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On the Singularity of Matrices

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Abstract

If B is a singular complex matrix, there is a singular C whose entries are the same magnitude as those of B, and all but two of C's entries are real.

Let M be an $n \times n$ matrix of nonnegative real entries m_{ij}. The Camion-Hoffman Theorem [1] gives necessary and sufficient conditions on M guaranteeing that any complex matrix B with $|b_{ij}| = m_{ij}$ must be nonsingular. Namely, any such B is nonsingular if and only if there is a permutation matrix P and a positive diagonal matrix D such that $M' = PMD$ is strongly diagonally dominant: $\forall i, m'_{ii} > \sum_{j \neq i} m_{ij}$.

If the Camion-Hoffman condition is not met, so that there is a singular B, we show that in fact there is such a singular C all but two of whose entries are real.

Theorem 1 *If B is a singular complex matrix, there is a singular C whose entries are the same magnitude as those of B, and either all entries of C are real, or two entries of C are complex and all others real.*

Proof : Let M be a nonnegative $n \times n$ real matrix and B a complex matrix satisfying $|b_{ij}| = m_{ij}$ and $\det(B) = 0$. Fix a nonzero n-vector $\mathbf{z}$ with $B\mathbf{z} = 0$.

We will define a sequence of matrices B^k and real vectors $\mathbf{x}^k$, $k = 0, 1, \ldots, n-1$, satisfying these conditions:

1. $B^k \mathbf{x}^k = 0$;
2. $|b^k_{ij}| = m_{ij}$;
3. $\mathbf{x}^k \neq 0$;
4. At least k of the rows of B^k are entirely real;
5. The other $n - k$ rows have at most two non-real entries each;
6. If $k > 0$ and b^{k-1}_{ij} is real, then $b^k_{ij} = b^{k-1}_{ij}$.

Define $\mathbf{x}^0$ by $x^0_j = |z_j|$. Define B' by $b'_{ij} = b_{ij} z_j / x^0_j$ if $x^0_j \neq 0$, and $b'_{ij} = |b_{ij}|$ if $x^0_j = 0$.

We first compute the matrix B^0. For each $i \in \{1, 2, \ldots, n\}$ the following procedure gives the ith row of B^0. Set $a_j = |b'_{ij} x_j|, j = 1, 2, \ldots, n$. Suppose the two largest entries are $a_\ell \geq a_m$. As

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j ranges sequentially though $\{1, 2, \ldots, n\}\backslash\{\ell, m\}$, we select $\epsilon_j \in \{-1, +1\}$ to bring the cumulative sum of $\epsilon_j a_j$ as close as possible to a_ℓ: if $\sum_{h<j, h\notin\{\ell,m\}} \epsilon_h a_h < a_\ell$ then $\epsilon_j = +1$, otherwise $\epsilon_j = -1$.

Because $0 = (B\mathbf{z})_i = \sum_j b_{ij} z_j$, we see that the cumulative sum will eventually get as large as $a_\ell - a_m$, and thereafter it must remain in the interval $[a_\ell - a_m, a_\ell + a_m]$ because the addends a_j are no larger than a_m.

The three quantities a_ℓ, a_m, and $\sum_{j\notin\{\ell,m\}} \epsilon_j a_j$ satisfy the triangle inequality. So if we set $b^0_{ij} = \epsilon_j b'_{ij}$ for $j \neq \ell, m$, we can find complex entries $b^0_{i\ell}$ and b^0_{im} satisfying $\sum_{1\leq j\leq n} b^0_{ij} x^0_j = 0$. Our conditions 1-5 are satisfied; 6 is inapplicable.

Now let $k \in \{1, 2, \ldots, n-1\}$, and suppose we have constructed B^{k-1}. We will construct B^k.

Let B_R consist of those rows of B^{k-1} whose entries are all real. If B_R has more than $k-1$ rows, set $B^k = B^{k-1}$ and $\mathbf{x}^k = \mathbf{x}^{k-1}$. Otherwise, consider the space Y of real vectors orthogonal to B_R; Y has dimension at least $n-k+1 \geq 2$. For each of the $n-k+1$ rows i not in B_R, consider the $4 = 2 \times 2$ possible rows $\mathbf{r}$ obtainable from B^{k-1}_{i*} by replacing its two non-real entries with real entries (either the corresponding entry of M or its negative). If $\mathbf{r}$ is orthogonal to all of Y, then $\mathbf{r}$ is in the linear span of the rows of B_R, and we can extend $B_R||\mathbf{r}$ to a singular matrix C with all real entries, finishing the problem. So assume that the subspace of Y orthogonal to $\mathbf{r}$ has co-dimension 1. Then we can select nonzero $\mathbf{y} \in Y$ avoiding all these $4(n-k+1)$ subspaces, as well avoiding the multiples of $\mathbf{x}^{k-1}$.

For each of the $4(n-k+1)$ rows $\mathbf{r}$ mentioned above, there is one value of the real parameter t making $\mathbf{x}^{k-1} + t * \mathbf{y}$ orthogonal to $\mathbf{r}$. Let t_0 be such a t with smallest absolute value, corresponding to row $\mathbf{r}$ obtained from row i by assigning signs s_1, s_2. Set $\mathbf{x}^k = \mathbf{x}^{k-1} + t_0 * \mathbf{y}$. Obtain B^k from B^{k-1} by replacing the ith row by $\mathbf{r}$, thereby replacing the two non-real entries of B^{k-1}_{i*} by real entries of the same magnitude. For each row h outside B_R other than the ith row, imagine the phases of the two non-real entries $b_{h\ell}, b_{hm}$ varying continuously with t to maintain orthogonality with $\mathbf{x}^{k-1} + t * \mathbf{y}$ as t ranges from 0 to t_0. At $t = 0$ the triangle inequality holds among the three values $|b^{k-1}_{h\ell} x^{k-1}_\ell|$, $|b^{k-1}_{hm} x^{k-1}_m|$, and $|\sum_{j\notin\{\ell,m\}} b^{k-1}_{hj} x^{k-1}_j|$. By minimality of $|t_0|$, we know that t does not pass through any of the four critical values of t for this row (which would turn a triangle inequality into an equality and then to a violation), so the triangle inequality continues to hold, and we can assign to these two entries complex values of the correct magnitude which maintain orthogonality of the new row B^k_{h*} to the vector $\mathbf{x}^k$. (Caveat: if several values of t tie for minimality, then each of several rows B^k may reach, but not pass through, a critical value, so that each will become totally real during this step.) The new $\mathbf{x}^k$ is trivially still orthogonal to all rows in B_R; it is also orthogonal to the newly changed row i. The other conditions are clearly satisfied.

When this procedure is over, $C = B^{n-1}$ is the desired matrix: C's entries have absolute value as prescribed by M, with at most two non-real entries, and C is singular.

If C has exactly one non-real entry, its cofactor must be zero (since $\det(C) = 0$ is real), so we could make that entry real without changing the determinant. So the number of non-real entries can be taken to be either 0 or 2. □

Remark For any $n \geq 3$, this result is best possible. Set $M = 2J - I$ of order n. Allowing complex entries, there is a singular C with $|c_{ij}| = m_{ij}$. (Since each row of M has $n-1$ entries of 2 and one entry of 1, the largest entry does not exceed the sum of the other entries; for each i we can build an n-gon with sides of length m_{ij}, interpret the sides as complex numbers c_{ij}, and remark that the resulting C annihilates the vector of all 1s.) If we insist on real entries, for any C with entries chosen from $\pm m_{ij}$, its determinant must be an odd integer, thus nonzero. If C has exactly one non-real entry, if $\det(C)$ is real then the cofactor of this entry must vanish, so that the determinant would not change if we replaced that non-real entry by a real one, reducing to the previous case. So any singular C with magnitudes of entries prescribed by this M, must have at least two non-real entries.

Remark We can compute the initial B from M by following some suggestions in [1]. We wish to find a positive diagonal matrix $D = diag(d_j)$ such that in each row of MD, no element is "dominant" (larger than the sum of the others in its row). This requirement can be expressed as a system of linear inequalities which d_j must satisfy. Let K be a matrix of order n with $k_{ii} = 1$, $k_{ij} = -1$ for $i \neq j$. Consider the system of n^2 inequalities in the positive variables $\{d_j\}$:

$$\sum_j m_{ij} k_{j\ell} d_j \leq 0, \quad i, \ell = 1, \ldots, n.$$

Linear programming finds such a solution $\{d_j\}$ if one exists. For each i, we can find complex numbers a_{ij} with $|a_{ij}| = m_{ij} d_j$ and $\sum_j a_{ij} = 0$ (since no $m_{ij} d_j$ is dominant). Then A is singular, since it annihilates the vector of all 1s. $B = AD^{-1}$ is also singular, and $|b_{ij}| = m_{ij}$ as required.

References

[1] Paul Camion and A. J. Hoffman, "On the nonsingularity of complex matrices," *Pacific J. Math.* **17** (1966) 211-214.