text{soc}}$ for $l \in \{k, \ldots, k+j\}$ with $j \in \{-1,0,1,2\}$, respectively.

Proof. Since λ_k^+ is uniformly bounded for all k with $x_k \in U_2$, we can find $\rho_1 > \|\lambda_*\|_D$ with

$$\rho_1 > \|\lambda_k^+\|_D \qquad (30)$$

for all k with $x_k \in U_2$. Defining now

$$\rho_2 := \frac{1+\gamma_\theta}{1-\gamma_\theta}\rho_1 + \frac{3\gamma_f}{1-\gamma_\theta}$$

and ρ_3 by (27a), it is then easy to verify that $\rho_2, \rho_3 \geq \rho_1 > \|\lambda_k^+\|_D$ and that (27b) and (27c) hold. Since $(1+\gamma_\theta) < 2$, Lemma 2 implies that there exists a neighborhood $U_3 \subseteq U_2$ of x_*, so that (28) holds for $x_k \in U_3$, since according to (c) and (d) on page 5 all choices of $\bar{d}_k^{\text{soc}}$ in (29) can be understood as second order correction steps to d_k. □

Before proceeding we will give a short graphical motivation of the remainder of the proof and introduce some more notation.

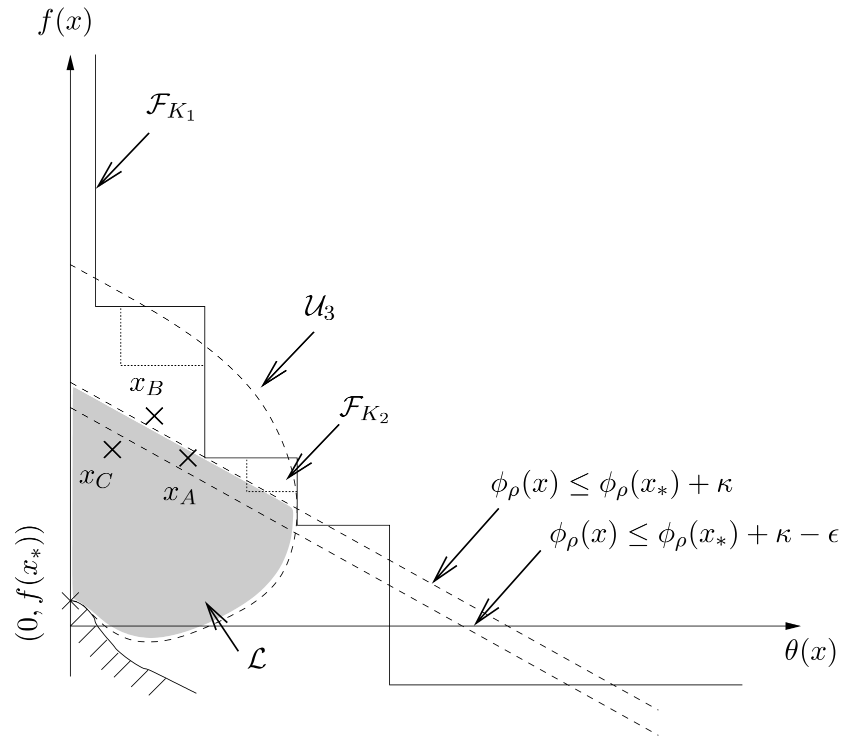

Figure 1: Basic idea of proof

Let U_3 and ρ_i be the neighborhood and constants from Lemma 5. Since $\lim_k x_k = x_*$, we can find $K_1 \in \mathbb{N}$ so that $x_k \in U_3$ for all $k \geq K_1$. In Figure 1 we see the (θ, f) half-plane with the current filter $\mathcal{F}_{K_1}$. Let us now define the level set

$$L := \{x \in U_3 : \phi_{\rho_3}(x) \leq \phi_{\rho_3}(x_*) + \kappa\}, \tag{31}$$

where $\kappa > 0$ is chosen so that for all $x \in L$ we have $(\theta(x), f(x)) \notin \mathcal{F}_{K_1}$. This is possible, since $\Theta_{K_1} > 0$ from (17), and since $\max\{\theta(x) : x \in L\}$ converges to zero as $\kappa \to 0$, because x_* is a strict local minimizer of ϕ_{ρ_3} [7]. Obviously, $x_* \in L$.

Let now K_2 be the first iteration $K_2 \geq K_1$ with $x_{K_2} \in L$. This means, that no iterate after K_1 and before K_2 will have been in L, and therefore also the filter $\mathcal{F}_{K_2}$ will only overlap with the region $\mathcal{L}$ corresponding to L in the (θ, f) graph by small corners of size $\gamma_\theta\theta(x_l) \times \gamma_f\theta(x_l)$. (θ, f)-pairs with constant value of the exact penalty function (19) correspond to straight (dashed) lines in the diagram, whose slope is determined by the penalty parameter ρ. The main trick of the proof will be to understand those straight lines as frontiers approaching $(0, f(x_*))$, so that the filter will always lie to the upper right side (except for small blocks coming from the sufficient decrease condition (12) in the filter update rule) of the lines, and at least every other iterate will lie on the lower left side (see (28)). For technical reasons we have to consider three of those frontiers corresponding to different values of the penalty parameter (in order to deal with sufficient descent with respect to the old filter entries, the current iterate (8), and new filter entries).

Let us finally define for $k \in \mathbb{N}$ the filter building blocks

$$\mathcal{G}_k := \Big\{(\theta, f) : \theta \geq (1 - \gamma_\theta)\theta(x_k) \quad \text{and} \quad f \geq f(x_k) - \gamma_f\theta(x_k)\Big\}$$

and index sets $I_{k_1}^{k_2} := \{l = k_1, \ldots, k_2 - 1 : l \in \mathcal{A}\}$ for $k_1 \le k_2$. Then it follows from the filter update rule (12) and the definition of $\mathcal{A}$ that for $k_1 \le k_2$

$$\mathcal{F}_{k_2} = \mathcal{F}_{k_1} \cup \bigcup_{l \in I_{k_1}^{k_2}} \mathcal{G}_l. \tag{32}$$

Also note, that $l \in I_{k_1}^{k_2} \subseteq \mathcal{A}$ implies $\theta(x_l) > 0$. Otherwise, we would have from (16) that $g_k^T d_k < 0$, so that (6) holds for all trial step sizes α, and the step must have been accepted in Step 4.4.1 or Step 4.7.1, hence satisfying (7) or (10). This would contradict the filter update condition in Step 6, respectively.

The last lemma will enable us to show in the main theorem of this section that, once the iterates have reached the level set L, the full step will always be acceptable to the current filter.

Lemma 6 *Suppose Assumptions L hold and let $l \ge K_1$ with $\theta(x_l) > 0$. Then the following statements hold.*

$$\left.\begin{array}{l} \textit{If } \phi_{\rho_2}(x_l) - \phi_{\rho_2}(x) \ge \frac{1+\gamma_\theta}{2}\left(q_{\rho_2}(x_l, 0) - q_{\rho_2}(x_l, d_l)\right), \\ \textit{then } (\theta(x), f(x)) \notin \mathcal{G}_l. \end{array}\right\} \tag{33}$$

$$\left.\begin{array}{l} \textit{If } x \in L \textit{ and } \phi_{\rho_2}(x_{K_2}) - \phi_{\rho_2}(x) \ge \frac{1+\gamma_\theta}{2}\left(q_{\rho_2}(x_{K_2}, 0) - q_{\rho_2}(x_{K_2}, d_{K_2})\right), \\ \textit{then } (\theta(x), f(x)) \notin \mathcal{F}_{K_2}. \end{array}\right\} \tag{34}$$

Proof. To (33): Since $\rho_1 > \|\lambda_l^+\|_D$ we have from Lemma 3 that $q_{\rho_1}(x_l, 0) - q_{\rho_1}(x_l, d_l) \ge 0$, and hence using definition for q_ρ (20) and $A_l^T d_l + c_l = 0$ (from (4)) that

$$\begin{aligned} \phi_{\rho_2}(x_l) - \phi_{\rho_2}(x) &\ge \frac{1+\gamma_\theta}{2}\left(q_{\rho_2}(x_l, 0) - q_{\rho_2}(x_l, d_l)\right) \\ &= \frac{1+\gamma_\theta}{2}\left(q_{\rho_1}(x_l, 0) - q_{\rho_1}(x_l, d_l) + (\rho_2 - \rho_1)\theta(x_l)\right) \\ &\ge \frac{1+\gamma_\theta}{2}(\rho_2 - \rho_1)\theta(x_l). \end{aligned} \tag{35}$$

If $f(x) < f(x_l) - \gamma_f \theta(x_l)$, the claim follows immediately. Otherwise, suppose that $f(x) \ge f(x_l) - \gamma_f \theta(x_l)$. In that case, we have together with $\theta(x_l) > 0$ that

$$\begin{aligned} \theta(x_l) - \theta(x) &\overset{(35),(19)}{\ge} \frac{1+\gamma_\theta}{2\rho_2}(\rho_2 - \rho_1)\theta(x_l) + \frac{1}{\rho_2}\left(f(x) - f(x_l)\right) \\ &\ge \frac{1+\gamma_\theta}{2\rho_2}(\rho_2 - \rho_1)\theta(x_l) - \frac{\gamma_f}{\rho_2}\theta(x_l) \\ &\overset{(27b)}{>} \gamma_\theta \theta(x_l), \end{aligned}$$

so that $(\theta(x), f(x)) \notin \mathcal{G}_l$.

To (34): Since $x \in L$, it follows by the choice of κ that $(\theta(x), f(x)) \notin \mathcal{F}_{K_1}$. Thus, according to (32) it remains to show that for all $l \in I_{K_1}^{K_2}$ we have $(\theta(x), f(x)) \notin \mathcal{G}_l$. Choose $l \in I_{K_1}^{K_2}$. As in (35) we can show that

$$\phi_{\rho_2}(x_{K_2}) - \phi_{\rho_2}(x) \ge \frac{1+\gamma_\theta}{2}(\rho_2 - \rho_1)\theta(x_{K_2}). \tag{36}$$

Since $x \in L$ it follows from the definition of K_2 (as the first iterate after K_1 with $x_{K_2} \in L$) and the fact that $l < K_2$ that

$$\begin{aligned}
\phi_{\rho_3}(x_l) &\overset{(31)}{>} \phi_{\rho_3}(x_{K_2}) \overset{(19)}{=} \phi_{\rho_2}(x_{K_2}) + (\rho_3 - \rho_2)\theta(x_{K_2}) \\
&\overset{(36)}{\geq} \phi_{\rho_2}(x) + \left(\rho_3 - \frac{1+\gamma_\theta}{2}\rho_1 - \frac{1-\gamma_\theta}{2}\rho_2\right)\theta(x_{K_2}) \\
&\overset{(27c)}{\geq} \phi_{\rho_2}(x).
\end{aligned} \tag{37}$$

If $f(x) < f(x_l) - \gamma_f\theta(x_l)$, we immediately have $(\theta(x), f(x)) \notin \mathcal{G}_l$. Otherwise we have $f(x) \geq f(x_l) - \gamma_f\theta(x_l)$ which yields

$$\begin{aligned}
\theta(x) &\overset{(37),(19)}{<} \frac{1}{\rho_2}\left(f(x_l) + \rho_3\theta(x_l) - f(x)\right) \\
&\leq \frac{\rho_3 + \gamma_f}{\rho_2}\theta(x_l) \\
&\overset{(27a)}{=} (1-\gamma_\theta)\theta(x_l),
\end{aligned}$$

so that $(\theta(x), f(x)) \notin \mathcal{G}_l$ which concludes the proof of (34). □

After these preparations we are finally able to show the main local convergence theorem.

Theorem 1 *Suppose Assumptions L hold. Then, for k sufficiently large full steps of the form* $x_{k+1} = x_k + d_k$ *or* $x_{k+1} = x_k + d_k + d_k^{\text{soc}}$ *will be taken, and* x_k *converges to* x_* *superlinearly.*

Proof. Recall that $K_2 \geq K_1$ is the first iteration after K_1 with $x_{K_2} \in L \subseteq U_3$. We now show by induction that the following statements are true for $k \geq K_2 + 2$:

$$(\text{i}_k) \qquad \phi_{\rho_i}(x_l) - \phi_{\rho_i}(x_k) \geq \frac{1+\gamma_\theta}{2}\left(q_{\rho_i}(x_l, 0) - q_{\rho_i}(x_l, d_l)\right) \qquad \text{for } i \in \{2,3\} \text{ and } K_2 \leq l \leq k-2$$

$$(\text{ii}_k) \qquad x_k \in L$$

$$(\text{iii}_k) \qquad x_k = x_{k-1} + d_{k-1} + \sigma_{k-1}d_{k-1}^{\text{soc}} \qquad \text{with } \sigma_{k-1} \in \{0,1\}.$$

We start by showing that these statements are true for $k = K_2 + 2$.

Suppose, the point $x_{K_2} + d_{K_2}$ is not accepted by the line search. In that case, define $\bar{x}_{K_2+1} := x_{K_2} + d_{K_2} + d_{K_2}^{\text{soc}}$. Then, from (28) with $i = 3$, $k = K_2$, and (29a), we see from $x_{K_2} \in L$ and the definition of L that $\bar{x}_{K_2+1} \in L$. After applying (28) again with $i = 2$ it follows from (34) that $(\theta(\bar{x}_{K_2+1}), f(\bar{x}_{K_2+1})) \notin \mathcal{F}_{K_2}$, i.e. $\bar{x}_{K_2+1}$ is not rejected in Step 4.6. Furthermore, if the switching condition (6) holds, we see from Lemma 4 that the Armijo condition (10) with $k = K_2$ is satisfied for the point $\bar{x}_{K_2+1}$. In the other case, i.e. if (6) is violated (note that then (16) and (6) imply $\theta(x_{K_2}) > 0$), we see from (28) for $i = 2$, $k = K_2$, and (29a), together with (33) for $l = K_2$, that (11) holds. Hence, $\bar{x}_{K_2+1}$ is also not rejected in Step 4.7 and accepted as next iterate. Summarizing the discussion in this paragraph we can write $x_{K_2+1} = x_{K_2} + d_{K_2} + \sigma_{K_2}d_{K_2}^{\text{soc}}$ with $\sigma_{K_2} \in \{0,1\}$.

Let us now consider iteration $K_2 + 1$. For $\sigma_{K_2+1} \in \{0,1\}$ we have from (28) for $k = K_2$ and (29b) that

$$\begin{aligned}
&\phi_{\rho_i}(x_{K_2}) - \phi_{\rho_i}(x_{K_2+1} + d_{K_2+1} + \sigma_{K_2+1}d_{K_2+1}^{\text{soc}}) \\
\geq\; &\frac{1+\gamma_\theta}{2}\left(q_{\rho_i}(x_{K_2}, 0) - q_{\rho_i}(x_{K_2}, d_{K_2})\right)
\end{aligned} \tag{38}$$

for $i = 2, 3$, which yields

$$x_{K_2+1} + d_{K_2+1} + \sigma_{K_2+1} d^{\rm soc}_{K_2+1} \in L. \qquad (39)$$

If $x_{K_2+1} + d_{K_2+1}$ is accepted as next iterate x_{K_2+2}, we immediately obtain from (38) and (39) that (i_{K_2+2})–(iii_{K_2+2}) hold. Otherwise, we consider the case $\sigma_{K_2+1} = 1$. From (38), (39), and (34) we have for $\bar{x}_{K_2+2} := x_{K_2+1} + d_{K_2+1} + d^{\rm soc}_{K_2+1}$ that $(\theta(\bar{x}_{K_2+2}), f(\bar{x}_{K_2+2})) \notin \mathcal{F}_{K_2}$. If $K_2 \notin I^{K_2+1}_{K_2}$ it immediately follows from (32) that $(\theta(\bar{x}_{K_2+2}), f(\bar{x}_{K_2+2})) \notin \mathcal{F}_{K_2+1}$. Otherwise, we have $\theta(x_{K_2}) > 0$. Then, (38) for $i = 2$ together with (33) implies $(\theta(\bar{x}_{K_2+2}), f(\bar{x}_{K_2+2})) \notin \mathcal{G}_{K_2}$, and hence with (32) we have $(\theta(\bar{x}_{K_2+2}), f(\bar{x}_{K_2+2})) \notin \mathcal{F}_{K_2+1}$, so that $\bar{x}_{K_2+2}$ is not rejected in Step 4.6. Arguing similarly as in the previous paragraph we can conclude that $\bar{x}_{K_2+2}$ is also not rejected in Step 4.7. Therefore, $x_{K_2+2} = \bar{x}_{K_2+2}$. Together with (38) and (39) this proves (i_{K_2+2})–(iii_{K_2+2}) for the case $\sigma_{K_2+1} = 1$.

Now suppose that (i_l)–(iii_l) are true for all $K_2 + 2 \le l \le k$ with some $k \ge K_2 + 2$. If $x_k + d_k$ is accepted by the line search, define $\sigma_k := 0$, otherwise $\sigma_k := 1$. Set $\bar{x}_{k+1} := x_k + d_k + \sigma_k d^{\rm soc}_k$. From (28) for (29c) we then have for $i = 2, 3$

$$\phi_{\rho_i}(x_{k-1}) - \phi_{\rho_i}(\bar{x}_{k+1}) \ge \frac{1+\gamma_\theta}{2} \left(q_{\rho_i}(x_{k-1}, 0) - q_{\rho_i}(x_{k-1}, d_{k-1}) \right) \ge 0. \qquad (40)$$

Choose l with $K_2 \le l < k - 1$ and consider two cases:

Case a): If $k = K_2 + 2$, then $l = K_2$, and it follows from (28) with (29d) that for $i = 2, 3$

$$\phi_{\rho_i}(x_l) - \phi_{\rho_i}(\bar{x}_{k+1}) \ge \frac{1+\gamma_\theta}{2} \left(q_{\rho_i}(x_l, 0) - q_{\rho_i}(x_l, d_l) \right) \ge 0. \qquad (41)$$

Case b): If $k > K_2 + 2$, we have from (40) that $\phi_{\rho_i}(\bar{x}_{k+1}) \le \phi_{\rho_i}(x_{k-1})$ and hence from (i_{k-1}) it follows that (41) also holds in this case.

In either case, (41) implies in particular that $\phi_{\rho_3}(\bar{x}_{k+1}) \le \phi_{\rho_3}(x_{K_2})$, and since $x_{K_2} \in L$, we obtain

$$\bar{x}_{k+1} \in L. \qquad (42)$$

If $x_k + d_k$ is accepted by the line search, (i_{k+1})–(iii_{k+1}) follow from (41), (40) and (42). If $x_k + d_k$ is rejected, we see from (42), (41) for $i = 2$ and $l = K_2$, and (34) that $(\theta(\bar{x}_{k+1}), f(\bar{x}_{k+1})) \notin \mathcal{F}_{K_2}$. Furthermore, for $l \in I^k_{K_2}$ we have from (40) and (41) with (33) that $(\theta(\bar{x}_{k+1}), f(\bar{x}_{k+1})) \notin \mathcal{G}_l$, and hence from (32) that $\bar{x}_{k+1}$ is not rejected in Step 4.6. We can again show as before that $\bar{x}_{k+1}$ is not rejected in Step 4.7, so that $x_{k+1} = \bar{x}_{k+1}$ which implies (i_{k+1})–(iii_{k+1}).

That $\{x_k\}$ converges to x_* with a superlinear rate follows from (14) (see e.g. [9]). □

Remark 1 *As can be expected, the convergence rate of x_k towards x_* is quadratic, if (14) is replaced by*

$$(W_k - H_k) d_k = O(\|d_k\|^2)$$

(see e.g. [3])

4 Fast Local Convergence of a Trust Region Filter SQP Method

The switching rule used in the trust region SQP-filter algorithm proposed by Fletcher et. al. [5] does not imply the relationship (23), and therefore the proof of Lemma 4 in our local convergence analysis does not hold for that method. However, it is easy to see that the global convergence analysis in [5] is still valid (in particular Lemma 3.7 and Lemma 3.10 in [5]), if the switching rule Eq. (2.19) in [5] is modified in analogy to (6) and replaced by

$$[m_k(d_k)]^{s_\varphi} \Delta_k^{1-s_\varphi} \ge \kappa_\theta \theta_k^\varphi,$$

where m_k is now the change of the objective function predicted by a quadratic model of the objective function, Δ_k the current trust region radius, $\kappa_\theta, \varphi > 0$ constants from [5] satisfying certain relationships, and the new constant $s_\varphi > 0$ satisfies $s_\varphi > 2\varphi$. Then the local convergence analysis in Section 3 is still valid (also for the quadratic model formulation), assuming that sufficiently close to a strict local solution the trust region is inactive, the trust region radius Δ_k is uniformly bounded away from zero, the (approximate) SQP steps s_k are computed sufficiently exactly, and a second order correction as discussed in Section 2 is performed.

5 Conclusions

We have shown that second order correction steps are able to overcome the Maratos effect within filter methods. Important for the success of our analysis is a particular switching rule (6), which differs from previous filter methods, such as the one proposed by Fletcher et. al. [5]. Also this method, however, can be adapted to overcome the Maratos effect.

References

[1] P. T. Boggs, J. W. Tolle, and P. Wang. On the local convergence of quasi-Newton methods for constrained optimization. *SIAM Journal on Control and Optimization*, 20:161–171, 1982.

[2] R. M. Chamberlain, C. Lemarechal, H. C. Pedersen, and M. J. D. Powell. The watchdog technique for forcing convergence in algorithms for constrained optimization. *Mathematical Programming Study*, 16:1–17, 1982.

[3] A. R. Conn, N. I. M. Gould, and Ph. L. Toint. *Trust-Region Methods*. SIAM, Philadelphia, PA, USA, 2000.

[4] J. E. Dennis and J. J. Moré. Quasi-Newton methods, motivation and theory. *SIAM Review*, 19(1):46–89, 1977.

[5] R. Fletcher, N. I. M. Gould, S. Leyffer, Ph. L. Toint, and A. Wächter. Global convergence of a trust-region SQP-filter algorithms for general nonlinear programming. Technical Report 99/03, Department of Mathematics, University of Namur, Belgium, May 1999. Revised October 2001. To appear in SIAM Journal on Optimization.

[6] R. Fletcher and S. Leyffer. Nonlinear programming without a penalty function. *Mathematical Programming*, 91(2):239–269, 2002.

[7] S.-P. Han. A globally convergent method for nonlinear programming. *Journal of Optimization Theory and Application*, 22:297–309, 1977.

[8] N. Maratos. *Exact Penalty Function Algorithms for Finite Dimensional and Control Optimization Problems*. PhD thesis, University of London, London, UK, 1978.

[9] J. Nocedal and M. Overton. Projected Hessian updating algorithms for nonlinearly constrained optimization. *SIAM Journal on Numerical Analysis*, 22:821–850, 1985.

[10] J. Nocedal and S. Wright. *Numerical Optimization*. Springer, New York, NY, USA, 1999.

[11] A. Wächter and L. T. Biegler. Line search filter methods for nonlinear programming: Motivation and glocal convergence. Technical report, Department of Chemical Engineering, Carnegie Mellon University, 2003.