RC23183 (W0404-092) April 14, 2004
Mathematics

IBM Research Report

Constructive Bounds on Ordered Factorizations

Don Coppersmith
IBM Research Division
Thomas J. Watson Research Center
P.O. Box 218
Yorktown Heights, NY 10598

Moshe Lewenstein
Bar Ilan University
Ramat Gan 52900
Israel

Research Division
Almaden - Austin - Beijing - Haifa - India - T. J. Watson - Tokyo - Zurich

Constructive bounds on ordered factorizations

Don Coppersmith[*] Moshe Lewenstein[†]

April 13, 2004

Abstract

The number of ways to factor a natural number into an ordered product of integers, each factor greater than one, is called the *ordered factorization of* n and is denoted $H(n)$. We show upper and lower bounds on $H(n)$ with explicit constructions.

1 Introduction

For $n \in \mathbb{Z}^+$, let $H(n)$ denote the number of *ordered factorizations* of n, by which we mean expressions of n as the product of integers $r_i \geq 2$ where the order of factors is essential. Equivalently, $H(n)$ is the number of tuples $(r_1, r_2, \ldots, r_k)$ with $r_i \geq 2$ and $\prod r_i = n$, without restrictions on k. $H(1) = 1$ by convention, the only factorization being () with $k = 0$. $H(20) = 8$, the factorizations being (20), (10,2), (5,4), (5,2,2), (4,5), (2,10), (2,5,2), (2,2,5). Newberg and Naor[3] use $H(n)$ as a lower bound for an application in computational biology.

Define

$$\rho = \zeta^{-1}(2) \approx 1.7264724,$$

where ζ is the Riemann zeta function, so that

$$\sum_{n=1}^{\infty} \frac{1}{n^\rho} = 2,$$

and more usefully,

$$\sum_{n=2}^{\infty} \frac{1}{n^\rho} = 1.$$

Hille [2] showed the existence of a constant c such that $H(n) \leq cn^\rho$; Chor *et al.* [1] improved this to $c = 1$:

$$H(n) \leq n^\rho. \tag{1}$$

Hille also gave an existential lower bound: for all $\epsilon > 0$,

$$\limsup_n \frac{H(n)}{n^{\rho-\epsilon}} = \infty. \tag{2}$$

Newberg and Naor show an explicit construction lower bounding $H(n)$ with $n \log^c n$ for some c. Chor *et al.* gave explicit constructions for certain values of ϵ.

In this note we give simplified proofs of both upper and lower bounds.

*IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598 USA. email: dcopper@us.ibm.com

†Bar Ilan University, Ramat Gan 52900, Israel. email: moshe@cs.biu.ac.il

2 Upper bound

The upper bound $H(n) \leq n^\rho$ is proven by induction on n. The base case $n = 1$ is satisfied. Suppose the result is true for all $n' < n$. We count the ordered factorizations of n according to their first element r_1, which is a factor of n larger than 1. The remainder $(r_2, \ldots, r_k)$ is an ordered factorization of n/r_1. So we have

$$H(n) = \sum_{d|n, d>1} H(n/d).$$

By induction,

$$H(n/d) \leq (n/d)^\rho,$$

so that

$$\begin{aligned} H(n) &= \textstyle\sum_{d|n,d>1} H(n/d) \leq \sum_{d|n,d>1} \frac{n^\rho}{d^\rho} < n^\rho \sum_{d>1} \frac{1}{d^\rho} \\ &= n^\rho(\zeta(\rho) - 1) = n^\rho(2-1) = n^\rho, \end{aligned}$$

completing the induction. In fact we see that the inequality is strict for $n > 1$.

3 Lower bound

For $\alpha = \rho - \epsilon$ we will give a family of integers n for which $\limsup_n H(n)/n^\alpha = \infty$.

Because $\zeta(t)$ is strictly monotone decreasing in t, we know

$$\zeta(\alpha) = \sum_1^\infty \frac{1}{n^\alpha} > 2.$$

There is a finite integer b for which already

$$\sum_1^b \frac{1}{n^\alpha} > 2.$$

Use monotonicity again to claim there is γ with $\alpha < \gamma < \rho$ satisfying

$$\sum_1^b \frac{1}{n^\gamma} = 2,$$

or, more usefully,

$$\sum_2^b \frac{1}{n^\gamma} = 1.$$

Fix such α, b, γ.

Now select a large integer t. For $k = 2, 3, \ldots, b$, we define

$$c_k = \lfloor t/k^\gamma \rfloor.$$

Set $u = \sum c_k$, so that $0 \leq t - u \leq b - 2$. Define

$$n = \prod_{k=2}^b k^{c_k}.$$

We will compare $H(n)$ to n^α. Among the ordered factorizations counted by $H(n)$ are the orderings of (c_2 copies of 2, ..., c_b copies of b). The number of such orderings is given by the multinomial coefficient

$$v(n) = \frac{u!}{\prod_{k=2}^{b} c_k!}.$$

From Stirling's approximation,

$$v(n) = \prod_k \left(\frac{u}{c_k}\right)^{c_k} \times \sqrt{\frac{2\pi u}{\prod(2\pi c_k)}} \times [1 + o(1)],$$

where the $o(1)$ term goes to 0 with increasing c_k and hence with increasing n.

To estimate the first product, recall $c_k \le t/k^\gamma$, so that

$$\prod_k \left(\frac{u}{c_k}\right)^{c_k} \ge \prod_k \left(\frac{uk^\gamma}{t}\right)^{c_k} = (u/t)^u (\prod_k k^{c_k})^\gamma.$$

We have $(u/t)^u \ge e^{-(t-u)} \ge e^{-b+2}$, while the other factor is simply n^γ. So our first product is at least $e^{-b+2} n^\gamma$.

The second product is

$$\sqrt{\frac{2\pi u}{\prod(2\pi c_k)}}.$$

Notice that $\log n = \sum c_k \log k$ which implies that $\log n < \sum (c_k \log b)$. Hence, $u = \sum c_k > (\log n / \log b)$. On the other hand, for any k, $c_k \le \sum (c_k \log k / \log 2) = \log n / \log 2$. Therefore, for some constant d_b we can lower bound the second product as follows

$$\sqrt{\frac{2\pi u}{\prod(2\pi c_k)}} > d_b (\log n)^{-(b-2)/2}.$$

Summarizing,

$$H(n) \ge v(n) \ge n^\gamma (\log n)^{-(b-2)/2} d_b (1 + o(1)).$$

Since $\gamma > \alpha$, we have

$$\limsup_n H(n)/n^\alpha = \infty,$$

as required.

References

[1] B. Chor, P. Lemke and Z. Mador. On the number of ordered factorizations of natural numbers. *Discrete Mathematics*, 214:123–133, 2000.

[2] E. Hille. A problem in factorisation numerorum. *Acta Arithmetica*, 2(1):134–144, 1936.

[3] L.A. Newberg and D. Naor. A lower bound on the number of solutiuons to the probed partial digestion problem. *Advanced Applied Math.*, 14:172–183, 1993.