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Support material for: "On Successive Refinement of the Binary Symmetric Markov Source"

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Support material for: On Successive Refinement of the Binary Symmetric Markov Source

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Abstract

In the paper of Lastras and Berger [1] the question of whether the Binary Symmetric Markov source is successively refinable [2], [3], [4] is addressed. This technical report contains the proof of a result not included in this paper due to space considerations.

1 Preliminaries

We refer the reader to [1] for notation and background information; we have mantained the same numbering for the theorem and lemmas. There is some overlap in the two publications to enhance the readability of this report.

2 Proof of optimality

Theorem 2 *Let n be a fixed integer. If $p > 1/2$, there exists an interval $(a_n^*, 1)$ such that for any $a \in (a_n^*, 1)$, the assignment $q^*_{00\cdots0} = q^*_{11\cdots1} = 1/2$ has the property that $c^*_{00\cdots0} = c^*_{11\cdots1} = 1$ and $\forall\, \mathbf{y} \in S^n - \{00\cdots0, 11\cdots1\}$, $c^*_{\mathbf{y}} < 1$. Hence, the channel generated by q^* at a is optimal.*

Proof. We shall find it useful to introduce the convention that a sequence $b_1 b_2 \cdots b_n$ will be associated with the integer $1 + \sum_{i=1}^{n} b_i 2^{i-1}$. We need to verify that $c_{00\cdots0} = 1, c_{11\cdots1} = 1$ and $c_k < 1$ for any other output reproduction sequence $k \notin \{1, 2^n\}$. The expression for λ_j is as follows:

$$\lambda_j^{-1} = \frac{1}{2} a^{d_{j,1}} + \frac{1}{2} a^{n-d_{j,1}}$$

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where $d_{j,1}$ denotes the Hamming weight of the sequence j, and hence

$$c_k = \sum_{j=1}^{2^n} \mathbf{p_{n,j}} \frac{a^{d_{j,k}}}{\frac{1}{2}(a^{d_{j,1}} + a^{n-d_{j,1}})}$$

Since the sequence indexed by $2^n - j + 1$ is the bit-wise complement of the sequence indexed by j, we have $\mathbf{p_{n,2^n-j+1}} = \mathbf{p_{n,j}}$ and $d_{2^n-j+1,k} = n - d_{j,k}$. Therefore, an alternative expression for c_k is:

$$c_k = \sum_{j=1}^{2^{n-1}} \mathbf{p_{n,j}} \frac{a^{d_{j,k}} + a^{n-d_{j,k}}}{\frac{1}{2}(a^{d_{j,1}} + a^{n-d_{j,1}})} \tag{1}$$

In particular,

$$\begin{aligned} c_{00\cdots0} &= \sum_{j=1}^{2^{n-1}} \mathbf{p_{n,j}} \frac{a^{d_{j,1}} + a^{n-d_{j,1}}}{\frac{1}{2}(a^{d_{j,1}} + a^{n-d_{j,1}})} \\ &= 2\sum_{j=1}^{2^{n-1}} \mathbf{p_{n,j}} = 2\frac{1}{2} = 1 \end{aligned}$$

It is clear that $c_{11\cdots1} = 1$ can be shown in a similar manner. This holds for all values of a, in particular, it is true for a in the interval $(a_n^*, 1)$ identified in the discussion that follows.

The arguments that we provide to demonstrate that $c_k < 1$ for $k \notin \{00\cdots0, 11\cdots1\}$ when $p > \frac{1}{2}$ are considerably longer. Evaluating (1) at $a = 1$ we see that for all k,

$$c_k\big|_{a=1} = 1 \tag{2}$$

Note that

$$\frac{d}{da}c_k = \sum_{j=1}^{2^{n-1}} \mathbf{p_{n,j}} \left(\frac{d}{da} \frac{a^{d_{j,k}} + a^{n-d_{j,k}}}{\frac{1}{2}(a^{d_{j,1}} + a^{n-d_{j,1}})} \right) \tag{3}$$

It is not hard to see that for all n and for all k,

$$\left.\frac{d}{da}c_k\right|_{a=1} = 0$$

Thus c_k has a critical point at $a = 1$, where we think of c_k as a function of a. Next, we establish

Lemma 2 *For all n, and for $k \notin \{1, 2^n\}$*

$$\left.\frac{d^2}{da^2}c_k\right|_{a=1} < 0$$

Remark: This Lemma shows that the critical point is a maximum. Combining this fact with Equation (2) allows us to conclude the existence of an open interval $(a_n, 1)$ in which $c_k < 1$, thus proving Theorem 2.

Proof. Let $\mathbf{D}_n$ represent the Hamming distortion matrix for sequences of length n (i.e.$\{\mathbf{D}_n\}_{j,k} = d_{j,k}$), $[A]^2$ denote the matrix with entries equal to the square of the corresponding element of the original matrix A, and $\mathbf{p_n}$ denote the probability vector of the 2^n source sequences. Let

$$\gamma_\mathbf{n} = [\mathbf{D}_n]^2 \, \mathbf{p_n}$$

Our first task will be to show that

$$\left.\frac{d^2}{da^2} c_k\right|_{a=1} = \gamma_{\mathbf{n,k}} - \gamma_{\mathbf{n,1}}$$

where $\gamma_{\mathbf{n,j}}$ represents the jth element of the $\gamma_\mathbf{n}$ vector.

The derivative that appears inside of the parenthesis in (3) is equal to

$$\frac{d}{da}\frac{a^{d_{j,k}} + a^{n-d_{j,k}}}{\frac{1}{2}(a^{d_{j,1}} + a^{n-d_{j,1}})} =$$
$$\frac{(a^{d_{j,1}} + a^{n-d_{j,1}})(d_{j,k}a^{d_{j,k}-1} + (n-d_{j,k})a^{n-d_{j,k}-1}) - (a^{d_{j,k}} + a^{n-d_{j,k}})(d_{j,1}a^{d_{j,1}-1} + (n-d_{j,1})a^{n-d_{j,1}-1})}{\frac{1}{2}(a^{d_{j,1}} + a^{n-d_{j,1}})^2}$$
$$\triangleq \frac{A}{\frac{1}{2}(a^{d_{j,1}} + a^{n-d_{j,1}})^2}$$

Taking a derivative again:

$$\frac{d}{da}\frac{A}{\frac{1}{2}(a^{d_{j,1}} + a^{n-d_{j,1}})^2} = \frac{(a^{d_{j,1}} + a^{n-d_{j,1}})^2 \frac{d}{da}A - A\frac{d}{da}(a^{d_{j,1}} + a^{n-d_{j,1}})^2}{\frac{1}{2}(a^{d_{j,1}} + a^{n-d_{j,1}})^4} \tag{4}$$

It is easy to verify that

$$A|_{a=1} = 0$$

Hence, in order to evaluate (4) at $a = 1$, we only need to evaluate $\frac{d}{da}A$ at $a = 1$:

$$\left.\frac{d}{da}A\right|_{a=1} = 2\left((d_{j,k}-n)^2 - (d_{j,1}-n)^2 + d_{j,k}^2 - d_{j,1}^2\right)$$

therefore

$$\left.\frac{d^2}{da^2}\frac{a^{d_{j,k}} + a^{n-d_{j,k}}}{\frac{1}{2}(a^{d_{j,1}} + a^{n-d_{j,1}})}\right|_{a=1} = (d_{j,k}-n)^2 - (d_{j,1}-n)^2 + d_{j,k}^2 - d_{j,1}^2$$

thus concluding that

$$\begin{aligned}
\left.\frac{d^2}{da^2}c_k\right|_{a=1} &= \sum_{j=1}^{2^{n-1}} \mathbf{p_{n,j}} \left((d_{j,k}-n)^2 - (d_{j,1}-n)^2 + d_{j,k}^2 - d_{j,1}^2\right) \\
&= \sum_{j=1}^{2^{n-1}} \mathbf{p_{n,j}} \left((d_{j,k}-n)^2 - (d_{j,1}-n)^2\right) + \sum_{j=1}^{2^{n-1}} \mathbf{p_{n,j}} \left(d_{j,k}^2 - d_{j,1}^2\right) \\
&= \sum_{j=2^{n-1}+1}^{2^n} \mathbf{p_{n,j}} \left(d_{j,k}^2 - d_{j,1}^2\right) + \sum_{j=1}^{2^{n-1}} \mathbf{p_{n,j}} \left(d_{j,k}^2 - d_{j,1}^2\right) \\
&= \sum_{j=1}^{2^n} \mathbf{p_{n,j}} \left(d_{j,k}^2 - d_{j,1}^2\right) \\
&= \sum_{j=1}^{2^n} \mathbf{p_{n,j}} d_{k,j}^2 - \sum_{j=1}^{2^n} \mathbf{p_{n,j}} d_{1,j}^2 \\
&= \gamma_{\mathbf{n,k}} - \gamma_{\mathbf{n,1}}
\end{aligned}$$

In order to prove the Lemma, we must show that

$$\gamma_{\mathbf{n,1}} > \gamma_{\mathbf{n,k}} \tag{5}$$

for all n and for all $k \notin \{1, 2^n\}$. With this goal in mind, let us introduce some definitions:

- We say that a k-vector $\mathbf{x}$ is top-dominant (resp. bottom-dominant) if $x_1 > x_j$ for $j \in \{2, \cdots, k\}$ (resp. $x_k > x_j$ for $j \in \{1, \cdots, k-1\}$).
- We say that a k-vector $\mathbf{x}$ is bi-dominant if $x_1 > x_j$ and $x_k > x_j$ for $j \in \{2, \cdots, k-1\}$.
- For k-vectors $\mathbf{x}$ and $\mathbf{y}$, we say that $\mathbf{x} > \mathbf{y}$ if $x_j > y_j$ for $j \in \{1, \cdots, k\}$

Therefore, if we show that $\gamma_{\mathbf{n}}$ is a bi-dominant vector, we will have achieved our goal (it would also suffice to show that $\gamma_{\mathbf{n}}$ is a top-dominant vector, but this is *not* true). We will show this using recursions.

Let $\Box$ denote a square matrix of all ones of the appropiate size. We will also denote by $\mathbf{p_n^0}$ and $\mathbf{p_n^1}$ the upper half and lower half of $\mathbf{p_n}$. It is not difficult to see that

$$\begin{aligned}
[\mathbf{D}_n]^2 \mathbf{p_n} &= \begin{bmatrix} [\mathbf{D}_{n-1}]^2 & [\mathbf{D}_{n-1} + \Box]^2 \\ [\mathbf{D}_{n-1} + \Box]^2 & [\mathbf{D}_{n-1}]^2 \end{bmatrix} \begin{bmatrix} \mathbf{p_n^0} \\ \mathbf{p_n^1} \end{bmatrix} \\
&= \begin{bmatrix} [\mathbf{D}_{n-1}]^2 \mathbf{p_n^0} + \left([\mathbf{D}_{n-1}]^2 + 2\mathbf{D}_{n-1} + \Box\right) \mathbf{p_n^1} \\ \left([\mathbf{D}_{n-1}]^2 + 2\mathbf{D}_{n-1} + \Box\right) \mathbf{p_n^0} + [\mathbf{D}_{n-1}]^2 \mathbf{p_n^1} \end{bmatrix}
\end{aligned}$$

We will make the following definitions:

$$\begin{aligned}\Gamma_n^0 &\triangleq [\mathbf{D}_{n-1}]^2\,\mathbf{p_n^0}\\ \Gamma_n^1 &\triangleq [\mathbf{D}_{n-1}]^2\,\mathbf{p_n^1}\\ \Pi_n^0 &\triangleq \mathbf{D}_{n-1}\mathbf{p_n^0}\\ \Pi_n^1 &\triangleq \mathbf{D}_{n-1}\mathbf{p_n^1}\end{aligned}$$

Also note that $\Box\mathbf{p_n^0} = \Box\mathbf{p_n^1} = \frac{1}{2}\mathbf{1}$ where $\mathbf{1}$ denotes a column vector of all ones of the appropiate size. Thus, $[\mathbf{D}_n]^2\,\mathbf{p_n}$ can be rewritten as

$$[\mathbf{D}_n]^2\,\mathbf{p_n} = \begin{bmatrix}\Gamma_n^0 + \Gamma_n^1 + 2\Pi_n^1 + \frac{1}{2}\mathbf{1}\\ \Gamma_n^0 + \Gamma_n^1 + 2\Pi_n^0 + \frac{1}{2}\mathbf{1}\end{bmatrix} \tag{6}$$

We now look for recursions for the Γ_{n-1}^0, Γ_{n-1}^1:

$$\begin{aligned}\Gamma_n^0 = [\mathbf{D}_{n-1}]^2\,\mathbf{p_n^0} &= \begin{bmatrix}[\mathbf{D}_{n-2}]^2 & [\mathbf{D}_{n-2}+\Box]^2\\ [\mathbf{D}_{n-2}+\Box]^2 & [\mathbf{D}_{n-2}]^2\end{bmatrix}\begin{bmatrix}\mathbf{p_{n-1}^0}p\\ \mathbf{p_{n-1}^1}q\end{bmatrix}\\ &= \begin{bmatrix}\Gamma_{n-1}^0 p + \Gamma_{n-1}^1 q + 2\Pi_{n-1}^1 q + \frac{1}{2}q\mathbf{1}\\ \Gamma_{n-1}^0 p + \Gamma_{n-1}^1 q + 2\Pi_{n-1}^0 p + \frac{1}{2}p\mathbf{1}\end{bmatrix}\end{aligned}$$

Similarly,

$$\Gamma_n^1 = [\mathbf{D}_{n-1}]^2\,\mathbf{p_n^1} = \begin{bmatrix}\Gamma_{n-1}^0 q + \Gamma_{n-1}^1 p + 2\Pi_{n-1}^1 p + \frac{1}{2}\mathbf{1}p\\ \Gamma_{n-1}^0 q + \Gamma_{n-1}^1 p + 2\Pi_{n-1}^0 q + \frac{1}{2}\mathbf{1}q\end{bmatrix}$$

It thus follows that

$$\Gamma_n^0 + \Gamma_n^1 = \begin{bmatrix}\Gamma_{n-1}^0 + \Gamma_{n-1}^1 + 2\Pi_{n-1}^1 + \frac{1}{2}\mathbf{1}\\ \Gamma_{n-1}^0 + \Gamma_{n-1}^1 + 2\Pi_{n-1}^0 + \frac{1}{2}\mathbf{1}\end{bmatrix}$$

Upon comparision with Equation (6), we see that

$$\Gamma_n^0 + \Gamma_n^1 = [\mathbf{D}_{n-1}]^2\,\mathbf{p_{n-1}} \tag{7}$$

Substituting (7) in (6), we obtain the recursion

$$[\mathbf{D}_n]^2\,\mathbf{p_n} = \begin{bmatrix}[\mathbf{D}_{n-1}]^2\,\mathbf{p_{n-1}} + 2\Pi_n^1 + \frac{1}{2}\mathbf{1}\\ [\mathbf{D}_{n-1}]^2\,\mathbf{p_{n-1}} + 2\Pi_n^0 + \frac{1}{2}\mathbf{1}\end{bmatrix}$$

Now we look for recursions for the Π_n^0, Π_n^1:

$$\Pi_n^0 = \mathbf{D}_{n-1}\mathbf{p_n^0} = \begin{bmatrix}\Pi_{n-1}^0 p + \Pi_{n-1}^1 q + \frac{1}{2}q\mathbf{1}\\ \Pi_{n-1}^0 p + \Pi_{n-1}^1 q + \frac{1}{2}p\mathbf{1}\end{bmatrix} \tag{8}$$

$$\Pi_n^1 = \mathbf{D}_{n-1}\mathbf{p_n^1} = \begin{bmatrix}\Pi_{n-1}^0 q + \Pi_{n-1}^1 p + \frac{1}{2}p\mathbf{1}\\ \Pi_{n-1}^0 q + \Pi_{n-1}^1 p + \frac{1}{2}q\mathbf{1}\end{bmatrix} \tag{9}$$

Let F be the "flip" operator, i.e. if $\mathbf{x}$ is a k-vector, $F\mathbf{x}$ is the k-vector whose jth element is defined by

$$\{F\mathbf{x}\}_j = \mathbf{x}_{k-j+1}$$

for $j \in \{1, \cdots, k\}$. Later, we shall make use of the following fact:

Lemma 6 $F\Pi_n^0 = \Pi_n^1$

Proof. (by induction). We know that

$$\Pi_2^0 = \begin{bmatrix} q/2 \\ p/2 \end{bmatrix}$$
$$\Pi_2^1 = \begin{bmatrix} p/2 \\ q/2 \end{bmatrix}$$

and therefore $F\Pi_2^0 = \Pi_2^1$. Assume that $F\Pi_{n-1}^0 = F\Pi_{n-1}^1$. From the recursions,

$$\begin{aligned} F\Pi_n^0 &= F \begin{bmatrix} \Pi_{n-1}^0 p + \Pi_{n-1}^1 q + \frac{1}{2} q\mathbf{1} \\ \Pi_{n-1}^0 p + \Pi_{n-1}^1 q + \frac{1}{2} p\mathbf{1} \end{bmatrix} \\ &= \begin{bmatrix} F\Pi_{n-1}^0 p + F\Pi_{n-1}^1 q + \frac{1}{2} p\mathbf{1} \\ F\Pi_{n-1}^0 p + F\Pi_{n-1}^1 q + \frac{1}{2} q\mathbf{1} \end{bmatrix} \\ &= \begin{bmatrix} \Pi_{n-1}^1 p + \Pi_{n-1}^0 q + \frac{1}{2} p\mathbf{1} \\ \Pi_{n-1}^1 p + \Pi_{n-1}^0 q + \frac{1}{2} q\mathbf{1} \end{bmatrix} \\ &= \Pi_n^1 \end{aligned}$$

♣

Using this Lemma, we can prove the following fact:

Lemma 7 $[\mathbf{D}_n]^2 \mathbf{p_n} = F [\mathbf{D}_n]^2 \mathbf{p_n}$

Proof. Note that in the case $n = 1$,

$$[D_1]^2 \mathbf{p_1} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = F \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Assume the Lemma is true for $n-1$. From the recursion for $[\mathbf{D}_n]^2\,\mathbf{p_n}$,

$$\begin{aligned} F\,[\mathbf{D}_n]^2\,\mathbf{p_n} &= F\begin{bmatrix} [\mathbf{D}_{n-1}]^2\,\mathbf{p_{n-1}} + 2\Pi_n^1 + \frac{1}{2}\mathbf{1} \\ [\mathbf{D}_{n-1}]^2\,\mathbf{p_{n-1}} + 2\Pi_n^0 + \frac{1}{2}\mathbf{1} \end{bmatrix} \\ &= \begin{bmatrix} F\,[\mathbf{D}_{n-1}]^2\,\mathbf{p_{n-1}} + 2F\Pi_n^0 + \frac{1}{2}\mathbf{1} \\ F\,[\mathbf{D}_{n-1}]^2\,\mathbf{p_{n-1}} + 2F\Pi_n^1 + \frac{1}{2}\mathbf{1} \end{bmatrix} \\ &= \begin{bmatrix} [\mathbf{D}_{n-1}]^2\,\mathbf{p_{n-1}} + 2\Pi_n^1 + \frac{1}{2}\mathbf{1} \\ [\mathbf{D}_{n-1}]^2\,\mathbf{p_{n-1}} + 2\Pi_n^0 + \frac{1}{2}\mathbf{1} \end{bmatrix} \\ &= [\mathbf{D}_n]^2\,\mathbf{p_n} \end{aligned}$$

which finalizes the proof of the Lemma. ♣

Lemma 8 Π_n^0 *is bottom-dominant*

Proof. Note that

$$\Pi_n^0 = \begin{bmatrix} \Pi_{n-1}^0 p + \Pi_{n-1}^1 q + \frac{1}{2} q\mathbf{1} \\ \Pi_{n-1}^0 p + \Pi_{n-1}^1 q + \frac{1}{2} p\mathbf{1} \end{bmatrix}$$

Since $p > q$,

$$\frac{1}{2}p\mathbf{1} > \frac{1}{2}q\mathbf{1}$$

Hence to prove that Π_n^0 is bottom-dominant it is sufficient to show that $\Pi_{n-1}^0 p + \Pi_{n-1}^1 q$ is also a bottom-dominant vector. Rewriting a bit, it is possible to see that

$$\Pi_n^0 p + \Pi_n^1 q = \left(p - \frac{1}{2}\right)\left(\Pi_n^0 - \Pi_n^1\right) + \frac{1}{2}\left(\Pi_n^0 + \Pi_n^1\right)$$

From the recursions in (8) and (9), we know that

$$\Pi_n^0 + \Pi_n^1 = \begin{bmatrix} \Pi_{n-1}^0 + \Pi_{n-1}^1 + \frac{1}{2}\mathbf{1} \\ \Pi_{n-1}^0 + \Pi_{n-1}^1 + \frac{1}{2}\mathbf{1} \end{bmatrix}$$

and $\Pi_2^0 + \Pi_2^1 = 0 + 0 = 0$. It thus follows that

$$\Pi_n^0 + \Pi_n^1 = \frac{n-1}{2}\mathbf{1}$$

Hence,

$$\Pi_n^0 p + \Pi_n^1 q = \left(p - \frac{1}{2}\right)\left(\Pi_n^0 - \Pi_n^1\right) + \frac{1}{2}(n-1)\mathbf{1}$$

Hence to show that $\Pi_n^0 p + \Pi_n^1 q$ is bottom-dominant, we only need to show that if $p > 1/2$, then $\Pi_n^0 - \Pi_n^1$ is bottom dominant. Note that

$$\Pi_2^0 - \Pi_2^1 = \begin{bmatrix} 1/2 - p \\ -1/2 + p \end{bmatrix}$$

and therefore for $n = 2$, $\Pi_n^0 - \Pi_n^1$ is clearly bottom-dominant. Assume that $\Pi_{n-1}^0 - \Pi_{n-1}^1$ is bottom-dominant. It can be shown that

$$\Pi_n^0 - \Pi_n^1 = \begin{bmatrix} (2p-1)\left(\Pi_{n-1}^0 - \Pi_{n-1}^1\right) - \frac{1}{2}(2p-1)\mathbf{1} \\ (2p-1)\left(\Pi_{n-1}^0 - \Pi_{n-1}^1\right) + \frac{1}{2}(2p-1)\mathbf{1} \end{bmatrix}$$

From the fact that $2p - 1 > 0$ and from the assumption that $\Pi_{n-1}^0 - \Pi_{n-1}^1$ is bottom-dominant, it is easy to see that $\Pi_n^0 - \Pi_n^1$ is bottom-dominant. It follows that $\Pi_n^0 p + \Pi_n^1 q$ is bottom-dominant and hence Π_n^0 is bottom-dominant, which proves the Lemma. ♣

Since $\Pi_n^1 = F\Pi_n^0$, we have also proved

Corollary 1 Π_n^1 *is top-dominant*

We proceed to establish that $[\mathbf{D}_n]^2 \mathbf{p_n}$ is a bi-dominant vector for all n. Recall that the recursion for $[\mathbf{D}_n]^2 \mathbf{p_n}$ is

$$[\mathbf{D}_n]^2 \mathbf{p_n} = \begin{bmatrix} [\mathbf{D}_{n-1}]^2 \mathbf{p_{n-1}} + 2\Pi_n^1 + \frac{1}{2}\mathbf{1} \\ [\mathbf{D}_{n-1}]^2 \mathbf{p_{n-1}} + 2\Pi_n^0 + \frac{1}{2}\mathbf{1} \end{bmatrix} \tag{10}$$

and

$$[D_1]^2 \mathbf{p_1} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Clearly, $[D_1]^2 \mathbf{p_1}$ is a bi-dominant vector. Assume that $[\mathbf{D}_{n-1}]^2 \mathbf{p_{n-1}}$ is a bi-dominant vector. Using this assumption, Corollary 1 and Lemma 7, it is easy to see that

$$[\mathbf{D}_{n-1}]^2 \mathbf{p_{n-1}} + 2\Pi_n^1 + \frac{1}{2}\mathbf{1}$$

is a top-dominant vector. Using Lemma 7 again, we conclude that $[\mathbf{D}_n]^2 \mathbf{p_n}$ is a bi-dominant vector.

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