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Constructive Bounds on Ordered Factorizations

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Abstract

The number of ways to factor a natural number into an ordered product of integers, each factor greater than one, is called the *ordered factorization of* n and is denoted $H(n)$. We show upper and lower bounds on $H(n)$ with explicit constructions.

1 Introduction

For $n \in \mathbb{Z}^+$, let $H(n)$ denote the number of *ordered factorizations* of n, by which we mean expressions of n as the product of integers $p_i \geq 2$ where the order of factors is essential. Equivalently, $H(n)$ is the number of tuples $(p_1, p_2, \ldots, p_k)$ with $p_i \geq 2$ and $\prod p_i = n$, without restrictions on k. $H(1) = 1$ by convention, the only factorization being () with $k = 0$. $H(20) = 8$, the factorizations being (20), (10,2), (5,4), (5,2,2), (4,5), (2,10), (2,5,2), (2,2,5). Newberg and Naor[3] use $H(n)$ as a lower bound for an application in computational biology.

Define

$$\rho = \zeta^{-1}(2) \approx 1.7264724,$$

where ζ is the Riemann zeta function, so that

$$\sum_{n=1}^{\infty} \frac{1}{n^\rho} = 2,$$

and more usefully,

$$\sum_{n=2}^{\infty} \frac{1}{n^\rho} = 1.$$

Hille [2] showed the existence of a constant c such that $H(n) \leq cn^\rho$; Chor *et al.* [1] improved this to $c = 1$:

$$H(n) \leq n^\rho. \tag{1}$$

Hille also gave an existential lower bound: for all $\epsilon > 0$,

$$\limsup \frac{H(n)}{n^{\rho-\epsilon}} = \infty. \tag{2}$$

Newberg and Naor show an explicit construction lower bounding $H(n)$ with $n \log^c n$ for some c. Chor *et al.* gave explicit constructions for certain values of ϵ.

In this note we give simplified proofs of both upper and lower bounds.

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2 Upper bound

The upper bound $H(n) \leq n^\rho$ is proven by induction on n. The base case $n = 1$ is satisfied. Suppose the result is true for all $n' < n$. We count the ordered factorizations of n according to their first element p_1, which is a factor of n larger than 1. The remainder $(p_2, \ldots, p_k)$ is an ordered factorization of n/p_1. So we have

$$H(n) = \sum_{d|n, d>1} H(n/d).$$

By induction,

$$H(n/d) \leq (n/d)^\rho,$$

so that

$$\begin{aligned} H(n) &= \textstyle\sum_{d|n,d>1} H(n/d) \leq \sum_{d|n,d>1} \frac{n^\rho}{d^\rho} < n^\rho \sum_{d>1} \frac{1}{d^\rho} \\ &= n^\rho(\zeta(\rho) - 1) = n^\rho(2-1) = n^\rho, \end{aligned}$$

completing the induction. In fact we see that the inequality is strict for $n > 1$.

3 Lower bound

For $\alpha = \rho - \epsilon$ we will give a family of integers n for which $\limsup H(n)/n^\alpha = \infty$.

Because $\zeta(t)$ is strictly monotone decreasing in t, we know

$$\zeta(\alpha) = \sum_1^\infty \frac{1}{n^\alpha} > 2.$$

There is a finite integer b for which already

$$\sum_1^b \frac{1}{n^\alpha} > 2.$$

Use monotonicity again to claim there is γ with $\alpha < \gamma < \rho$ satisfying

$$\sum_1^b \frac{1}{n^\gamma} = 2,$$

or, more usefully,

$$\sum_2^b \frac{1}{n^\gamma} = 1.$$

Fix such α, b, γ.

Now select a large integer t. For $k = 2, 3, \ldots, b$, we define

$$c_k = \lfloor t/k^\gamma \rfloor.$$

Set $u = \sum c_k$, so that $0 \leq t - u \leq b - 2$. Define

$$n = \prod_{k=2}^b k^{c_k}.$$

We will compare $H(n)$ to n^α. Among the ordered factorizations counted by $H(n)$ are the orderings of (c_2 copies of 2, ..., c_b copies of b). The number of such orderings is given by the multinomial coefficient

$$v(n) = \frac{u!}{\prod_{k=2}^{b} c_k!}.$$

From Stirling's approximation,

$$v(n) = \prod_k \left(\frac{u}{c_k}\right)^{c_k} \times \sqrt{\frac{2\pi u}{\prod(2\pi c_k)}} \times [1 + o(1)],$$

where the $o(1)$ term goes to 0 with increasing c_k and hence with increasing n.

To estimate the first product, recall $c_k \le t/k^\gamma$, so that

$$\prod_k \left(\frac{u}{c_k}\right)^{c_k} \ge \prod_k \left(\frac{uk^\gamma}{t}\right)^{c_k} = (u/t)^u (\prod_k k^{c_k})^\gamma$$

We have $(u/t)^u \ge e^{-(t-u)} \ge e^{-b+2}$, while the other factor is simply n^γ. So our first product is at least $e^{-b+2}n^\gamma$.

The second product is

$$\sqrt{\frac{2\pi u}{\prod(2\pi c_k)}}.$$

Notice that u and each c_k vary linearly with t and hence with $\log n$, the coefficients depending on b but not n. So the second product is

$$\Omega_b((\log n)^{-(b-2)/2}).$$

To estimate the implied coefficient:

$$\log n = \sum c_k \log k \approx t \sum k^{-\gamma} \log k$$

so that

$$u \approx \frac{\log n}{\sum_2^b k^{-\gamma} \log k},$$

$$c_j \approx \frac{\log n}{\sum_2^b k^{-\gamma} \log k} j^{-\gamma},$$

and we get that the second product is about

$$(2\pi)^{-(b-2)/2} (\prod_2^b j^{\gamma/2}) \left(\sum k^{-\gamma} \log k\right)^{(b-2)/2} (\log n)^{-(b-2)/2}$$

$$= b!^{\gamma/2} \left(\frac{\sum k^{-\gamma} \log k}{2\pi \log n}\right)^{(b-2)/2}.$$

Summarizing,

$$H(n) \ge v(n) \ge n^\gamma (\log n)^{-(b-2)/2} c_b (1 + o(1))$$

with

$$c_b = e^{-b+2} b!^{\gamma/2} \left(\frac{\sum k^{-\gamma} \log k}{2\pi}\right)^{(b-2)/2}.$$

Since $\gamma > \alpha$, we have

$$\limsup_n H(n)/n^\alpha = \infty,$$

as required.

References

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[3] L.A. Newberg and D. Naor. A lower bound on the number of solutiuons to the probed partial digestion problem. *Advanced Applied Math.*, 14:172–183, 1993.