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## Polynomial-Time Separation of a Superclass of Simple Comb Inequalities

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# Polynomial-Time Separation of a Superclass of Simple Comb Inequalities[1]

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**Abstract**

The *comb* inequalities are a well-known class of facet-inducing inequalities for the Travelling Salesman Problem, defined in terms of certain vertex sets called the *handle* and the *teeth*. We say that a comb inequality is *simple* if the following holds for each tooth: either the intersection of the tooth with the handle has cardinality one, or the part of the tooth outside the handle has cardinality one, or both. The simple comb inequalities generalize the classical *2-matching* inequalities of Edmonds, and also the so-called *Chvátal comb* inequalities.

In 1982, Padberg and Rao [34] gave a polynomial-time algorithm for *separating* the 2-matching inequalities — i.e., for testing if a given fractional solution to an LP relaxation violates a 2-matching inequality. We extend this significantly by giving a polynomial-time algorithm for separating a class of valid inequalities which includes all simple comb inequalities.

**Key Words**: travelling salesman problem, cutting planes, separation.

## 1 Introduction

The famous *Symmetric Travelling Salesman Problem* (STSP) is the $\mathcal{NP}$-hard problem of finding a minimum cost Hamiltonian cycle (or *tour*) in a complete undirected graph. The most successful optimization algorithms at present (e.g.,

[1] An extended abstract of this paper appeared in the IPCO proceedings [23]. However, the abstract gives a running time of $\mathcal{O}(n^3|E^*|^3 \log n)$. Here, we describe an improved run time of $\mathcal{O}(n^2|E^*|^2 \log(n^2/|E^*|))$. Moreover, we correct an error in [23] by showing that our separation algorithm detects not only violated simple comb inequalities, but also inequalities in a slightly extended class.

Padberg & Rinaldi [36], Applegate, Bixby, Chvátal & Cook [1]), are based on an integer programming formulation of the STSP due to Dantzig, Fulkerson & Johnson [9], which we now describe.

Let $G$ be a complete graph with vertex set $V$ and edge set $E$. For each edge $e \in E$, let $c_e$ be the cost of traversing edge $e$. For any $S \subset V$, let $\delta(S)$ (respectively, $E(S)$), denote the set of edges in $G$ with exactly one end-vertex (respectively, both end vertices) in $S$. Then, for each $e \in E$, define the 0-1 variable $x_e$ taking the value 1 if $e$ is to be in the tour, 0 otherwise. Finally let $x(F)$ for any $F \subset E$ denote $\sum_{e \in F} x_e$. Then the formulation is:

Minimise $\sum_{e \in E} c_e x_e$

Subject to:

$$x(\delta(\{i\})) = 2 \qquad \forall i \in V, \tag{1}$$
$$x(E(S)) \leq |S| - 1 \qquad \forall S \subset V : 2 \leq |S| \leq |V| - 2, \tag{2}$$
$$x_e \geq 0 \qquad \forall e \in E, \tag{3}$$
$$x \in Z^{|E|}. \tag{4}$$

Equations (1) are called *degree equations*. The inequalities (2) are called *subtour elimination constraints* (SECs) and the inequalities (3) are simple non-negativity conditions. Note that an SEC with $|S| = 2$ is a mere upper bound of the form $x_e \leq 1$ for some edge $e$.

The convex hull in $\mathbb{R}^{|E|}$ of vectors satisfying (1) - (4) is called a *Symmetric Travelling Salesman Polytope*. The polytope defined by (1) - (3) is called a *Subtour Elimination Polytope*. These polytopes are denoted by STSP$(n)$ and SEP$(n)$ respectively, where $n := |V|$. Clearly, STSP$(n) \subseteq$ SEP$(n)$, and containment is strict for $n \geq 6$.

The polytopes STSP$(n)$ have been studied in great depth and many classes of valid and facet-inducing inequalities are known; see the surveys by Jünger, Reinelt & Rinaldi [19, 20] and Naddef [25]. Here we are primarily interested in the *comb* inequalities of Grötschel & Padberg [16, 17], which are defined as follows. Let $t \geq 3$ be an odd integer. Let $H \subset V$ and $T_j \subset V$ for $j = 1, \ldots, t$ be such that $T_j \cap H \neq \emptyset$ and $T_j \setminus H \neq \emptyset$ for $j = 1, \ldots, t$, and also let the $T_j$ be vertex-disjoint. (See Figure 1 for an illustration.) The comb inequality is:

$$x(E(H)) + \sum_{j=1}^{t} x(E(T_j)) \leq |H| + \sum_{j=1}^{t} |T_j| - \lceil 3t/2 \rceil. \tag{5}$$

The set $H$ is called the *handle* of the comb and the $T_j$ are called *teeth*.

Comb inequalities induce facets of STSP$(n)$ for $n \geq 6$ [16, 17]. The validity of comb inequalities in the special case where $|T_j \cap H| = 1$ for all $j$ was proved by Chvátal [7]. For this reason inequalities of this type are sometimes referred to as *Chvátal comb* inequalities. If, in addition, $|T_j \setminus H| = 1$ for all $j$, then the inequalities reduce to the classical *2-matching* inequalities of Edmonds [10].

In this paper we are concerned with a class of inequalities which is intermediate in generality between the class of comb inequalities and the class of Chvátal

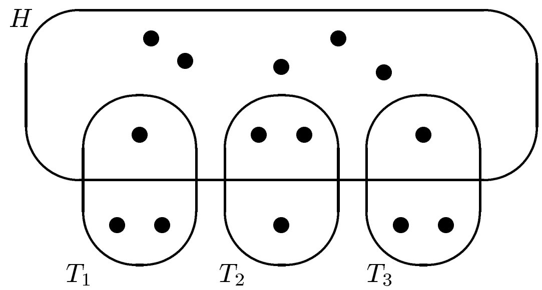


Figure 1: A comb with three teeth.

comb inequalities. For want of a better term, we call them *simple* comb inequalities, although the reader should be aware that the term *simple* is used with a different meaning in Padberg & Rinaldi [35], and with yet another meaning in Naddef & Rinaldi [28, 29].

**Definition 1** *A comb (and its associated comb inequality) will be said to be* simple *if, for all $j$, either $|T_j \cap H| = 1$ or $|T_j \setminus H| = 1$ (or both).*

So, for example, the comb shown in Figure 1 is simple because $|T_1 \cap H|$, $|T_2 \setminus H|$ and $|T_3 \cap H|$ are all equal to 1. Note however that it is not a Chvátal comb, because $|T_2 \cap H| = 2$.

For a given class of inequalities, a *separation algorithm* is a procedure which, given a vector $x^* \in \mathbb{R}^{|E|}$ as input, either finds an inequality in the class which is violated by $x^*$, or proves that none exists (see Grötschel, Lovász & Schrijver [15]). A desirable property of a separation algorithm is that it runs in polynomial time.

In 1982, Padberg & Rao [34] discovered a polynomial-time separation algorithm for the *2-matching* inequalities. First, they showed that the separation problem is equivalent to the problem of finding a *minimum weight odd cut* in a certain weighted labelled graph. This graph has $\mathcal{O}(|E^*|)$ vertices and edges, where $E^* := \{e \in E : x^*_e > 0\}$. Then, they proved that the desired cut can be found by solving a sequence of $O(|E^*|)$ max-flow problems. Using the well-known *pre-flow push* algorithm (Goldberg & Tarjan [13]) to solve the max-flow problems, along with some implementation tricks given in Grötschel & Holland [14], the Padberg - Rao separation algorithm can be implemented to run in $\mathcal{O}(n|E^*|^2 \log(n^2/|E^*|))$ time, which is $\mathcal{O}(n^5)$ in the worst case, but $\mathcal{O}(n^3 \log n)$ if the support graph is sparse. [2]

In Padberg & Grötschel [33], page 341, it is conjectured that there also exists a polynomial-time separation algorithm for the more general *comb* inequalities. This conjecture is still unsettled, and in practice many researchers resort to

[2]The *support graph*, denoted by $G^*$, is the subgraph of $G$ induced by $E^*$.

*heuristics* for comb separation (see for example Padberg & Rinaldi [35], Applegate, Bixby, Chvátal & Cook [1], Naddef & Thienel [30]). Nevertheless, some progress has recently been made on the theoretical side. In chronological order:

- Carr [6] showed that, for a *fixed* value of $t$, the comb inequalities with $t$ teeth can be separated by solving $\mathcal{O}(n^{2t})$ maximum flow problems, i.e., in $\mathcal{O}(n^{2t+1}|E^*|\log(n^2/|E^*|))$ time using the pre-flow push algorithm.

- Fleischer & Tardos [11] gave an $\mathcal{O}(n^2 \log n)$ algorithm for detecting *maximally violated* comb inequalities. (A comb inequality is maximally violated if it is violated by $\frac{1}{2}$, which is the largest violation possible if $x^* \in SEP(n)$.) However this algorithm only works when $G^*$ is planar.

- Caprara, Fischetti & Letchford [4] showed that the comb inequalities were contained in a more general class of inequalities, called $\{0, \frac{1}{2}\}$-*cuts*, and showed how to detect maximally violated $\{0, \frac{1}{2}\}$-cuts in $\mathcal{O}(n^2|E^*|)$ time.

- Letchford [21] defined a different generalization of the comb inequalities, called *domino-parity* inequalities, and showed that the associated separation problem can be solved in $\mathcal{O}(n^3)$ time when $G^*$ is planar.

- Caprara & Letchford [5] showed that, if the *handle* $H$ is fixed, then the separation problem for a class of inequalities including all $\{0, \frac{1}{2}\}$-cuts, called *split cuts*, can be solved in polynomial time. They did not analyse the running time, but the order of the polynomial is likely to be very high.

In this paper we make another step forward in this line of research, by proving the following theorem:

**Theorem 1** *There is a class of valid inequalities, containing all simple comb inequalities, that can be separated in polynomial time, provided that $x^* \in SEP(n)$.*

This is a significant extension of the Padberg - Rao result. As in [21], the proof is based on some results of Caprara & Fischetti [3] concerning $\{0, \frac{1}{2}\}$-cuts, together with arguments which enable one to restrict attention to a small (polynomial-sized) collection of *candidate teeth*.

The structure of the paper is as follows. In Section 2 we summarize the results given in [3] about $\{0, \frac{1}{2}\}$-cuts and show how they relate to the simple comb inequalities. In Section 3 we analyse the structure of candidate teeth. In Section 4 we describe a simple version of the separation algorithm and analyse its running time, which turns out to be very high at $\mathcal{O}(n^9 \log n)$. In Section 5, we show that the running time can be reduced to $\mathcal{O}(n^2|E^*|^2 \log(n^2/|E^*|))$. Conclusions are given in Section 6.

## 2 Simple Comb Inequalities as $\{0, \frac{1}{2}\}$-cuts

As mentioned above, we will need some definitions and results from Caprara & Fischetti [3]. We begin with the definition of $\{0, \frac{1}{2}\}$-*cuts*:

**Definition 2** *Given an integer polyhedron* $P_I := conv\{x \in Z_+^q : Ax \leq b\}$, *where* $A$ *is a* $p \times q$ *integer matrix and* $b$ *is a column vector with* $p$ *integer entries, a* $\{0, \frac{1}{2}\}$-cut *is a valid inequality for* $P_I$ *of the form*

$$\lfloor \lambda A \rfloor x \leq \lfloor \lambda b \rfloor, \tag{6}$$

*where the multiplier vector* $\lambda \in \{0, \frac{1}{2}\}^p$ *is chosen so that* $\lambda b$ *is not integral.*

(Actually, Caprara & Fischetti give a more general definition, applicable when variables are not necessarily required to be non-negative; but the definition given here is more appropriate for our purposes. Also note that an equation can easily be represented by two inequalities.)

Define the $i$th inequality in the system $Ax \leq b$ to be *used* if $\lambda_i = \frac{1}{2}$. The non-negativity inequality for a given variable $x_j$ is *used* if the $j$th coefficient of the vector $\lambda A$ is fractional. (Rounding down the coefficient of $x_j$ on the left hand side of (6) is equivalent to adding one half of the non-negativity inequality $-x_j \leq 0$.) Using this terminology, we have:

**Proposition 1** *Let* $x^* \in \mathbb{R}_+^q$ *be a point to be separated. Then a given* $\{0, \frac{1}{2}\}$-*cut is violated by* $x^*$ *if and only if the sum of the slacks of the inequalities used, computed with respect to* $x^*$, *is less than* 1.

Under the (reasonable) assumption that $Ax^* \leq b$, all slacks are non-negative and Proposition 1 also implies that the slack of each inequality used must be less than 1.

Using this, Caprara & Fischetti [3] show that the separation problem for $\{0, \frac{1}{2}\}$-cuts is strongly $\mathcal{NP}$-hard in general, but polynomially solvable in certain special cases. One of these special cases is of interest for this paper and to present it, we need two more definitions:

**Definition 3** *The* mod-2 support *of an integer matrix* $A$, *denoted by* $\bar{A}$, *is the matrix obtained by replacing each entry in* $A$ *by its parity (0 if even, 1 if odd).*

The mod-2 support of a single inequality is defined analogously.

**Definition 4** *A* $p \times q$ *binary matrix* $A$ *is called an* edge-path incidence matrix of a tree *(EPT for short), if there is a tree* $T$ *on* $p+1$ *vertices such that each row of* $M$ *corresponds to an edge of* $T$ *and each column* $c$ *of* $A$ *is the incidence vector of edges of a path* $P_c$ *in* $T$.

Given an EPT matrix $A$ and corresponding tree $T$, define the graph $G$ on $p+1$ vertices to contain $T$ and an edge $e_c$ for each column of $A$ with the same end points as $P_c$. Then $A$ is the matrix whose rows correspond to the incidence vectors of the fundamental cuts of $G$ with respect to $T$. We call the pair $(G, T)$ the *witness* for $A$.

We use the following result from [3].

**Theorem 2 (Caprara & Fischetti [3])** *The separation problem for $\{0, \frac{1}{2}\}$-cuts for a system $Ax \leq b$, $x \geq 0$ can be solved in polynomial time if $\bar{A}$ is EPT.*

The separation algorithm for the case where $\bar{A}$ is EPT is essentially an extension of the Padberg-Rao [34] algorithm, which, as mentioned above, is based on the computation of a minimum weight odd cut in a suitable weighted labelled graph. In this case, the graph is $G$. The edges of $T$ are in one-to-one correspondence with the inequalities in the system $Ax \leq b$; the edges of $G \backslash T$ are in one-to-one correspondence with the non-negativity inequalities. Each edge in the resulting graph is labelled *odd* or *even* according to whether the right hand side of the associated inequality is odd or even, and is given a weight equal to the slack of the associated inequality (computed with respect to $x^*$). Then, there is a one-to-one correspondence between odd cuts in this graph and $\{0, \frac{1}{2}\}$-cuts for the original problem, and every odd cut of weight less than 1 yields a violated $\{0, \frac{1}{2}\}$-cut.

The reason that these results are of relevance is that the comb inequalities (and certain more general inequalities such as the *extended* comb inequalities of Naddef & Rinaldi [27]) can be derived as $\{0, \frac{1}{2}\}$-cuts from the degree equations and SECs; see Caprara, Fischetti & Letchford [4] for details. In fact, as pointed out in Letchford [21], to derive the comb inequalities as $\{0, \frac{1}{2}\}$-cuts it suffices to use, together with the degree equations, a certain *weakened version* of the SECs, as expressed in the following propositions and definition:

**Proposition 2 (Letchford [21])** *Let $S \subset V$ and $T \subset V$ be disjoint, non-empty vertex sets such that $S \cup T \neq V$. Summing together the SECs on $S$, $T$ and $S \cup T$ yields the following inequality:*

$$2x(E(S)) + 2x(E(T)) + x(E(S:T)) \leq 2|S| + 2|T| - 3, \qquad (7)$$

*where $E(S:T)$ denotes the set of edges with one end-node in $S$ and the other end-node in $T$.*

Inequalities of the form (7) are called *tooth* inequalities in Letchford [22].

**Definition 5 (Letchford [21])** *A* domino-parity *(DP) inequality is a valid inequality for the TSP which can be derived as a $\{0, \frac{1}{2}\}$-cut from the degree equations (1) and the tooth inequalities (7).*

**Proposition 3 (Letchford [21])** *The DP inequalities are a proper generalization of the comb inequalities, in the sense that every comb inequality is a DP inequality, yet there are facet-inducing DP inequalities which are not comb inequalities.*

For more results on domino parity inequalities see Boyd et al. [2] and Naddef [26].

In this paper, we restrict our attention to *s*imple comb inequalities. To derive these as $\{0, \frac{1}{2}\}$-cuts, it suffices to use a subclass of the tooth inequalities, as expressed in the following definition and proposition:

**Definition 6** *A tooth inequality is* simple *if* $|T| = 1$. *It takes the form*

$$2x(E(S)) + x(E(i:S)) \leq 2|S| - 1, \tag{8}$$

*where* $S \subset V$ *satisfies* $1 \leq |S| \leq |V| - 2$, $i \in V \setminus S$, *and where* $E(i:S)$ *denotes the set of edges with* $i$ *as one end-node and the other end-node in* $S$. *The vertex* $i$ *is the* root *of the tooth, and the vertex set* $S$ *the* body.

**Proposition 4** *Simple comb inequalities can be derived as* $\{0, \frac{1}{2}\}$*-cuts from the degree equations (1) and the simple tooth inequalities (8).*

**Proof:** First, sum together the degree equations for all $i \in H$ to obtain:

$$2x(E(H)) + x(\delta(H)) \leq 2|H|. \tag{9}$$

Now suppose, without loss of generality, that there is some $1 \leq k \leq t$ such that $|T_j \cap H| = 1$ for $j = 1, \ldots, k$, and $|T_j \setminus H| = 1$ for $k+1, \ldots, t$. For $j = 1, \ldots, k$, associate a simple tooth inequality of the form (8) with tooth $T_j$, by setting $\{i\} := T_j \cap H$ and $S := T_j \setminus H$. Similarly, for $j = k+1, \ldots, t$, associate a simple tooth inequality with tooth $T_j$, by setting $\{i\} := T_j \setminus H$ and $S := T_j \cap H$. Add all of these simple tooth inequalities to (9) to obtain:

$$2x(E(H)) + x(\delta(H)) + \sum_{j=1}^{k} \left(2x(E(T_j \setminus H)) + x(E(T_j \cap H : T_j \setminus H))\right)$$

$$+ \sum_{j=k+1}^{t} \left(2x(E(T_j \cap H)) + x(E(T_j \cap H : T_j \setminus H))\right) \leq 2|H| + 2\sum_{j=1}^{t} |T_j| - 3t.$$

This can be re-arranged to give:

$$2x(E(H)) + 2\sum_{j=1}^{t} x(E(T_j)) + x\left(\delta(H) \setminus \bigcup_{j=1}^{t} E(T_j \cap H : T_j \setminus H)\right)$$

$$\leq 2|H| + 2\sum_{j=1}^{t} |T_j| - 3t.$$

Dividing by two and rounding down yields (5). □

Let us call the inequalities which can be derived as $\{0, \frac{1}{2}\}$-cuts from the degree equations and simple tooth inequalities, *simple* DP inequalities. Then the above proposition states that every simple comb inequality is a simple DP inequality. It is interesting to note that the SECs (2) can themselves be regarded as simple DP inequalities, obtained by dividing a single simple tooth inequality by two and rounding down. In the conference version of this paper [23], it was conjectured that every simple DP inequality is equivalent to, or dominated by, SECs and simple comb inequalities. However, this is false. A counter-example with nine vertices is displayed in Figures 2 and 3.

The violated simple DP inequality is derived using the degree equations for $\{1,2,3,4,5\}$ and 5 tooth inequalities: the first 4 teeth $\{1,8\}, \{4,6\}, \{5,7\}, \{3,9\}$, are just edges, while the fifth tooth $\{2,4,5,6,7\}$ has vertex 2 as root and the set $\{4,5,6,7\}$ as body. This structure differs from simple comb inequalities in two respects: some teeth are nested inside another teeth, and the body of one tooth crosses the handle, which is defined as the set of vertices for which the degree constraints are included in the derivation.

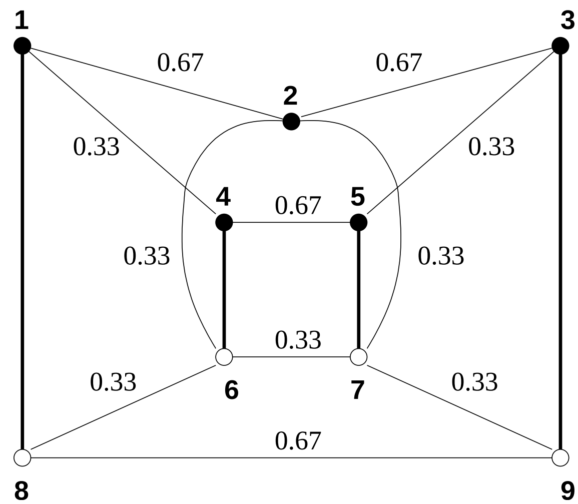


Figure 2: A point inside the subtour polytope on 9 vertices for which there is a violated simple DP inequality but no violated comb inequalities. The unmarked, dark lines have weight 1.

The resulting simple DP inequality is

$$x(E(H)) + \sum_i x(E(T_i)) - x_{24} - x_{25} \leq 10.$$

The point depicted in Figure 2 has left-hand-side value of 10.33. It is a vertex of the polytope that is described by $x_{18} = x_{39} = x_{46} = x_{57} = 1$, $x_e = 0$ if $e$ is not displayed in Figure 2, all degree constraints, the subtour constraint on $\{2,4,5,6,7\}$, and the two comb inequalities implied by the following sets of handles and teeth: $H1 = \{1,2,3\}, T1a = \{1,8\}, T1b = \{3,9\}, T1c = \{2,4,5,6,7\}$, and $H2 = \{2,6,7\}, T2a = \{4,6\}, T2b = \{5,7\}, T2c = \{1,2,3,8,9\}$.

Interestingly, this inequality is not facet-inducing, although it induces a face of high dimension. (It can be made into a facet by increasing the left hand side coefficients of $x_{48}$ and $x_{58}$ from zero to one, but the resulting inequality is not a $\{0, \frac{1}{2}\}$ cut.) Indeed, results of Naddef [26] imply that the simple comb inequalities are the only *facet-inducing* simple DP inequalities.

In this paper, then, we actually give a separation algorithm for simple DP inequalities, which include the simple comb inequalities as a special case. To aid

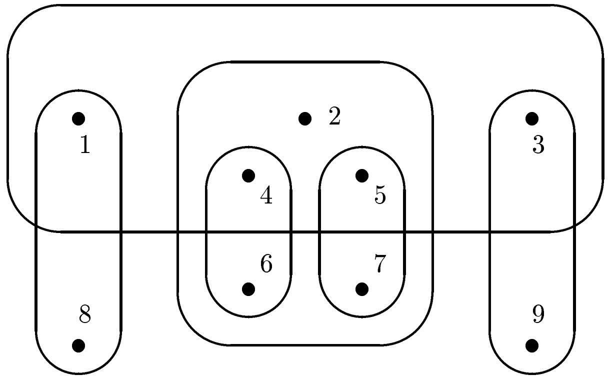


Figure 3: A schematic diagram of a simple DP inequality which is neither an SEC nor a simple comb inequality.

the reader, we display in Figure 4 the relationships between all of the inequalities discussed so far. An arrow from one class to another means that the former is a proper generalization of the latter.

# 3 The Structure of Candidate Teeth

Our goal in this paper is to apply the results of Caprara & Fischetti [3] to yield a polynomial-time separation algorithm for simple DP inequalities. However, a problem which immediately presents itself is that there is an exponential number of simple tooth inequalities, and therefore the system $Ax \leq b$ defined by the degree and simple tooth inequalities is of exponential size.

Fortunately, Proposition 1 tells us that we can restrict our attention to simple tooth inequalities whose slack is less than 1, without losing any violated $\{0, \frac{1}{2}\}$-cuts. Such tooth inequalities are polynomial in number, as shown in the following two lemmas.

**Lemma 1** *Suppose that $x^* \in SEP(n)$. Then the number of sets whose SECs have slack less than $\frac{1}{2}$ is $\mathcal{O}(n^2)$, and these sets can be found in $\mathcal{O}(n|E^*|(|E^*| + n \log n))$ time.*

**Proof:** The degree equations can be used to show that the slack of the SEC on a set $S$ is less than $\frac{1}{2}$ if and only if $x^*(\delta(S)) < 3$. Since the minimum cut in $G^*$ has weight 2, we require that the cut-set $\delta(S)$ has a weight strictly less than $\frac{3}{2}$ times the weight of the minimum cut. It is known (Hensinger & Williamson [18]) that there are $\mathcal{O}(n^2)$ such sets, and that the algorithm of Nagamochi, Nishimura & Ibaraki [31] finds them in $\mathcal{O}(n|E^*|(|E^*| + n \log n))$ time. □

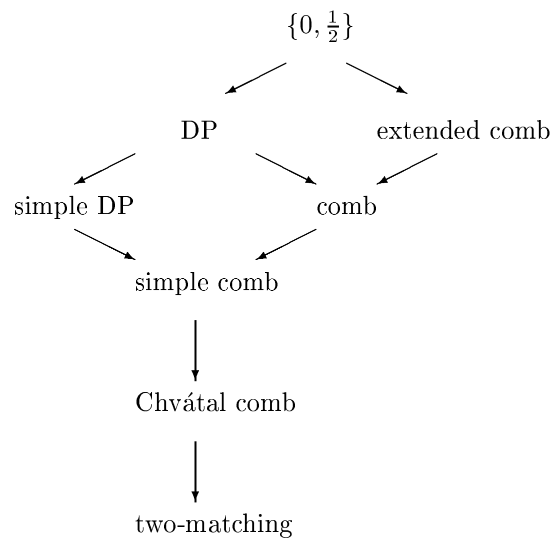


Figure 4: Relationships between various valid inequalities.

**Lemma 2** *Suppose that $x^* \in SEP(n)$. Then the number of distinct simple tooth inequalities with slack less than* 1 *is $\mathcal{O}(n^3)$, and these teeth can be found in $\mathcal{O}(n|E^*|(|E^*| + n \log n))$ time.*

**Proof:** The slack of the tooth inequality is equal to the slack of the SEC for $S$ plus the slack of the SEC for $\{i\} \cup S$. For the tooth inequality to have slack less than 1, the slack for at least one of these SECs must be less than $\frac{1}{2}$. So we can take each of the $\mathcal{O}(n^2)$ sets mentioned in Lemma 1 and consider them as candidates for either $S$ or $\{i\} \cup S$. For each candidate, there are only $n$ possibilities for the vertex $i$. The time bottleneck is easily seen to be the Nagamochi, Nishimura & Ibaraki algorithm. □

Now consider the system of inequalities $Ax \leq b$ formed by the degree equations and the $\mathcal{O}(n^3)$ simple tooth inequalities mentioned in Lemma 2. If we could show that the mod-2 support of $A$ is always an EPT matrix, then we would be done. Unfortunately this is *not* the case. (It is easy to produce counter-examples even for $n = 6$.)

Therefore we must use a more involved argument if we wish to separate simple DP inequalities via $\{0, \frac{1}{2}\}$-cut arguments. It turns out that the key is to pay special attention to simple tooth inequalities whose slack is strictly less than $\frac{1}{2}$. This leads us to the following definitions and lemma:

**Definition 7** *A tooth in a comb is said to be* light *if the slack of the associated tooth inequality (computed with respect to $x^*$) is less than $\frac{1}{2}$. If the slack is at least $\frac{1}{2}$, but less than* 1*, it is said to be* heavy. *For a given root $i \in V$, a vertex set $S \subset V \setminus \{i\}$ is said to be $i$-*light *if the slack of the tooth inequality with root $i$*

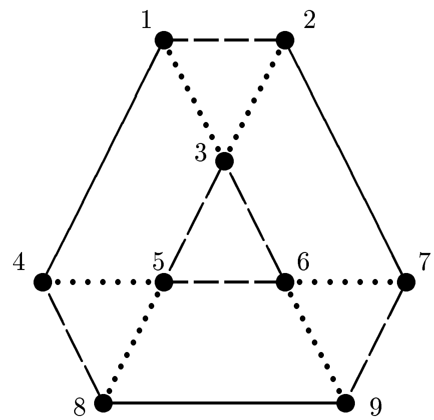


Figure 5: A fractional point contained in SEP(9).

*and body $S$ has slack strictly less than $\frac{1}{2}$. If the slack is at least $\frac{1}{2}$, but less than 1, it is said to be $i$-*heavy.

**Lemma 3** *If a simple DP inequality is violated by a given $x^* \in SEP(n)$, then at most one of its teeth can be heavy and the others are light.*

**Proof:** If two of the teeth are heavy, the slacks of the associated tooth inequalities sum to at least $\frac{1}{2} + \frac{1}{2} = 1$. Then, by Proposition 1, the DP inequality is not violated. □

We illustrate these ideas on a small example.

**Example:** Figure 5 shows the support graph $G^*$ for a vector $x^*$ which lies in SEP(9). The solid lines, dashed lines and dotted lines show edges with $x^*_e = 1$, 2/3 and 1/3, respectively. The 1-light sets are $\{2\}$, $\{4\}$, $\{2,7\}$, $\{3,4,5,6,8,9\}$, $\{3,\ldots,9\}$ and $\{2,3,5,6,7,8,9\}$; the 3-light sets are $\{5\}$, $\{6\}$, $\{5,6\}$, $\{1,2,4,7,8,9\}$, $\{1,2,4,5,7,8,9\}$ and $\{1,2,4,6,7,8,9\}$. The 1-heavy sets are $\{3\}$, $\{4,8\}$, $\{3,5,6\}$, $\{2,4,7,8,9\}$, $\{2,3,5,6,7,9\}$ and $\{2,4,5,6,7,8,9\}$. The 3-heavy sets are $\{1\}$, $\{2\}$, $\{1,4\}$, $\{2,7\}$, $\{2,5,6,7,8,9\}$, $\{1,4,5,6,8,9\}$, $\{2,4,5,6,7,8,9\}$ and $\{1,4,5,6,7,8,9\}$. The reader can easily identify light and heavy sets for other roots by exploiting the symmetry of the fractional point.

The light sets have an interesting structure, as expressed in the following definition and theorem:

**Definition 8** *Let $i \in V$ be a fixed root. Two vertex sets $S_1, S_2 \subset V \setminus \{i\}$ are said to $i$-*cross *if each of the four sets $S_1 \cap S_2$, $S_1 \setminus S_2$, $S_2 \setminus S_1$ and $V \setminus (S_1 \cup S_2 \cup \{i\})$ is non-empty.*

**Theorem 3** *Let $i \in V$ be a fixed root. If $x^* \in SEP(n)$, it is impossible for two $i$-light sets to $i$-cross.*

**Proof:** If we sum together the degree equations (1) for all $j \in S_1 \cap S_2$, along with the SECs on the four vertex sets $i \cup S_1 \cup S_2$, $S_1 \setminus S_2$, $S_2 \setminus S_1$ and $S_1 \cap S_2$, then (after some re-arranging) we obtain the inequality:

$$x^*(E(i : S_1)) + 2x^*(E(S_1)) + x^*(E(i : S_2)) + 2x^*(E(S_2)) \leq 2|S_1| + 2|S_2| - 3. \quad (10)$$

On the other hand, the sum of the tooth inequality with root $i$ and body $S_1$ and the tooth inequality with root $i$ and body $S_2$ is:

$$x^*(E(i : S_1)) + 2x^*(E(S_1)) + x^*(E(i : S_2)) + 2x^*(E(S_2)) \leq 2|S_1| + 2|S_2| - 2. \quad (11)$$

Comparing (11) and (10) we see that the sum of the slacks of these two tooth inequalities is at least 1. Since $x^* \in SEP(n)$, each of the individual slacks is non-negative. Hence at least one of the slacks must be $\geq \frac{1}{2}$. That is, at least one of $S_1$ and $S_2$ is $i$-heavy. □

The following lemma shows that we can eliminate half of the $i$-light sets from consideration.

**Lemma 4** *A tooth inequality with root $i$ and body $S$ is equivalent to the tooth inequality with root $i$ and body $V \setminus (S \cup \{i\})$.*

**Proof:** The latter inequality can be obtained from the former by subtracting the degree equations for the vertices in $S$, and adding the degree equations for the vertices in $V \setminus (S \cup \{i\})$. □

A set $\mathcal{S}$ of subsets of $V$ are *laminar* if at least one of the following three sets are empty $S \cap T$, $S \backslash T$, $T \backslash S$. Together, Theorem 3 and Lemma 4 imply the following Corollary.

**Corollary 1** *For a given root $i$, the bodies of $i$-light sets may be chosen to be* laminar*. Thus, there are only $\mathcal{O}(n)$ $i$-light vertex sets.*

## 4 Separation

Our separation algorithm has two stages. In the first stage, we search for a violated simple DP inequality in which all of the teeth are light. If this fails, then we proceed to the second stage, where we search for a violated simple DP inequality in which one of the teeth is heavy. Lemma 3 in the previous section shows that this approach is valid.

We will need the following lemma:

**Lemma 5** *Let $i$ be an arbitrary root, let $j \in V \setminus \{i\}$ be an arbitrary vertex and let $Ax \leq b$ be the inequality system formed by the degree equation on $i$ (written in less-than-or-equal-to form), and the tooth inequalities whose bodies form a laminar set $\mathcal{S}$. Then the mod-2 support of the matrix $A$ is an EPT matrix.*

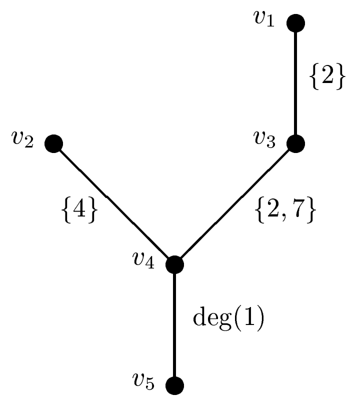


Figure 6: Tree in illustration of Lemma 5.

**Proof:** We show how to construct a tree $T$ such that the mod-2 support of $A$ is an edge-path incidence matrix of $T$. Suppose $|\mathcal{S}| = m$. The tree will have $m+2$ nodes, numbered $v_1, \ldots, v_{m+2}$, and $m+1$ edges (one for each body in $\mathcal{S}$ plus an extra one for the degree equation). For $\S_1, S_2 \in \mathcal{S}$, $S_2$ is the *parent* of body $S_1$ if $S_1 \subset S_2$ but there is no third set $S_3 \in \mathcal{S}$ with $S_1 \subset S_3 \subset S_2$. We construct our tree as follows: if the $p$th body in the family has the $q$th body as parent, we connect vertex $v_p$ to vertex $v_q$ by an edge. If the $p$th body has no parent, then we connect vertex $v_p$ to vertex $v_{m+1}$ by an edge. In either case the edge added represents the tooth inequality with root $i$ and body $p$. Finally we connect vertex $v_{m+1}$ to vertex $v_{m+2}$ by an edge, which represents the degree equation. To see that the mod-2 support of $A$ is an edge-path incidence matrix of $T$, note that, if the variable $x_e$ receives an odd coefficient in the tooth inequality with root $i$ and body $S$, and $S'$ is the parent of $S$, then $x_e$ also receives an odd coefficient in the tooth inequality with root $i$ and body $S'$, and also in the degree equation for $i$. Hence a column of $A$ either consists of zeroes and twos (when the associated edge $e \in E \setminus \delta(\{i\})$), or is the characteristic vector of a path in $T$ ending at vertex $m+2$. □

**Example (continued):** The 1-light sets which do not include vertex 9 are $\{2\}$, $\{4\}$ and $\{2,7\}$. The third set, $\{2,7\}$, is the parent of the first, $\{2\}$. The associated tooth inequalities are $x_{12} \leq 1$, $x_{14} \leq 1$ and $2x_{27} + x_{12} + x_{17} \leq 1$. The corresponding tree is shown in Figure 6. It can be seen, for example, that the column of $A$ associated with variable $x_{12}$ is the incidence vector of the path from vertex $v_1$ to vertex $v_5$ in the tree.

We are now in a position to state an important theorem, which is at the heart of our separation algorithm:

**Theorem 4** *Let $A'x \leq b'$ be the inequality system formed by the degree equations (written in less-than-or-equal-to form), and the tooth inequalities in a set $\mathcal{S}$ such that for each $i$, the set of bodies of teeth with root $i$ form a laminar set,* *then the mod-2 support of the matrix $A'$ is an EPT matrix.*

**Proof:** The inequality system $A'x \leq b'$ is the union of $n$ inequality systems of the form given in Lemma 5, one for each root $i$. We already know that each of these inequality systems can be represented by a tree. Moreover, in each of these trees, the edge representing the degree equation is incident on a leaf vertex (called $v_{m+2}$ in Lemma 5). Take each of the $n$ trees and form a single larger tree by identifying each of these leaf vertices to form a single vertex $v^*$. Note that a variable $x_{ij}$ has an odd coefficient in exactly *two* of the $n$ smaller inequality systems, namely the ones associated with the roots $i$ and $j$. This means that the mod-2 support of the associated column of $A'$ is the incidence vector of a path in exactly two sub-trees. But, each of the paths ends at $v^*$, because $x_{ij}$ has an odd coefficient in the degree equation for $i$ and $j$. Hence these two paths form a single larger path in the large tree, passing through $v^*$. So each column of $A'$ is the incidence vector of a path in the larger tree. □

**Example (continued):** There are six $i$-light sets for each root. Applying Lemma 4 we can eliminate half of these from consideration. So suppose we choose:

- 1-light sets: $\{2\}, \{4\}, \{2, 7\}$;
- 2-light sets: $\{1\}, \{7\}, \{1, 4\}$;
- 3-light sets: $\{5\}, \{6\}, \{5, 6\}$;
- 4-light sets: $\{1\}, \{8\}, \{8, 9\}$;
- 5-light sets: $\{3\}, \{6\}, \{3, 6\}$;
- 6-light sets: $\{3\}, \{5\}, \{3, 5\}$;
- 7-light sets: $\{2\}, \{9\}, \{8, 9\}$;
- 8-light sets: $\{4\}, \{9\}, \{1, 4\}$;
- 9-light sets: $\{7\}, \{8\}, \{2, 7\}$.

This leads to 27 light tooth inequalities in total. However, there are some duplicates: a tooth inequality with root $i$ and body $\{j\}$ is identical to a tooth inequality with root $j$ and body $\{i\}$ (in both cases the inequality is a simple upper bound, $x_{ij} \leq 1$). In fact there are only 18 distinct inequalities, namely:

$$\begin{array}{lll} 2x_{27} + x_{12} + x_{17} \leq 3 & 2x_{14} + x_{12} + x_{24} \leq 3 & 2x_{56} + x_{35} + x_{36} \leq 3 \\ 2x_{89} + x_{48} + x_{49} \leq 3 & 2x_{36} + x_{35} + x_{56} \leq 3 & 2x_{35} + x_{36} + x_{56} \leq 3 \\ 2x_{89} + x_{78} + x_{79} \leq 3 & 2x_{14} + x_{18} + x_{48} \leq 3 & 2x_{27} + x_{29} + x_{79} \leq 3, \end{array}$$

plus the upper bounds on $x_{12}, x_{14}, x_{27}, x_{35}, x_{36}, x_{48}, x_{56}, x_{79}$ and $x_{89}$. Therefore the matrix $A'$ has 36 columns (one for each variable), and 27 rows (18 tooth inequalities plus 9 degree equations). The single large tree is shown in Figure

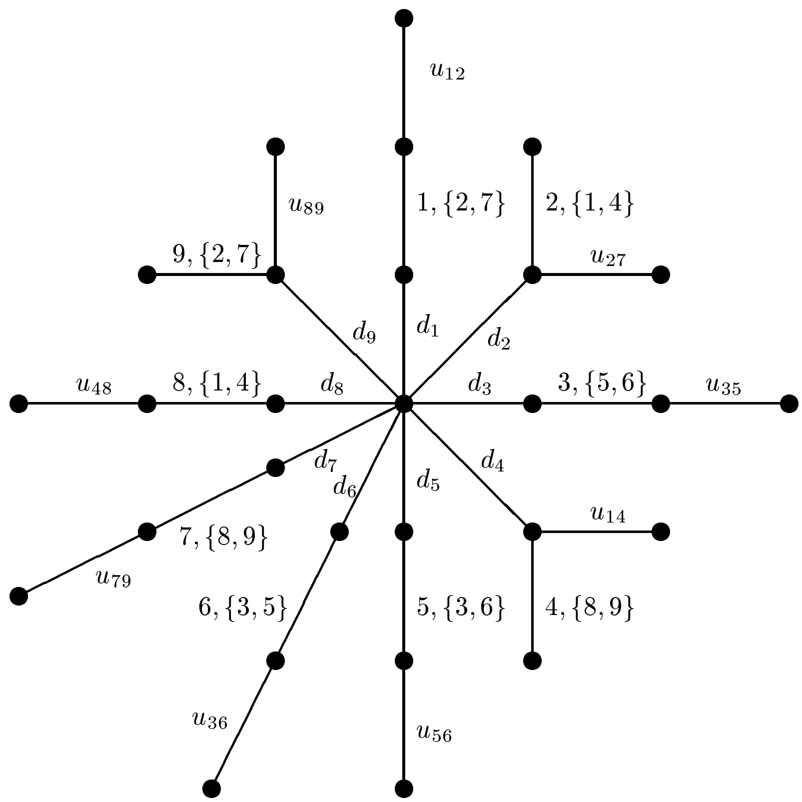


Figure 7: Tree in demonstration of Theorem 4.

7. The edge associated with the $i$th degree equation is labelled $d_i$. The edge associated with the upper bound $x_{ij} \leq 1$ is labelled $u_{ij}$. The edges associated with the remaining nine light tooth inequalities are labelled with the root and body. The vertex at the centre of the tree, incident on the edges labelled $d_1, \ldots, d_9$, is the vertex $v^*$ mentioned in the proof of Theorem 4. Note that many other compatible trees can be formed by moving the edges representing the upper bounds. For example, the edge marked $u_{12}$ could be moved to the right, making it adjacent to the edge marked $2, \{1, 4\}$.

Given the set $\mathcal{S}$ of teeth such that the tooth inequalities mod 2 of teeth in $\mathcal{S}$ together with the degree constraints form an EPT matrix $A$, we call the subroutine that builds the witness $(G_{\mathcal{S}}, T_{\mathcal{S}})$ as $\mathsf{buildT}(\mathcal{S})$. It is easy to see that $\mathsf{buildT}(\mathcal{S})$ runs in $O(|S|m)$ time.

Theorem 4 has an important corollary:

**Corollary 2** *A simple DP inequality derived from tooth inequalities in a set $\mathcal{S}$ such that the bodies with root $i$ form a laminar set, degree constraints, and nonnegativity inequalities is violated by a point in the subtour polytope if and* *only if the corresponding edges form an odd cutset in $G_{\mathcal{S}}$ with weight less than 1.*

**Corollary 3** *If $x^* \in SEP(n)$, then a violated simple DP inequality which uses only light teeth can be found in polynomial time, if any exists.*

**Proof:** In the previous section we showed that the desired tooth inequalities are polynomial in number and that they can be found in polynomial time. The necessary system $A'x \leq b$, the EPT matrix, and the associated tree can easily be constructed in polynomial time. The result then follows from Theorem 2. □

**Example (continued):** Applying stage 1 to the fractional point shown in Figure 3, we find a violated simple comb inequality with $H = \{1, 2, 3\}$, $T_1 = \{1, 4\}$, $T_2 = \{2, 7\}$ and $T_3 = \{3, 5, 6\}$. The inequality is

$$x_{12} + x_{13} + x_{23} + x_{14} + x_{27} + x_{35} + x_{36} + x_{56} \leq 5,$$

and it is violated by one-third.

We now proceed to describe stage 2 of the algorithm, in which we search for a violated simple DP inequality in which one used tooth is heavy. The key to this stage is the following lemma:

**Lemma 6** *Let $x^*$ be a given fractional point lying in the subtour polytope. If there is a violated simple DP inequality, then there is an simple DP inequality, violated by at least as much, such that no used teeth share the same root.*

**Proof:** Suppose that two *used* teeth have the same root $i$, and bodies $S_1$ and $S_2$. Without loss of generality we can assume that $S_1 \setminus S_2$ and $S_2 \setminus S_1$ are non-empty. (If $S_1 \subset S_2$, then we can replace $S_2$ by its complement $V \setminus (\{i\} \cup S_2$, and similarly if $S_2 \subset S_1$.)

Our claim is that a $\{0, \frac{1}{2}\}$-cut with at least the same amount of violation can be obtained by reducing the multipliers for the two teeth from $\frac{1}{2}$ to zero, and increasing the multipliers for the degree equation on $i$ (in less-than-or-equal-to form), and for the non-negativity inequalities on $E(i : V \setminus (\{i\} \cup (S_1 \setminus S_2) \cup (S_2 \setminus S_1))$, by $\frac{1}{2}$.

Note that this change causes the left hand side of the $\{0, \frac{1}{2}\}$-cut to reduce by $x(E(S_1)) + x(E(S_2)) + x(E(i : S_1 \cap S_2))$, and the right hand side to reduce by $|S_1| + |S_2| - 2$. The change in the amount of violation, computed with respect to $x^*$, is therefore $|S_1| + |S_2| - 2 - x^*(E(S_1)) - x^*(E(S_2)) - x^*(E(i : S_1 \cap S_2))$. This can be re-written as the sum of three terms:

a) $|S_1 \setminus S_2| - x^*(E(S_1 \setminus S_2)) - 1$,
b) $|S_2 \setminus S_1| - x(E(S_2 \setminus S_1)) - 1$,
c) $2|S_1 \cap S_2| - 2x(E(S_1 \cap S_2)) - x(E(S_1 \cap S_2 : (S_1 \setminus S_2) \cup (S_2 \setminus S_1) \cup \{i\}))$.

The first of these terms is non-negative because, by assumption, $x^*$ satisfies the SEC on $S_1 \setminus S_2$. Similarly, the second term is non-negative because of the

SEC on $S_2 \setminus S_1$. Finally, the third term is non-negative because of the degree equations on $S_1 \cap S_2$. Thus, the total amount of violation is either unchanged or increased.

The resulting $\{0, \frac{1}{2}\}$-cut may not be a simple DP inequality, because some of the multipliers may have increased from $\frac{1}{2}$ to 1. However, we can obtain a still stronger $\{0, \frac{1}{2}\}$-cut by changing any such multipliers to zero. The resulting $\{0, \frac{1}{2}\}$-cut is now a simple DP inequality.

This procedure can be repeated until no two teeth share the same root. □

We are now finally in a position to prove Theorem 1. The algorithm is summarized in Figure 8.

$\mathsf{SimpleDPSep}(G, x)$

Find all simple tooth inequalities with $x$-slack less than 1.
$(G_{\mathcal{L}}, T_{\mathcal{L}}) \leftarrow \mathsf{buildT}(\mathcal{L})$.
Find a minimum odd cut in $G_{\mathcal{L}}$.
If weight of cut is $< 1$, output candidate inequality.
For each root $i$,
    $L_i \leftarrow$ the set of light teeth with root $i$.
    For each heavy tooth $S$ with root $i$,
        $\mathcal{K} \leftarrow (\mathcal{L} - \mathcal{L}_i) \cup \{S\}$.
        $(G_{\mathcal{K}}, T_{\mathcal{K}}) \leftarrow \mathsf{buildT}(\mathcal{K})$.
        Find the minimum odd cut in $G_{\mathcal{K}}$.
        If weight of cut is $< 1$, output candidate inequality.

Figure 8: Separation algorithm for simple domino parity inequalities.

**Proof of Theorem 1:** We already know that the first stage, separation of simple DP inequalities in which all teeth are light, can be accomplished in polynomial time. So we now need to deal with the second stage, i.e., the case where one of the teeth involved is heavy. From Lemma 2 we know that there are $\mathcal{O}(n^3)$ candidates for this heavy tooth. We do the following for each of these candidates: we take the collection of $\mathcal{O}(n^2)$ light tooth inequalities, and eliminate the ones whose teeth have the same root as the heavy tooth under consideration. (Lemma 6 shows that this is a valid operation.) It is easy to show that the resulting modified matrix is still an EPT matrix: one of the sub-trees is removed from the larger tree and replaced by a single edge representing the heavy tooth inequality. The minimum odd cut procedure can then be repeated on this modified graph. □

Now let us analyse the running time of this separation algorithm. Stage 1 involves computing a minimum weight odd cut in a labelled weighted graph with $\mathcal{O}(n^2)$ vertices and $\mathcal{O}(n^2)$ edges (because there are $\mathcal{O}(n^2)$ light tooth inequalities). Using the Padberg - Rao algorithm, this means solving $\mathcal{O}(n^2)$ max-flow problems in this graph. Using the pre-flow push algorithm [13] to solve the max-flow problems, stage 1 takes $\mathcal{O}(n^6 \log n)$ time. This is bad enough, but an even bigger running time is needed for stage 2, which involves $\mathcal{O}(n^3)$ minimum odd cut computations on graphs of a similar size. This leads to a running time of $\mathcal{O}(n^9 \log n)$ for stage 2, which, though polynomial, is totally impractical.

In the next section, we show that this running time can be reduced to $\mathcal{O}(n^2|E^*|^2 \log(n^2/|E^*|))$.

# 5 Improving the Running Time

In this section we prove two theorems that allow us to reduce the complexity of our separation algorithm. The first theorem, implies that it is sufficient to consider a set of light teeth of size $O(n)$. The second theorem implies that it is sufficient to consider a set of heavy teeth of size $O(mn)$ that has a special structure. The proofs are contained respectively in Sections 5.1 and 5.2.

**Theorem 5** *There exists a set of light teeth $\mathcal{L}$ of size $O(n)$ such that if there exists a violated simple DP inequality, then there exists one with light teeth from the set $\mathcal{L}$ only. The set $\mathcal{L}$ can be found in $O(n^3m)$ time.*

Let $|\delta(i)|$ denote the degree of node $i$ in $G^*$.

**Theorem 6** *There is a set of heavy teeth of size $O(mn)$, such that the heavy teeth with root i can be partitioned into $|\delta(i)|$ laminar subsets; and if there exists a violated simple DP inequality derived using a heavy tooth, then there exists one with its heavy tooth in this set. This set can be found in $O(n^3m)$ time.*

We use these two theorems to modify SimpleDPSep to obtain a faster separation algorithm described in Figure 9. As before, the algorithm first looks for simple DP inequalities that are derived using light teeth only. Then, it looks for simple DP inequalities that use one heavy tooth. By Theorem 6, Lemma 2, and and the following Lemma 6, this can be done as follows: for each $i \in V$, consider at one time all light teeth with roots in $V - \{i\}$ and a subset of heavy teeth with root $i$, and check the corresponding graph $G_S$ for minimum odd cut.

**Theorem 7** FastSimpleDPSep *is a separation algorithm for simple domino parity inequalities that runs in $O(n^2m^2 \log \frac{n^2}{m})$ time.*

**Proof:** If there is a violated simple DP inequality using only teeth in $\mathcal{L}$, then by Lemma 2 and Theorem 5, FastSimpleDPSep finds it in line 4. Otherwise, using Lemmas 6 and 2 and Theorems 6 and 5, if there is a violated simple DP inequality using a heavy tooth with root $i$, FastSimpleDPSep finds it in line 13. This establishes correctness. Now we focus on run time.

FastSimpleDPSep$(G, x)$

1 Find all simple tooth inequalities with $x$-slack less than 1.<br>
2 Reduce the subset of light teeth to a set $\mathcal{L}$ of size $O(n)$ by<br>
    uncrossing. *(Theorem 5)*<br>
3 $(G_{\mathcal{L}}, T_{\mathcal{L}}) \leftarrow$ buildT$(\mathcal{L})$.<br>
4 Find a minimum odd cut in $G_{\mathcal{L}}$.<br>
5 If weight of cut is $< 1$, output candidate inequality.<br>
6 Reduce the subset of heavy teeth to $O(m)$ laminar sets. *(Theorem 6)*<br>
7 For each root $i$,<br>
8     $L_i \leftarrow$ the set of light teeth with root $i$.<br>
9     Partition the set of heavy teeth with root $i$ into $|\delta(i)|$ laminar<br>
        subsets, $\{\mathcal{H}_1^i, \ldots, \mathcal{H}_{|\delta(i)|}^i\}$. *(Theorem 6)*<br>
10     For each such subset $\mathcal{H}_j^i$,<br>
11         $\mathcal{K} \leftarrow (\mathcal{L} - \mathcal{L}_i) \cup \mathcal{H}_j^i$.<br>
12         $(G_{\mathcal{K}}, T_{\mathcal{K}}) \leftarrow$ buildT$(\mathcal{K})$.<br>
13         Find the minimum odd cut in $G_{\mathcal{K}}$.<br>
14         If weight of cut is $< 1$, output candidate inequality.

Figure 9: Separation algorithm for simple domino parity inequalities.

As discussed in Section 3, the algorithm of Nagamochi, Nishimura, and Ibaraki [31] finds all sets $S$ with $x(\delta(S)) < 3$ in $O(m^2n + mn^2 \log n)$ time [18]. The number of such sets is at most $O(n^2)$ [18]. Each of the sets can be the body or union of body and root of at most $n$ light or heavy teeth. These $O(n^3)$ teeth can then be found and sorted according to root in $O(n^4)$ additional time. Thus the time required for step 1 is $O(n^4 + m^2n + mn^2 \log n)$.

Theorem 5 implies that time required for Step 2 is $O(n^3m)$. Details of buildT imply that the total time spent in this subroutine over the course of the algorithm (lines 3 and 12) is at most $O(nm^2)$.

For the light teeth, this results in a graph $G_{\mathcal{L}}$ containing a tree $T_{\mathcal{L}}$ with a root with $n$ branches, and a total number of nodes of $O(n)$. There is an edge for every edge in the support graph, so the number of edges is $O(m)$. On this graph, it takes $O(n^2 m \log \frac{n^2}{m})$ time to find a minimum odd cut using a Gomory-Hu tree (step 4).

By Theorem 6, steps 7-9 take total time at most $O(n^3m)$. Lines 10-11 are not a bottleneck. To find violated inequalities that use a heavy tooth with root $i$, the graph $G_{\mathcal{K}}$ still contains $O(n)$ nodes (nodes from the original $O(n)$ light teeth plus $O(n)$ nodes for the laminar teeth with root $i$) and $O(m)$ edges. By Theorem 6, it suffices to consider $O(m)$ of these graphs. Thus the time spent on these graphs is $O(n^2m^2 \log \frac{n^2}{m})$ (step 13).

Given an odd cut in $G_{\mathcal{S}}$, the corresponding inequality can be recovered in time $O(n)$ times the size of the cut, i.e. in $O(nm)$ time (steps 5 and 14). $\square$

## 5.1 Proof of Theorem 5

The proof of Theorem 5 works in two steps. In the first step, we show that the number of roots $i$ that form a light tooth with any fixed body $S$ is at most 3. In the second step, we show that it is possible to obtain a laminar set of bodies such that all light teeth we consider have a body in this set. This implies that the number of bodies we consider is $O(n)$.

**Lemma 7** *At most 3 distinct light teeth share the same body.*

**Proof:** Consider a body $S$. The subtour constraint for $S$ imposes that $x^*(E(S)) \leq |S|-1$. The total weight leaving $S$, $x^*(\delta(S))$, equals $2|S|-2x^*(E(S))$, which can be seen by summing degree constraints on nodes in $S$. The tooth inequality with body $S$ and root $i$ imposes that $x^*(E(i:S)) \leq 2|S|-1-2x^*(E(S))$. Thus if $(i,S)$ is a light tooth, then $x^*(E(i:S)) > 2|S|-\frac{3}{2}-2x^*(E(S))$. Let $\ell(S)$ be the number of light teeth with body $S$. Summing over all roots of light teeth, we have that

$$\ell(S)(2|S|-\frac{3}{2}-2x^*(E(S))) < \sum_{i:(i,S)\text{ light}} x^*(E(i:S)) \leq x^*(\delta(S)) = 2|S|-2x^*(E(S)),$$

so that

$$\ell(S) < \frac{2|S|-2x^*(E(S))}{2|S|-\frac{3}{2}-2x^*(E(S))}.$$

The right hand side of this expression is maximized when $x^*(E(S))$ is at its upper bound of $|S|-1$, and hence equals $\frac{2}{1/2}=4$. Since $\ell(S)<4$ and integral, we have that $\ell(S) \leq 3$. $\square$

Let $(i,X)$ and $(i,Y)$ be two teeth with the property that $X \subseteq Y$. If the slack of the tooth inequality on $(i,Y)$ is at least the slack of the tooth inequality on $(i,X)$ plus the slack of the nonnegativity constraints for $E(i:Y-X)$, then we say that $(i,X)$ *improves* $(i,Y)$.

Sets $S,T \subset V$ are said to *cross* if $S \cap T$, $S-T$, $T-S$, and $V-(S \cup T)$ are all nonempty.

Instead of multiplying inequalities in the derivation of a $\{0,\frac{1}{2}\}$ inequality by $\frac{1}{2}$, we can simply add inequalities together and consider the derived inequality modulo 2. The $\{0,\frac{1}{2}\}$ inequalities are then inequalities with odd right hand side and even coefficients on the left hand side. To obtain even coefficients on the left, for a fixed set of tooth inequalities and degree constraints, it may be necessary to add nonnegativity inequalities.

If the tooth inequality for $(i,S)$ is used in the derivation of some $\{0,\frac{1}{2}\}$ inequality, it contributes an odd amount to the right hand side, and an odd amount to the coefficients of all edges in $E(i:S)$. Thus, removing $(i,S)$ and replacing it with $(i, S-T)$ changes the parity only of coefficients of edges in $E(i : S \cap T)$. If we then add (or remove) the nonnegativity constraints for these edges, no parities are changed, and the inequality remains valid. The next lemma describes how this can be useful to uncross bodies of teeth.

**Lemma 8** *If $(i,S)$ and $(j,T)$ are two teeth and $S$ and $T$ cross, then either one of the following four conditions holds strictly, or two conditions hold exactly:*
*i) the slack on tooth inequality $(i, S-T)$ plus the slack for the nonnegativity constraints for $E(i : S \cap T)$ is at most the slack of tooth inequality $(i,S)$,*
*ii) the slack on tooth inequality $(i, S \cup T)$ plus the slack for the nonnegativity constraints for $E(i : T-S)$ is at most the slack of tooth inequality $(i,S)$,*
*iii) the slack on tooth inequality $(j, T-S)$ plus the slack of the nonnegativity constraints for $E(j : S \cap T)$ is at most the slack of tooth inequality $(j,T)$.*
*iv) the slack on tooth inequality $(j, S \cup T)$ plus the slack of the nonnegativity constraints for $E(j : S-T)$ is at most the slack of tooth inequality $(j,T)$.*

**Proof:** We first consider the case $i \notin T$ and $j \notin S$. The slack for $(i,S)$ is

$$\begin{aligned} & 2\,|S| - 1 - 2x^*(E(S)) - x^*(E(i:S)) \\ = \; & [\; 2\,|S-T| - 1 - x^*(E(i:S-T)) - 2x^*(E(S-T)) \;] \\ & + [\; 2\,|S\cap T| - 2x^*(E(S-T : S\cap T)) - 2x^*(E(S\cap T) - x^*(E(i : S\cap T)) \;] \\ = \; & [\; 2\,|S-T| - 1 - 2x^*(E(S-T)) - x^*(E(i:S-T)) + x^*(E(i:S\cap T) \;] \\ & + [\; 2\,|S\cap T| - 2x^*(E(\{i\} \cup (S-T) : S\cap T)) - 2x^*(E(S\cap T)) \;], \end{aligned}$$

where the last expression is obtained by adding and subtracting the term $x^*(E(i : S \cap T))$. Note that the first bracketed term in this last expression is the sum of the slack on tooth inequality $(i, S-T)$ plus the slack on nonnegativity constraints for $E(i : S \cap T)$. The slack for $(j,T)$ is

$$\begin{aligned} & 2\,|T| - 1 - 2x^*(E(T)) - x^*(E(j:T)) \\ = \; & [\; 2\,|T-S| - 1 - x^*(E(j:T-S)) - 2x^*(E(T-S)) \;] \\ & + [\; 2\,|S\cap T| - 2x^*(E(T-S : S\cap T)) - 2x^*(E(S\cap T) - x^*(E(j : S\cap T)) \;] \\ = \; & [\; 2\,|T-S| - 1 - 2x^*(E(T-S)) - x^*(E(j:T-S)) + x^*(E(j:S\cap T) \;] \\ & + [\; 2\,|S\cap T| - 2x^*(E(\{j\} \cup (T-S) : S\cap T)) - 2x^*(E(S\cap T)) \;], \end{aligned}$$

By summing degree constraints on $S \cap T$, we have that $x^*(E(\{i\} \cup (S-T) : S \cap T)) + x^*(E(\{j\} \cup (T-S) : S \cap T)) + 2x^*(E(S \cap T)) \leq 2\,|S \cap T|$. Thus either at least one of $[\; 2\,|S\cap T| - 2x^*(E(\{i\} \cup (S-T) : S\cap T)) - 2x^*(E(S\cap T))]$ and $[\; 2\,|S\cap T| - 2x^*(E(\{j\} \cup (T-S) : S\cap T)) - 2x^*(E(S\cap T))]$ must be positive, or both terms equal 0. In conjunction with the above expressions for the slack of inequalities for $(i,S)$ and $(j,T)$, this implies that either (i) or (iii) holds strictly, or both hold exactly.

If $i \notin T$, $j \in S$, then apply the above argument to $(i, V-S-\{i\})$ and $(j,T)$ to get that either (ii) or (iii) holds.

If $i \in T$, $j \notin S$, then apply the above argument to $(i,S)$ and $(j, V-T-\{j\})$ to get that either (i) or (iv) holds.

If $i \in T$, $j \in S$, then apply the above argument to $(i, V-S-\{i\})$ and $(j, V-T-\{j\})$ to get that either (ii) or (iv) holds. □

The next lemma describes why uncrossing teeth is useful in bounding $\mathcal{L}$.

**Lemma 9** *Let $\mathcal{L}$ be a set of light teeth that satisfies the following property. For all pairs $(i,S)$ and $(j,T)$ in $\mathcal{L}$, at most one of the following pairs of bodies cross: $(S,T)$, $(S,T'_j)$, $(S'_i,T)$, $(S'_i,T'_j)$. Then, the size of $\mathcal{L}$ is $O(n)$.*

**Proof:** We first remove all teeth in $\mathcal{L}$ of the form $(i,j)$ or $(i, V - \{i,j\})$. By complementing with respect to some $i \in V$, this set of teeth has bodies in $V$. Let $\mathcal{L}'$ be the set of teeth in $\mathcal{L}$ that are not of this form.

For the teeth in $\mathcal{L}'$, we construct the following graph: There are $|\mathcal{L}'|$ pairs of vertices. Each pair corresponds to a tooth in $\mathcal{L}'$. For tooth $(i,S)$ the first vertex corresponds to $S$, the second to $S'_i$. There is an edge joining vertices for $S$ and $S'_i$. There is also an edge joining each pair of vertices that correspond to bodies that cross. A maximal independent set $I$ in this graph corresponds to a set of laminar bodies $\mathcal{K}$. Thus, this set $\mathcal{K}$ has size $O(n)$. Associate each $S \in \mathcal{K}$ with the light teeth in $\{(i,S)|i \in V\}$. Let $\mathcal{L}'' = L' \bigcap \cup_{S\in\mathcal{K}}\{(i,S)|i \in V\}$. By Lemma 7, the size of $\mathcal{L}''$ is $O(n)$. Define $\mathcal{K}'$ to be the set $\{S' \subset V | S' = V - S - \{i\}$ for $(i,S) \in \mathcal{L}''\}$. Since $|\mathcal{K}'| \leq |\mathcal{L}''|$, we have that $|\mathcal{K}'| = O(n)$.

Suppose that for tooth $(i,S)$, neither the vertex for $S$ nor the vertex for $S'$ are in $I$. Then both $S$ and $S'$ each cross some set in $\mathcal{K}$. Suppose $S$ crosses $T$ for a tooth $(j,T) \in \mathcal{L}''$, and $S'$ crosses $U$ for some tooth $(k,U) \in \mathcal{L}''$. We claim that either $T' = S'$ or $U' = S$.

If this claim is true, then every tooth in $\mathcal{L}$ has a body or complement body in $\mathcal{K}$ or $\mathcal{K}'$ or $V$. Since $|\mathcal{K} \cup \mathcal{K}' \cup V| = O(n)$, Lemma 7 implies that $|\mathcal{L}| = O(n)$.

We now establish the claim. Since $S$ crosses $T$ it cannot cross $T'$, thus either $S \cap T' = \emptyset$ or $T' \subset S$. In the former case, $S - T = \{j\}$; in the latter, $T' \cap S' = \emptyset$. Since also $S'$ does not cross $T$, either $S' \cap T = \emptyset$ or $S' \subset T$. The former implies that $T - S = \{i\}$; the latter that $S' \cap T' = \emptyset$.

If both $S - T = \{j\}$ and $T - S = \{i\}$, then $S' = T'$, and we are done. Otherwise $S' \cap T' = \emptyset$. If in addition, either $S \cap T' = \emptyset$ or $S' \cap T = \emptyset$, then either $S' = \{j\}$ or $T' = \{i\}$. But we removed all such teeth from $\mathcal{L}'$ at the start, so this case cannot occur. If neither $T' = S'$ nor $U' = S$, then it must be that $T' \subset S$, $S' \subset T$, $U' \subset S'$, and $S \subset U$. In this case, since $T$ crosses $S$ and contains $S'$, which contains $U'$, we then have that $T$ crosses $U$. But this contradicts $T, U \in \mathcal{K}$. Thus either $T' = S'$ or $U' = S$, and the claim is established. $\square$

We say that tooth $(i,S)$ *t-crosses* tooth $(j,T)$ if either $S$ or $S'_i$ crosses both $T$ and $T'_j$ **or** either $T$ or $T'_j$ crosses both $S$ and $S'_i$. Since $i$ can be in at most one of $T$ and $T'$, this implies that $(i,S)$ and $(j,T)$ do not cross if and only if at most one following pairs of bodies cross: $(S,T)$, $(S,T'_j)$, $(S'_i,T)$, and $(S'_i,T'_j)$. If two teeth $t$-cross, we can apply Lemma 8 to uncross them.

**Lemma 10** *There is a set of light teeth $\mathcal{L}$ with $|\mathcal{L}| = O(n)$ such that if there is a violated simple DP inequality derived using tooth inequalities of light teeth only, then there is a simple DP inequality derived using tooth inequalities from* *the set $\mathcal{L}$ only. Given laminar sets of $i$-teeth for all $i$, the set $\mathcal{L}$ can be found in $O(n^3m)$ time.*

**Proof:** We begin with the set of teeth $\mathcal{L}'$ given by Lemma 2 and Corollary 1. We will create a new set $\mathcal{L}$ such that each tooth in $\mathcal{L}'$ is improved by some tooth in $\mathcal{L}$. This is done on an incremental basis, until all teeth in $\mathcal{L}'$ are improved by some tooth in $\mathcal{L}$. We start with $\mathcal{L} = \emptyset$. Throughout, we maintain that no two teeth in $\mathcal{L}$ t-cross. Then, by Lemma 9 the size of $\mathcal{L}$ is $O(n)$ throughout the algorithm.

We begin with $\mathcal{L} = \{(1,S)|(1,S) \in \mathcal{L}'\}$. By Theorem 3, no two teeth in $\mathcal{L}$ t-cross. Then, for roots $i = 2$ through $n$, we orient all teeth $(j,T) \in \mathcal{L}$ (by perhaps replacing $(j,T)$ with $(j,T'_j)$) so that $T$ does not contain $i$. We consider one by one teeth in $\{(i,S)|S \subset V\} \cap \mathcal{L}'$; and for one such $(i,S)$ we apply Lemma 8 to $(i,S)$ and the teeth in $\mathcal{L}$ starting with the teeth with the smallest bodies. With such a procedure, each uncrossing produces a new tooth that does not t-cross any previously uncrossed tooth: If the bodies of $(j,T) \in \mathcal{L}$ and $(k,U) \in \mathcal{L}$ are nested before uncrossing one of them with $(i,S)$, then after uncrossing, their respective bodies are either still nested, or completely disjoint. Thus we end up with a final set of teeth that do not t-cross.

The time to uncross one pair of teeth is $O(m)$. Since the size of $\mathcal{L}$ is at most $O(n)$, the number of uncrossings per new tooth in $\mathcal{L}'$ is at most $O(n)$. The initial size of $\mathcal{L}'$ may be $O(n^2)$. Thus the total time taken by this routine is $O(n^3m)$. □

The final piece of the proof involves establishing the time it takes to go from the list of light teeth sorted by root obtained in line (1) of the algorithm to an organized laminar set of bodies for each root. For the proof of Lemma 10, all that is needed is that the bodies be sorted according to size. Naively, this takes at most $O(n^2)$ time per root, or $O(n^3)$ overall.

## 5.2 Proof of Theorem 6

We begin the proof with a theorem about the structure of *heavy* teeth. Although the $i$-heavy sets need not be nested, they satisfy a certain 'circular' property.

**Theorem 8** *Let $i \in V$ be a given root. There is a cyclic ordering of the vertices in $V \setminus \{i\}$ such that each $i$-heavy set is the union of consecutive vertices in the ordering.*

The full proof is given in the Appendix. It is based on the following: Let $j \in V \setminus \{i\}$ be an arbitrary vertex, and let $M$ be a 0-1 matrix whose columns correspond to the vertices in $V \setminus \{i,j\}$, and whose rows are the incidence vectors of the $i$-heavy sets which do not include $j$. Due to Lemma 4, the theorem is true if and only if the columns of $M$ can be permuted so that, in every row of the resulting matrix, the 1s occur consecutively. Then, from Theorem 9 of Tucker [37] on matrices with the consecutive 1s property, it suffices to prove five claims that disallow certain arrangements of $i$-heavy teeth.

**Example (continued):** The 1-light sets are $\{2\}$, $\{4\}$, $\{2,7\}$, $\{3,4,5,6,8,9\}$, $\{3,\ldots,9\}$ and $\{2,3,5,6,7,8,9\}$; The 1-heavy sets are $\{3\}$, $\{4,8\}$, $\{3,5,6\}$, $\{2,4,7,8,9\}$, $\{2,3,5,6,7,9\}$ and $\{2,4,5,6,7,8,9\}$. A suitable ordering of $V \setminus \{1\}$ is $3,5,9,8,4,2,7,6$. The 3-light sets are $\{5\}$, $\{6\}$, $\{5,6\}$, $\{1,2,4,7,8,9\}$, $\{1,2,4,5,7,8,9\}$ and $\{1,2,4,6,7,8,9\}$. The 3-heavy sets are $\{1\}$, $\{2\}$, $\{1,4\}$, $\{2,7\}$, $\{2,5,6,7,8,9\}$, $\{1,4,5,6,8,9\}$, $\{2,4,5,6,7,8,9\}$ and $\{1,4,5,6,7,8,9\}$. A suitable ordering of $V \setminus \{3\}$ is $5,6,1,4,2,7,8,9$.

An important corollary of Theorem 8 is the following.

**Corollary 4** *For a fixed root $i$, the $i$-heavy sets can be partitioned into $\mathcal{O}(n)$ nested families.*

**Proof:** Without loss of generality, assume that $i = 1$ and that a suitable ordering of $V \setminus \{1\}$ is simple $2,\ldots,n$. Then the first nested family includes all sets which contain 2 but not $n$, the second includes all those which contain 3 but not 2, and so on. □

This immediately enables us to save a factor of $\mathcal{O}(n)$ in phase 2:

**Corollary 5** *Only $\mathcal{O}(n^2)$ minimum odd cut computations are needed in phase 2.*

**Proof:** Instead of performing one minimum odd cut calculation for each heavy tooth, we need only perform one minimum odd cut calculation for each of the $\mathcal{O}(n^2)$ nested families. □

To improve the running time further, we need to exploit the sparsity of the support graph $G^*$. To this end, we now describe a simple lemma which enables us to eliminate teeth from consideration. We will then prove that, after applying the lemma, the number of light and heavy teeth is significantly reduced.

**Lemma 11** *Suppose a violated $\{0,\frac{1}{2}\}$-cut can be derived using the tooth inequality with root $i$ and body $S$. If there exists a set $S' \subset V \setminus \{i\}$ such that*

- $E(i:S) \cap E^* = E(i:S') \cap E^*$,
- $2|S'| - 2x^*(E(S')) - x^*(E(i:S')) \leq 2|S| - 2x^*(E(S)) - x^*(E(i:S))$,

*then we can obtain a $\{0,\frac{1}{2}\}$-cut violated by at least as much by replacing the body $S$ with the body $S'$ (and adjusting the set of used non-negativity inequalities accordingly).*

**Proof**: By Proposition 1, we have to consider the net change in the sum of the slacks of the *used* inequalities. The second condition in the lemma simply says that the slack of the tooth inequality with root $i$ and body $S'$ is not greater than the slack of the tooth inequality with root $i$ and body $S$. Therefore replacing $S$

with $S'$ causes the sum of the slacks to either remain the same or decrease. Now we consider the *used* non-negativity inequalities. The only variables to receive an *odd* coefficient in a tooth inequality with root $i$ and body $S$ are those which correspond to edges in $E(i:S)$, and a similar statement holds for $S'$. So, for the edges in $E(i:(S \setminus S') \cup (S' \setminus S))$, the non-used non-negativity inequalities must now be used and vice-versa. But this has no effect on the sum of the slacks, because $E(i:(S \setminus S') \cup (S' \setminus S)) \subset E \setminus E^*$ by assumption and the slack of a non-negativity inequality for an edge in $E \setminus E^*$ is zero. Hence, the total sum of slacks is either unchanged or decreased and the new $\{0, \frac{1}{2}\}$-cut is violated by at least as much as the original. □

The next theorem shows that, after Lemma 11 is applied, relatively few heavy teeth remain.

**Theorem 9** *After applying the elimination criterion of Lemma 11, only $\mathcal{O}(nm)$ heavy tooth inequalities remain, and these can be partitioned into $\mathcal{O}(m)$ nested families.*

**Proof:** By the circular property of $i$-heavy teeth (Theorem 8), the sets $E(i:S)$ also have a circular property. After applying the elimination criterion, there can be at most $|\delta(i)|^2$ $i$-heavy teeth. So the total number of heavy tooth inequalities is at most $\sum_{i \in V}(|\delta(i)|)^2 \leq n \sum_{i \in V} |\delta(i)| = 2nm$. Moreover the $i$-heavy sets partition naturally into $|\delta(i)|$ nested families, giving $\sum_{i \in V} |\delta(i)| = 2m$ nested families in total. □

To complete the proof of Theorem 6, it remains to show that the reduction and reorganization of heavy teeth from $O(n^3)$ teeth in lists sorted by root to $O(m)$ nested families can be accomplished in $O(n^3 m)$ time.

**Completion of proof of Theorem 6:** From the sorted lists, we first apply Lemma 11. There are at most $O(n^2)$ $i$-heavy teeth in $i$'s list. For each tooth $(i:S)$, we create an ordered list of size at most $|\delta(i)|$ of the edges in $E(i:S)$ in $O(|\delta(i)|)$ time per set, and compute the value on the right of the expression of the second criteria of Lemma 11. We can sort these $O(n^2)$ lists with $O(n^2 \log n)$ comparisons, each comparison taking $O(|\delta(i)|)$ time. The value computation takes at most $O(n^2 m)$ time. After sorting, elimination of sets is not a bottleneck. Thus, over all roots $i$, the time spent in elimination is $O(n^3 m)$.

For each $i$, it then takes $O(n)$ time per tooth to place it in a sorted family. To find the nesting order of each nested family for teeth with root $i$ then takes $O(n|\delta(i)|^2 \log n)$ time, or $O(n^2 m \log n)$ overall. This completes the proof. □

# 6 Concluding Remarks

We have given a polynomial-time separation algorithm for the simple DP inequalities, which include the simple comb inequalities as a special case. This is a significant extension of the results of Padberg and Rao [34] and forms the latest in a series of positive results concerned with comb separation (Padberg & Rao [34], Carr [6], Fleischer & Tardos [11], Caprara, Fischetti & Letchford [4], Letchford [21], Caprara & Letchford [5]).

Since our work first appeared in [23], Letchford, Reinelt & Theis [24] have described an improved algorithm for the separation of 2-matching inequalities, which runs in $\mathcal{O}(n^2|E^*|\log(n^2/|E^*|))$ time.

A number of open questions immediately spring to mind. The main one is, of course, whether there exists a polynomial-time separation algorithm for *general* comb inequalities, or perhaps a generalization of them such as the *domino-parity* inequalities [21]. For some further discussion of this issue see [22].

We can also consider special classes of graphs. For a given graph $G$, let us denote by $\mathrm{S}(G)$ the polytope defined by the degree equations, the SECs, and the non-negativity and simple DP inequalities. (Now we only define variables for the edges in $G$.) Let us say that a graph $G$ is *S-perfect* if $\mathrm{S}(G)$ is an integral polytope. Clearly, the TSP is polynomially-solvable on S-perfect graphs. It would be desirable to know which graphs are S-perfect. Similarly, let us say that a graph is *S-Hamiltonian* if $\mathrm{S}(G)$ is non-empty. Obviously, every Hamiltonian graph is S-Hamiltonian, but the reverse does not hold. (The famous *Peterson graph* is S-Hamiltonian, but not Hamiltonian.) It would be desirable to establish structural properties for the S-Hamiltonian graphs, just as Chvátal [7] did for the so-called *weakly Hamiltonian* graphs.

Finally, we would like to make an observation about *lower bounds*. The lower bound obtained by optimizing over $\mathrm{SEP}(n)$ is good in practice, and it is conjectured (e.g. Goemans [12]) that it is always at least 3/4 of the optimal value when the costs satisfy the triangle inequality. We would expect the addition of the simple DP inequalities to lead to even stronger bounds in practice. However, consider the family of fractional extreme points of $\mathrm{SEP}(4k)$, with $k \geq 2$, shown in Figure 10. Points of this type violate many comb inequalities, but no simple DP inequalities. The following *path inequality* is valid for $\mathrm{STSP}(4k)$, see Cornuéjols, Fonlupt & Naddef [8]:

$$\sum_{i=1}^{k-1} x(\delta(H_i)) + \sum_{i=1}^{3} x(\delta(T_i)) \geq 4k+2,$$

where $H_i := \{1,\ldots,4i\}, T_1 := \{1,5,\ldots,4k-3\}, T_2 := \{2,3,6,7,\ldots,4k-2,4k-1\}$ and $T_3 := \{4,8,\ldots,4k\}$. Moreover the left hand side coefficients of the path inequality are easily seen to satisfy the triangle inequality. Now, the left hand side of this inequality, computed with respect to the fractional point, is only $3k+3$. Thus, even when simple DP inequalities are used, the ratio between lower bound and optimum can be as bad as $(3k+3)/(4k+2)$, which approaches 3/4 as $k$ approaches infinity.

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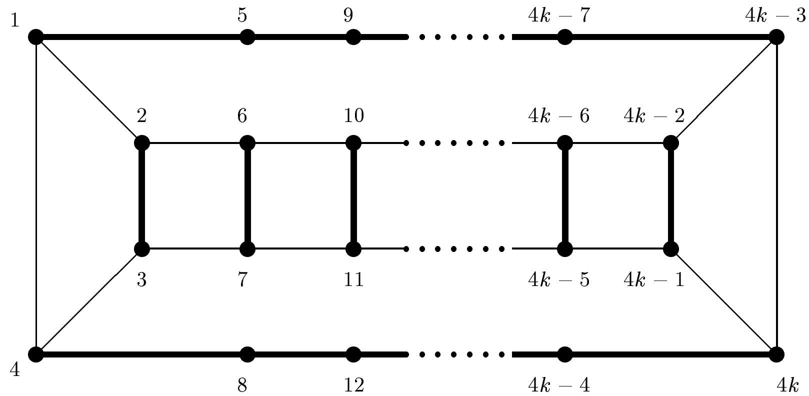


Figure 10: Fractional vertex of subtour polytope. Bold lines have $x_e^* = 1$, narrow lines have $x_e^* = 1/2$.

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# Appendix

**Proof of Lemma 8:** Let $j \in V \setminus \{i\}$ be an arbitrary vertex, and let $M$ be a 0-1 matrix whose columns correspond to the vertices in $V \setminus \{i, j\}$, and whose rows are the incidence vectors of the $i$-heavy sets which do not include $j$. Due to Lemma 4, the theorem is true if and only if the columns of $M$ can be permuted so that, in every row of the resulting matrix, the 1s occur consecutively. Then, from Theorem 9 of Tucker [37] on matrices with the consecutive 1s property, it suffices to prove the following five claims:

**Claim 1:** There cannot exist $i$-heavy sets $S_1, \ldots, S_m \subset V \setminus \{i,j\}$, with $m \geq 3$, and a set $R \subseteq V \setminus \{i,j\}$ consisting of distinct vertices $v_1, \ldots, v_m$, such that, for all $k$, $S_k \cap R = \{v_k, v_{k+1}\}$ (with subscripts taken modulo $m$).

**Claim 2:** There cannot exist $i$-heavy sets $S_1, \ldots, S_m \subset V \setminus \{i,j\}$, with $m \geq 4$, and a set $R \subseteq V \setminus \{i,j\}$ consisting of distinct vertices $v_1, \ldots, v_m$, such that $S_k \cap R = \{v_k, v_{k+1}\}$ for $k = 1, \ldots, m-2$, $S_{m-1} \cap R = \{v_1, \ldots, v_{m-2}, v_m\}$ and $S_m \cap R = \{v_2, \ldots, v_m\}$.

**Claim 3:** There cannot exist $i$-heavy sets $S_1, \ldots, S_m \subset V \setminus \{i,j\}$, with $m \geq 3$, and a set $R \subseteq V \setminus \{i,j\}$ consisting of distinct vertices $v_1, \ldots, v_{m+1}$, such that $S_k \cap R = \{v_k, v_{k+1}\}$ for $k = 1, \ldots, m-1$ and $S_m \cap R = \{v_2, \ldots, v_{m-1}, v_{m+1}\}$.

**Claim 4:** There cannot exist four $i$-heavy sets $S_1, \ldots, S_4 \subset V \setminus \{i,j\}$ such that $S_2, S_3$ and $S_4$ are mutually disjoint and, for $i = 2, \ldots 4$, $S_1 \cap S_i$ and $S_i \setminus S_1$ are both non-empty.

**Claim 5:** There cannot exist four $i$-heavy sets $S_1, \ldots, S_4 \subset V \setminus \{i,j\}$ such that $S_2 \cap S_3 = \emptyset$, $S_2 \cup S_3 \subset S_4$, $S_1 \setminus S_4 \neq \emptyset$, $V \setminus (\{i,j\} \cup S_1 \cup S_4) \neq \emptyset$ and, for $i \in \{2,3\}$, $S_1 \cap S_i$ and $S_i \setminus S_1$ are both non-empty.

**Proof of Claim 1:** Here we assume that indices are taken modulo $m$. We sum together the following subtour elimination constraints:

- the SEC on $\{i\} \cup S_1 \cup \cdots \cup S_m$ (two times);
- the SECs on $S_k \cap S_{k+1}$ for $k = 1, \ldots, m$;
- the SECs on $(S_k \cap S_{k+1}) \setminus (S_{k+2} \cup \cdots \cup S_{k+m-1})$ for $k = 1, \ldots, m$;
- the SEC on $\{i\} \cup (S_1 \cap \cdots \cap S_m)$ ($m-2$ times).

Simple but tedious checking shows that the left hand side of the resulting inequality is greater than or equal to $2\sum_{k=1}^m x(E(S_k)) + \sum_{k=1}^m x(E(i : S_k))$, and that the right hand side is $2\sum_{k=1}^m |S_k| - 2m$. Hence we have:

$$2\sum_{k=1}^{m} x(E(S_k)) + \sum_{k=1}^{m} x(E(i : S_k)) \leq \sum_{k=1}^{m} 2|S_k| - 2m. \tag{12}$$

But the sum from $k = 1, \ldots, m$ of the tooth inequality with root $i$ and body $S_k$ is:

$$2\sum_{k=1}^{m} x(E(S_k)) + \sum_{k=1}^{m} x(E(i : S_k)) \leq \sum_{k=1}^{m} 2|S_k| - m. \tag{13}$$

Comparing (12) and (13) we conclude that, when $x^* \in SEP(n)$, at least one of the $m$ tooth inequalities has slack $\geq 1$ at $x^*$. Hence at least one of the sets

$S_1, \ldots, S_m$ is not $i$-heavy.

**Proof of Claim 2:** Lemma 4 shows that $S_m$ is $i$-heavy if and only if $V \setminus (\{i\} \cup S_m)$ is $i$-heavy, and a similar statement holds for $S_{m-1}$. If we replace $S_m$ with $V \setminus (\{i\} \cup S_m)$ and replace $S_{m-1}$ with $V \setminus (\{i\} \cup S_{m-1})$, we obtain the configuration described in Claim 1, which we have already proved cannot exist.

**Proof of Claim 3:** If we replace $S_m$ with $V \setminus (\{i\} \cup S_m)$, we again obtain the configuration described in Claim 1.

**Proof of Claim 4:** If $S_2, S_3$ and $S_4$ are $i$-heavy, then the sum of the slacks of the three associated tooth inequalities must be less than 3. Equivalently,

$$\sum_{j=2}^{4} (2x(E(S_j)) + x(E(i : S_j))) > 2 \sum_{j=2}^{4} |S_j| - 6.$$

Subtracting from this the degree equation on $i$, and weakening gives:

$$2 \sum_{j=2}^{4} x(E(S_j) > 2 \sum_{j=2}^{4} |S_j| - 8;$$

or, equivalently,

$$\sum_{j=2}^{4} x(E(S_j) > \sum_{j=2}^{4} |S_j| - 4.$$

Subtracting the degree equations on $S_j \cap S_1$ for $j = 2, 3, 4$, and adding the SECs on $S_j \cap S_1$ and $S_j \setminus S_1$ for $j = 2, 3, 4$ yields (after some tedious rearranging):

$$2x(E(S_1) + x(i : S_1) < 2|S_j| - 2,$$

thus showing that $S_1$ cannot be $i$-heavy.

**Proof of Claim 5:** If we replace $S_4$ with $V \setminus (\{i\} \cup S_4)$, we obtain the configuration described in Claim 4. □