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Abstract

This paper presents and evaluates alternative methods for multi-step forecasting using univariate and multivariate functional coefficient autoregressive (FCAR) models. The methods include a simple "plug-in" approach, a bootstrap-based approach, and a multi-stage smoothing approach, where the functional coefficients are updated at each step to incorporate information from the time series captured in the previous predictions. The three methods are applied to a series of U.S. GNP and unemployment data to compare performance in practice. We find that the bootstrap-based approach out-performs the other two methods for nonlinear prediction, and that little forecast accuracy is sacrificed using any of the methods if the underlying process is actually linear.

Key Words: Bootstrap prediction, multi-step prediction, smoothing, vector nonlinear time series

1 Introduction

Nonlinear models for time series have become fairly common in the last ten years, with research primarily focused on characterization of the nonlinear process behavior using parametric models, such as Threshold Autoregressive (TAR) or Exponential AR (EXPAR), or nonparametric methods, such as nonlinear AR models estimated using kernel regression techniques. However, forecasting with nonlinear time series models is not necessarily straightforward, even in the parametric case. For a time series $\{Y_t\}$, the assumed nonlinear mean structure of the process complicates the derivation of the expected value of Y_{t+k} given $Y_t, \ldots, Y_1$ when $k > 1$. See Pemberton (1987) for initial discussion of this issue in the case of a TAR model, and a proposed numerical approach. A number of papers have since appeared which find that Monte Carlo or bootstrap approaches are feasible for multi-step forecasting with parametric nonlinear models. See, for example, Clements and Smith (1993). When nonparametric methods are used to model $\{Y_t\}$, the multi-step prediction issue is further complicated. Chen, Yang, and Hafner (2004) proposed a multi-stage smoothing approach for k-step-ahead prediction in the case of a nonlinear AR(p) model estimated non-parametrically, and showed their method is more efficient than a direct smoother.

In this paper, we investigate methods for obtaining multi-step forecasts from a Functional Coefficient AR (FCAR) model, an AR model in which the AR coefficients are allowed to vary as a function of another variable, such as a lagged value of the time series itself or a variable exogeneous to the time series. The functional form is usually left unspecified and estimated nonparametrically using kernel methods. In this sense, an FCAR model might be thought of as a hybrid of parametric and nonparametric models. These types of models were first introduced by Chen and Tsay (1993) and have been further investigated by, for example, Chen and Liu (2001) and Cai, Fan, and Yao (2000). However, little work has been done to assess the accuracy of different prediction methods in the context of these models.

We investigate this issue by outlining and assessing three different multi-step forecast methods that may be used with univariate and multivariate FCAR models. While there have been a few limited examples of forecasting with a univariate FCAR model (see, for example, Chen and Tsay, 1993), systematic assessment of forecast methods for FCAR models remains largely unexplored. In particular, no work has looked at FCAR models for forecasting vector time series. The structure of the paper is as follows. In Section 2, we formally define an FCAR model and briefly discuss model properties and estimation techniques. In Section 3, we detail the proposed FCAR forecast methods and compare the methods based on a small simulation study. Section 4 provides an application to forecasting U.S. GNP and unemployment rates using a vector FCAR model, and compares the results to those obtained from prior attempts at modeling this data. Section 5 concludes.

2 Definition and Estimation of a Functional Coefficient AR Model

For generality, we present the discussion in terms of a vector FCAR (VFCAR) model, which was defined and discussed in Harvill and Ray (2004). A brief outline of that work is included here to facilitate describing forecasting methods in Section 3. A vector FCAR model of order p, defined in (1), is one where the coefficient *matrices* are allowed to vary as a function of a specified variable Z which may be a variable exogeneous to the series, such as time, or lagged values of the series. Specifically, the model is

$$\boldsymbol{Y}_t = \boldsymbol{f}^{(0)}(\boldsymbol{Z}_t) + \sum_{j=1}^{p} \boldsymbol{f}^{(j)}(\boldsymbol{Z}_t)\boldsymbol{Y}_{t-j} + \boldsymbol{\varepsilon}_t, \qquad t = s+1, \ldots, T, \tag{1}$$

where $s = \max(p, d)$, $\boldsymbol{Z}$ is the functional variable of dimension $m \geq 1$, which may be an exogeneous predictor or lagged value(s) of the series, $\boldsymbol{Z} = \boldsymbol{Y}_{t-d}$, the $\boldsymbol{\varepsilon}_t$ are independent, identically distributed random variables having mean vector $\mathbf{0}$ and $k \times k$ covariance matrix $\boldsymbol{\Sigma}$, independent of $\boldsymbol{Y}_s$ and $\boldsymbol{Z}_s$ for all $s < t$; $\boldsymbol{f}^{(j)}$, $j = 1, \ldots, p$ are $k \times k$ matrices with elements $[f_{il}^{(j)}]$ that are real-valued measurable functions that change as a function of $\boldsymbol{Z}_t$, and which have continuous second derivatives. The model (1) includes a functional intercept term in each component of the series, as specified by the vector $\boldsymbol{f}^{(0)}$. In the case where $\boldsymbol{Z}$ is a lagged value of $\boldsymbol{Y}$ with functional delay $d \leq p$, inclusion of the intercept term results in a non-identifiable model. In such cases, one of either the intercept term or the lag d term in the autoregression should be omitted. When $k = 1$, this model reduces to the univariate FCAR model of Cai, Fan, and Yao (2000).

2.1 Estimation of VFCAR model

The elements of the matrices $\boldsymbol{f}^{(j)}$, $j = 1, \ldots, p$ in model (1) are functions that can be estimated from the observations $\{\boldsymbol{Z}_t, \boldsymbol{Y}_t\}_{t=1}^{T}$ using locally constant or locally linear multi-variable regression in a neighborhood of $\boldsymbol{Z}_t$ determined by a specified kernel and bandwidth matrix. Let p^* represent the autoregressive fit order, and at time t, denote the kp^*-vector of predictors by $\boldsymbol{X}_t$; that is, let

$$\boldsymbol{X}_t = [\boldsymbol{Y}_{t-1}, \boldsymbol{Y}_{t-2}, \cdots \boldsymbol{Y}_{t-p^*}]'$$

where $\boldsymbol{Y}_{t-j} = [Y_{1,t-j}, Y_{2,t-j}, \ldots, Y_{k,t-j}]$, for $j = 1, \ldots, p^*$. Define $\boldsymbol{f}(\boldsymbol{Z}_t)$ by

$$\boldsymbol{f}(\boldsymbol{Z}_t) = [\boldsymbol{f}^{(1)}(\boldsymbol{Z}_t), \ldots, \boldsymbol{f}^{(p^*)}(\boldsymbol{Z}_t)]'.$$

Then model (1) can be written as

$$\boldsymbol{Y}_t = \boldsymbol{f}(\boldsymbol{Z}_t)\boldsymbol{X}_t + \boldsymbol{\varepsilon}_t, \quad t = s^*+1, \ldots, T, \quad s^* = \max(p^*, d).$$

Since all elements of $\boldsymbol{f}$ have continuous second-order derivatives, each $f_{il}^{(j)}(\cdot)$ may be approximated locally at z_0 by a linear function $f_{il}^{(j)}(\boldsymbol{z}) = \alpha_{il}^{(j)} + \beta_{il}^{(j)}(\boldsymbol{z} - \boldsymbol{z}_0)$. If the coefficient matrices are partitioned as $[\boldsymbol{\alpha} \,|\, \boldsymbol{\beta}]$, then the local linear least squares kernel estimator of $\boldsymbol{f}(\boldsymbol{Z})$ is $\hat{\boldsymbol{f}}(\boldsymbol{z}_0) = \hat{\boldsymbol{\alpha}}$, where $[\hat{\boldsymbol{\alpha}} \,|\, \hat{\boldsymbol{\beta}}]$ is the solution to $[\boldsymbol{\alpha} \,|\, \boldsymbol{\beta}]$ minimizing the sum of weighted squares

$$\sum_{t=s^*+1}^{T} \left[\boldsymbol{Y}_t - [\boldsymbol{\alpha} \,|\, \boldsymbol{\beta}] \begin{pmatrix} \boldsymbol{X}_t \\ \boldsymbol{U}_t \end{pmatrix}\right] \left[\boldsymbol{Y}_t - [\boldsymbol{\alpha} \,|\, \boldsymbol{\beta}] \begin{pmatrix} \boldsymbol{X}_t \\ \boldsymbol{U}_t \end{pmatrix}\right]' K_{\boldsymbol{H}}(\boldsymbol{Z}_t - \boldsymbol{z}_0), \tag{2}$$

where the first kp^* rows of $\boldsymbol{U}_t$ are the element-by-element product of $\boldsymbol{X}_t$ and $(Z_{1,t} - z_{1,0})$, the second kp^* rows are that of $\boldsymbol{X}_t$ and $(Z_{2,t} - z_{2,0})$, etc., K is a specified m-variate kernel function, $\boldsymbol{H}^{1/2}$ is the bandwidth matrix, and $K_{\boldsymbol{H}}(\boldsymbol{u}) = |\boldsymbol{H}|^{-1/2} K(\boldsymbol{H}^{-1/2}\boldsymbol{u})$. It follows from least squares theory that

$$\hat{\boldsymbol{f}}(\boldsymbol{z}_0) = (\boldsymbol{U}'\boldsymbol{W}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{W}\boldsymbol{Y},$$

where

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{X}_{p^*+1} & \boldsymbol{X}_{p^*+1}(Z_{1,p^*+1} - z_{1,0}) & \boldsymbol{X}_{p^*+1}(Z_{2,p^*+1} - z_{2,0}) & \cdots & \boldsymbol{X}_{m,p^*+1}(Z_{p^*+1} - z_{m,0}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{X}_T & \boldsymbol{X}_T(Z_{1,T} - z_{1,0}) & \boldsymbol{X}_T(Z_{2,T} - z_{2,0}) & \cdots & \boldsymbol{X}_T(Z_{m,T} - z_{m,0}) \end{bmatrix},$$

$\boldsymbol{U}'\boldsymbol{W}\boldsymbol{U}$ is non-singular, and $\boldsymbol{W} = \text{diag}\{K_h(Z_{p^*+1} - z_0), \ldots, K_h(Z_n - z_0)\}$.

2.2 Selection of bandwidth, functional variable, and model order

Several parameters affect the ultimate forecast performance of the estimated FCAR model, including the smoothing bandwidth, the functional variable, and the selected order of the autoregression.

We use a modified multi-fold cross-validation technique for determining an optimal bandwidth. Let r and Q be two positive integers such that $T > rQ$. To find an optimal value for the bandwidth, the first Q subseries of lengths $T - rq$ $(q = 1, \ldots, Q)$ are used to estimate the unknown coefficient functions. The one-step forecasting errors are computed based on the next section of the time series of length r using the estimated models. The selected bandwidth, h_{opt} is that value of h which minimizes the accumulated prediction error (APE)

$$APE(h) = \sum_{q=1}^{Q} APE_q(h), \tag{3}$$

where for $q = 1, \ldots Q$, $APE_q(h)$ is trace of the estimated error covariance matrix. This method was proposed in Cai, Fan, and Yao (2000) in the context of bandwidth selection for univariate FCAR models and was used in De Gooijer and Ray (2003) for model selection in the context of adaptive spline threshold-type models for vector time series. In our applications, $r = [0.1T]$ and $Q = 4$.

For selecting the delay variable, $\boldsymbol{Z}$, we recommend using knowledge of the underlying physical process, if available. Otherwise, the smoothing variable can be chosen by data-driven methods, such as to minimize the Akaike Information Criteria (AIC) or the APE criterion defined in (3). Similar methods are recommended for selecting the AR model order.

3 Forecasting with the FCAR Model

There is little work in the literature on multi-step forecasting with FCAR models. Chen and Tsay (1993) use their FCAR model to obtain multi-step ahead forecasts without specifying exactly how this is done, but note that FCAR models can substantially improve post-sample multi-step forecast accuracy compared to other linear and nonlinear time series models. Fan and Yao (2003) discuss direct and iterative, "plug-in" methods for forecasting with FCAR models. The direct method forecasts Y_{t+k} as a function of Y_t based on a model that ignores any relationship between Y_{t+k} and $Y_{t+k-j}, j = 1, \ldots, k-1$ and hence is potentially misspecified. The iterative method forecasts $Y_{t+k}, k > 1$ by naively substituting previously forecasted values, $\hat{Y}_{t+k-j}, j = 1, \ldots, k-1$ into the FCAR mean function, without taking account of the fact that computation of $E(Y_{t+k}|Y_t)$ is not a linear operation in the case of an FCAR model.

Here we present two alternative methods to the "plug-in" method for multi-step prediction using an FCAR model. The first is the bootstrap method, which has been found to perform well for multi-step prediction with parametric nonlinear time series models. The second is a multi-stage smoothing method, which has been proposed in the context a general nonlinear AR model by Chen (1996). For the sake of discussion, we will restrict our discussion to the univariate FCAR model of order p having functional variable Z of dimension $m = 1$. Extensions of the method to the vector framework are straightforward and will be illustrated in Section 4.

For the model given by

$$Y_t = f_0(Z_t) + \sum_{j=1}^{p} f_j(Z_t) Y_{t-j} + \varepsilon_t, \quad t = 1, 2, \ldots T, \tag{4}$$

the goal of prediction is to find an estimator of the conditional expectation

$$\begin{aligned} \mathrm{E}[Y_{T+M} \,|\, Y_T, \ldots, Y_{T-p}] &= \mathrm{E}\left[\sum_{j=1}^{p} f_j(Z_{T+M}) Y_{T+M-j} \,\middle|\, Y_T, \ldots, Y_{T-p} \right] && (5) \\ &= \sum_{j=1}^{p} f_j(Z_{T+M}) \mathrm{E}[Y_{T+M-j} \,|\, Y_T, \ldots, Y_{T-p}] && (6) \end{aligned}$$

$$= \sum_{j=1}^{p} f_j(Z_{T+M})\hat{Y}_{T+M-j}, \tag{7}$$

assuming $f_j(\cdot)$ is known and Z_t is exogeneous. When $f_j(\cdot)$ is estimated from $\{Y_t\}$ using $Z_t = Y_t - d$, the expectation in (5) is no longer a simple linear operation. The three methods described below for finding an M-step predictive estimate hinge on different ways of dealing with this complication.

3.1 Naive Plug-in Predictor

The naive plug-in predictor ignores the fact that the expectation in (5) is not a linear function of Y_{t+k-j} for $k \geq 2$ and simply plugs $\hat{Y}_{t+k-j}$ into the forecasted equation. The form of the functional coefficient is determined using only the within-sample series values. In other words, for $Z_t = Y_{t-d}$,

$$\hat{Y}_{T+M} = \sum_{j=1}^{p^*} \hat{f}_j(\hat{Y}_{T+M-d})\hat{Y}_{T+M-j}, \tag{8}$$

where $\hat{Y}_t = Y_t,\ t \leq T$ and $\hat{f}_j(\cdot)$ are the values $\hat{\alpha}_j$ minimizing (2).

3.2 Bootstrap Predictor

Like the plug-in estimator, the bootstrap prediction method uses only within-sample values to compute the functional coefficients and evaluates these coefficients at the predicted values. However, the predicted values are obtained as $\hat{Y}_{T+M} = \sum_{j=1}^{p^*} \hat{f}_j(\hat{Y}_{T+M-d})\hat{Y}_{T+M-j} + \epsilon^b$, where ϵ^b is a bootstrapped value of the within-sample residuals from the fitted FCAR model. The bootstrapped forecast is obtained for $b = 1, \ldots, B$ and the average across all bootstrap predictions is used as the M-step ahead point forecast. The predictive density of Y_{t+k} can be obtained using the complete set of bootstrap predictions. A similar idea was proposed in Huang and Shen (2004) for univariate FCAR models obtained using polynomial splines to estimate the functional coefficients. They note that care must be taken when $\hat{Y}_{t+M-d}$ falls outside or near the boundary of the range of the original Y_{t-d}, as the estimated functional coefficients may be very unreliable in this case.

3.3 Multi-stage Predictor

The multi-stage predictor is a modification of the naive predictor in which the functional coefficients are updated at each step to incorporate the information from Y_t encoded in the predicted response at time $T + j, j = 1, \ldots, M - 1$. Specifically,

$$\hat{Y}_{T+M} = \sum_{j=1}^{p^*} \hat{f}_j^M(\hat{Y}_{T+M-d})\hat{Y}_{T+M-j}, \tag{9}$$

where $\hat{Y}_t = Y_t, t \leq T$ and $\hat{f}_j^M$ are the values $\hat{\alpha}_j$ minimizing

$$\sum_{t=s^*+1}^{T+M-1} \left\{ Y_t - \sum_{j=1}^{p} \Big[\alpha_j + \beta_j(Z_t - z)\Big] Y_{t-j} \right\} K_h(Z_t - z). \tag{10}$$

Chen, Yang, and Hafner (2004) examine multi-step ahead prediction for a nonlinear AR(p) model fit using local polynomial estimation, and show their multi-stage smoother is more efficient than a direct smoother. A similar method for Markovian structures estimated using locally constant methods was studied by Chen (1996), who showed that multi-stage smoothing improves the estimation of the conditional mean. To the best of our knowledge, use of the multi-stage predictor in the context of FCAR models has not been previously considered.

3.4 Empirical Investigation of Forecasting Methods

A small empirical study of the three forecasting methods was conducted to compare their performance. The methods were applied to two models; a nonlinear, univariate smoothed threshold autoregressive (STAR) model

$$X_t = \varepsilon_t + 0.6X_{t-1}\left[1 - \frac{1}{1+\exp(-5X_{t-1})}\right] - 0.4X_{t-1}\left[\frac{1}{1+\exp(-5X_{t-1})}\right],$$

and a linear vector autoregressive (VAR) model of order 2

$$\begin{aligned} X_{1,t} &= \varepsilon_{1,t} - 0.4X_{1,t-1} + 0.1X_{2,t-1} - 0.15X_{1,t-2} + 0.2X_{2,t-2} \\ X_{2,t} &= \varepsilon_{2,t} - 0.3X_{1,t-1} + 0.1X_{2,t-1} + 0.15X_{1,t-2} - 0.2X_{2,t-2}. \end{aligned}$$

The STAR model was selected to gain insight into the performance of the FCAR forecasting methods for a process that is truly nonlinear. The VAR model was selected to gauge the potential loss in accuracy resulting from using the nonlinear FCAR model to fit and forecast a model that is actually linear. In both models, the innovations, $\{\varepsilon_t\}$, are normal with mean zero and variance 1. For the VAR(2), the error cross-correlations included in the study were $\rho = 0.0, 0.4$, and 0.8. For each model, 500 replications were run for forecasting seven steps ahead for three sample sizes $T = 75, 150$, and 250. The number of bootstrap replications for the bootstrap forecasting method was 400. A correctly specified FCAR model, having order 1 and functional variable X_{t-1}, was fit to each of the STAR realizations. An FCAR model of order 2 and functional variable $X_{1,t-1}$ was fit to the VAR(2) data.

Numerical results for the STAR model are summarized in Table 1. The bias of the forecasts is reported in the top of each cell and the root mean square error (RMSE) in parentheses below the bias. The values in bold font are those forecasts with the smallest bias, although not necessarily the smallest RMSE. Figures 1 and 2 are graphical displays of the bias and RMSE of the forecasts in the $n = 150$ case for the two processes respectively.

TABLE 1 ABOUT HERE.
FIGURES 1 AND 2 ABOUT HERE.

All methods should result in the same forecasts at one-step-ahead. However, an examination of the table and plots for the STAR model shows that the bootstrap method is biased in this case. This could possibly be due to the influence of a few large residuals from the fitted FCAR model, in particular from residuals computed from observations near the boundaries of the range of Z_t. The estimation method described in Section 2 does not allow for boundary effects in the estimation of the functional coefficients. The bias persists until $k \geq 4$, when the bootstrap method begins to out-perform the other two methods. The multi-stage predictor is competitive with the bootstrap method in terms of RMSE, but tends to have a larger bias. Intuitively, the multi-stage predictor should provide less variable results than the bootstrap method. Failure to account for the nonlinearity of the expectation in computing the forecasts results in increased bias, although less than that of the naive method.

For the VAR(2) model, all methods perform about the same, which is expected given that the model is linear. Different values of error cross-correlation had little effect on forecasting methods. In the interest of space, only results for $\rho = 0.0$ are presented. The interested reader is referred to

`http://www.erc.msstate.edu/$\sim$harvill/FCARforecast/`

for the complete set of tables and figures. The next section examines the usefulness of forecasts from an FCAR model in practice.

4 Forecasting U.S. GNP and Unemployment

Much literature has been devoted to U.S. gross national product (GNP) and unemployment rate. The more well-known papers on modeling the U.S. GNP include Tiao and Tsay (1994) and Potter (1995). Both papers model U.S. GNP using a threshold-type model. Montgomery, *et. al.* (1998) analyze U.S. unemployment rate using a variety of techniques, and come to a set of rather involved conclusions and implications for an optimal way to model and predict unemployment. Van Dijk (1999) uses a STAR model for capturing the cyclical behavior of unemployment. Since STAR and TAR models may be thought of as special cases of FCAR models, we investigate characterizing and forecasting U.S. GNP and unemployment rates using a vector FCAR model.

To illustrate the effectiveness of the three forecasting methods presented in Section 3, we apply them to U.S. GNP and unemployment from January 1, 1959 through October 1, 2003. This data were obtained from the Federal Reserve Economic Data II (FRED II) web site affiliated with the Federal Reserve Bank of St. Louis Economic Research (`http://research.stlouisfed.org/fred2`). The source of the unemployment data is the U.S. Department of Labor: Bureau of Labor Statistics, and is the seasonally adjusted monthly unemployment rate (for people 16 years and older) from January 1, 1949 through March 1, 2004. The GNP is from the U.S. Department of Commerce: Bureau of Economic Analysis, and is the seasonally adjusted GNP (in billions of dollars) collected quarterly from January 1, 1959 to October 1, 2003. The unemployment from January 1, 1959 through October 1, 2003 was averaged across quarters so that the unemployment rates would correspond to the U.S. GNP. The product of 100 and the first difference of the logarithms of the variables was analyzed, with $\boldsymbol{Y}_1$ representing unemployment and $\boldsymbol{Y}_2$ GNP. A plot of the transformed bivariate time series is seen in Figure 3.

FIGURE 3 ABOUT HERE.

To assess the presence of nonlinear structure in the data, the multivariate nonlinearity tests of Harvill and Ray (1999, 2004) were applied. The test of Harvill and Ray (1999) assesses nonlinearity in a vector time series using a likelihood ratio-type statistic, comparing a linear VAR model to a nonlinear VAR model with interactions between lagged variables. The test of Harvill and Ray (2004) is a bootstrap-based test to compare residual sums of squared errors (SSEs) from a fitted vector FCAR model to residual SSEs from a fitted linear VAR model. Both tests reject linearity for the unemployment and GNP data with p-values < 0.001.

Using the first $T = 173$ points, FCAR models of order 1 were fit to each series individually and to the bivariate series. For unemployment, the functional variable that minimized the accumulated prediction error was lag 2 unemployment. For the GNP series, the selected functional variable was lag two GNP. This is in accordance with prior studies. The bivariate FCAR model was fit using lag 2 GNP as the functional variable, as in Harvill and Ray (2004). The estimated functional coefficients are shown in Figure 4.

FIGURE 4 ABOUT HERE.

We see that the correlation between current and previous unemployment rates increases with a rise in GNP two quarters before, whereas the correlation between current unemployment and previous quarter's GNP is minimal. Current growth in GNP shows increasing correlation with previous quarter's unemployment rate as growth in GNP two quarters previous rises to a level of around 2.3, and then starts to decline again. Correlation between current GNP and previous quarter's GNP is fairly constant when GNP growth rates two quarters prior are smaller than 2.3. Higher growth rates in GNP in prior quarters tend to reduce this correlation. The fitted model is consistent with asymmetry in the business cycle, with periods of strong and not-so-strong correlation between unemployment and GNP.

Based on the fitted FCAR models, forecasts were computed for the last five values of the series using each of the three methods described in Section 2. Tables 2 and 3 contain the actual values of the time series along with the forecasts. The numbers in bold are the forecasts that are closest to the true value of the series. For bootstrap forecasting, 400 bootstrap replications were used.

TABLES 2 AND 3 ABOUT HERE.

Figure 5 contains transformed U.S. GNP and Unemployment since the first quarter of 2000 with bootstrap forecasts and 90% bootstrap prediction intervals for the last five quarters superimposed. The prediction intervals easily capture the realized value of both GNP and unemployment for all steps.

FIGURE 5 ABOUT HERE.

A comparison of corresponding values in the two tables illustrates the improvement in forecasting using a bivariate model. This agrees in spirit with one of the conclusions in Montgomery, et. al. (1998), who use initial claims for unemployment insurance as a secondary variable to aid in predicting U.S. unemployment. The main message of their Conclusion (3) is that a bivariate linear (AR) model, although not superior in overall quality, outperforms the univariate benchmark linear model during periods of rapidly increasing unemployment. In our study, we see that including both variables in the nonparametric VFCAR model results in forecasts that outperform either of the univariate FCAR models. Of the three forecast methods, the bootstrap method appears to provide more accurate predictions for this data.

5 Summary and Directions for Future Work

We have explored three methods for multi-step prediction using an FCAR model, finding the bootstrap method to be somewhat preferred among the three. The multi-stage method tends to have a larger bias, especially for forecasting beyond two- or three-steps ahead. Application of a vector FCAR model to U.S. GNP and unemployment data shows that allowing the data to indicate the nonlinear relationship through nonparametric estimation of the AR coefficients provides improved predictability over previous modeling attempts, suggesting that FCAR models provide powerful tools both for characterization and forecasting of vector nonlinear time series. Future research might investigate FCAR models along other dimensions, such as impulse response analysis or evaluation of density forecasts, such as explored in Clements and Smith (2000) in the context of parametric nonlinear models.

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Table 1: Empirical results for FCAR Forecasting: STAR model.

	Naive			Multistage			Bootstrap		
	$n = 75$	$n = 150$	$n = 250$	$n = 75$	$n = 150$	$n = 250$	$n = 75$	$n = 150$	$n = 250$
X_{T+1}	**−0.08052**	**−0.02695**	**−0.01426**	−0.08052	−0.02695	−0.01426	−0.33150	−0.24717	−0.29439
	(1.003)	(1.020)	(0.996)	(1.003)	(1.020)	(0.996)	(1.048)	(1.038)	(1.031)
X_{T+2}	−0.29531	**−0.27004**	**−0.23172**	**−0.28699**	−0.28339	−0.25453	−0.36717	−0.36364	−0.36068
	(1.145)	(1.101)	(1.106)	(1.248)	(1.103)	(1.119)	(1.137)	(1.099)	(1.098)
X_{T+3}	−0.43548	−0.45629	−0.35844	**−0.35451**	**−0.41254**	**−0.31314**	−0.45463	−0.46652	−0.38447
	(1.174)	(1.065)	(1.062)	(1.489)	(1.065)	(1.086)	(1.164)	(1.057)	(1.077)
X_{T+4}	−0.49045	−0.50061	−0.40758	**−0.41074**	−0.47635	**−0.37790**	−0.47028	**−0.47403**	−0.38208
	(1.085)	(1.039)	(1.077)	(1.965)	(1.058)	(1.079)	(1.063)	(1.046)	(1.075)
X_{T+5}	−0.46717	−0.50231	−0.48233	**−0.30997**	−0.47405	−0.44897	−0.43247	**−0.45890**	**−0.43835**
	(1.182)	(1.088)	(1.155)	(3.178)	(1.100)	(1.163)	(1.157)	(1.097)	(1.150)
X_{T+6}	−0.47787	−0.56683	−0.51506	**−0.24531**	−0.55203	−0.49659	−0.43388	**−0.51146**	**−0.45803**
	(1.105)	(1.067)	(1.110)	(5.368)	(1.066)	(1.110)	(1.103)	(1.075)	(1.112)
X_{T+7}	−0.53222	−0.55448	−0.49059	**−0.10115**	−0.53991	−0.47606	−0.48135	**−0.49432**	**−0.43233**
	(1.085)	(1.101)	(1.100)	(9.268)	(1.098)	(1.099)	(1.075)	(1.097)	(1.101)

Table 2: Values and Nonparametric FCAR Forecasts of U.S. Unemployment.

		Univariate forecast method			Vector forecast method		
Step	Unemployment	Naive	Multistage	Bootstrap	Naive	Multistage	Bootstrap
1	−1.409788	4.778727	4.778727	5.009502	**3.777976**	**3.777976**	4.152555
2	2.457165	3.023621	2.894699	3.133962	2.169781	1.920381	**2.505596**
3	−0.383157	1.917916	**1.062150**	2.155153	1.355409	1.062217	1.687443
4	5.609801	1.223648	1.062150	**1.711035**	0.905614	0.650342	1.200728
5	−0.215038	0.780571	0.643394	1.098573	0.641637	**0.440280**	0.762362

Table 3: Values and Nonparametric FCAR Forecasts of U.S. GNP.

		Univariate forecast method			Vector forecast method		
Step	GNP	Naive	Multistage	Bootstrap	Naive	Multistage	Bootstrap
1	1.186884	0.323810	0.323810	0.652339	0.989269	0.989269	**1.062834**
2	0.952788	0.235829	0.240663	**0.892840**	1.078551	1.107441	1.311016
3	0.953086	0.169344	0.178866	1.063526	**1.048188**	1.053645	1.458029
4	1.074688	0.121486	0.132937	1.259711	**0.972075**	0.940860	1.547236
5	2.404678	0.087755	0.098802	1.337264	0.878695	0.816124	**1.606438**

Figure 1: Comparison of Forecasting Methods: STAR Model, $n = 150$.

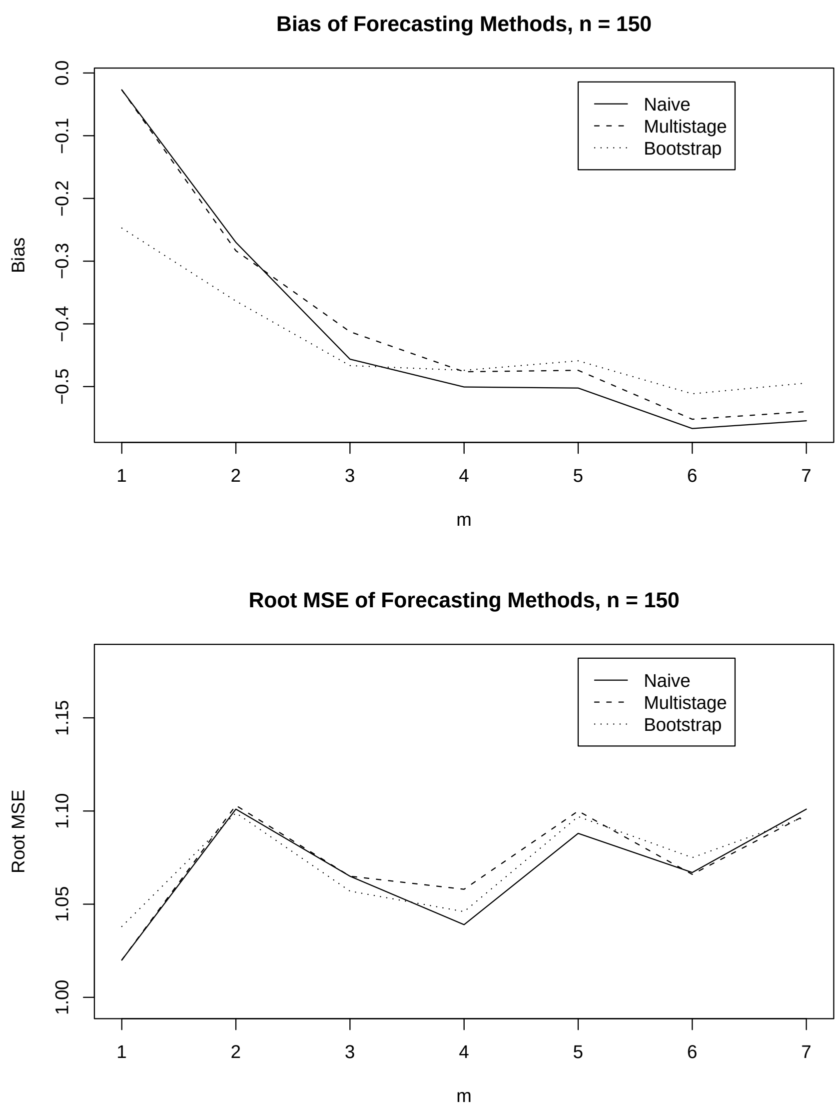


Figure 2: Comparison of Forecasting Methods: VAR(2) Model, $n = 150$.

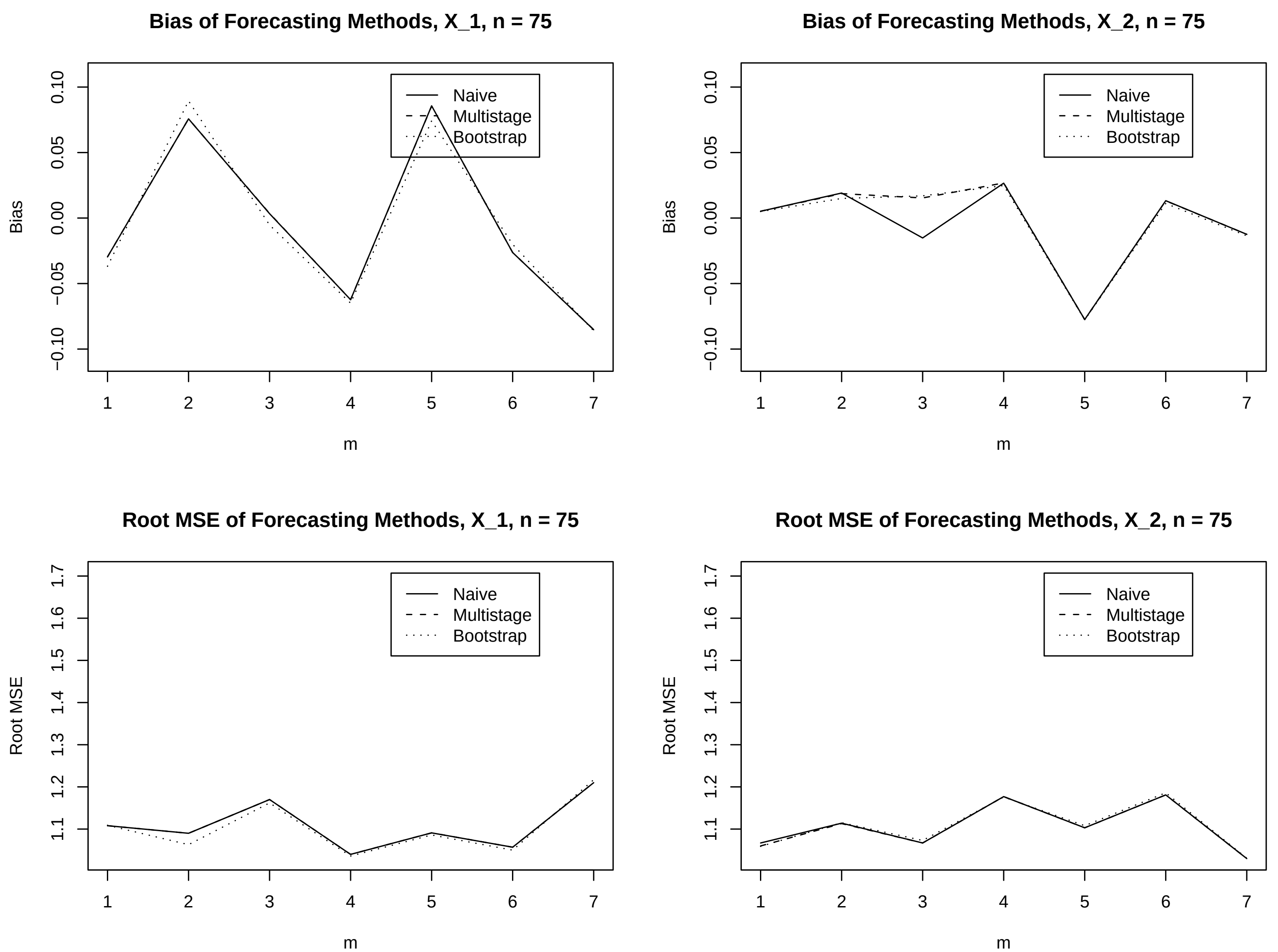
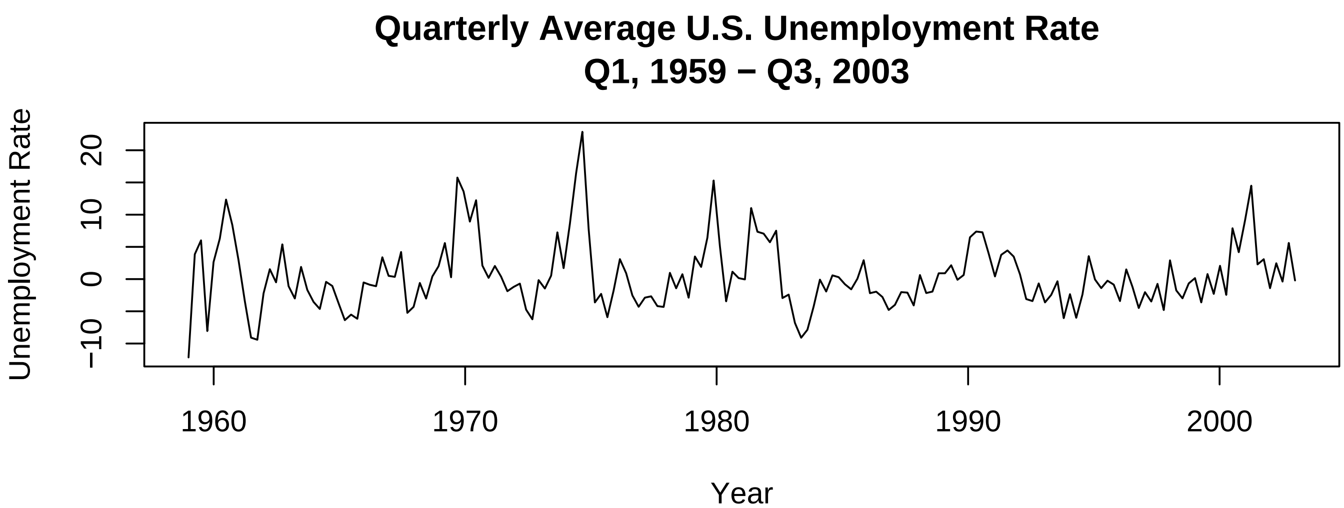


Figure 3: Transformed U.S. Unemployment and GNP (1959 - 2003).



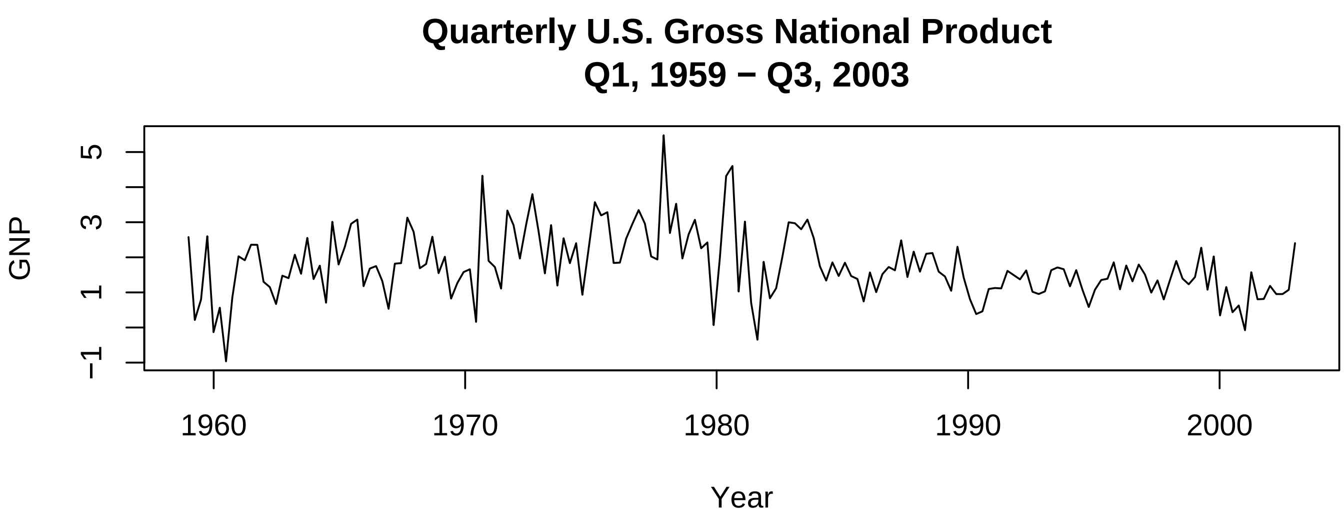


Figure 4: Estimates of Functional Coefficients for Modeling U.S. Unemployment Rate and GNP (1959 - 2003).

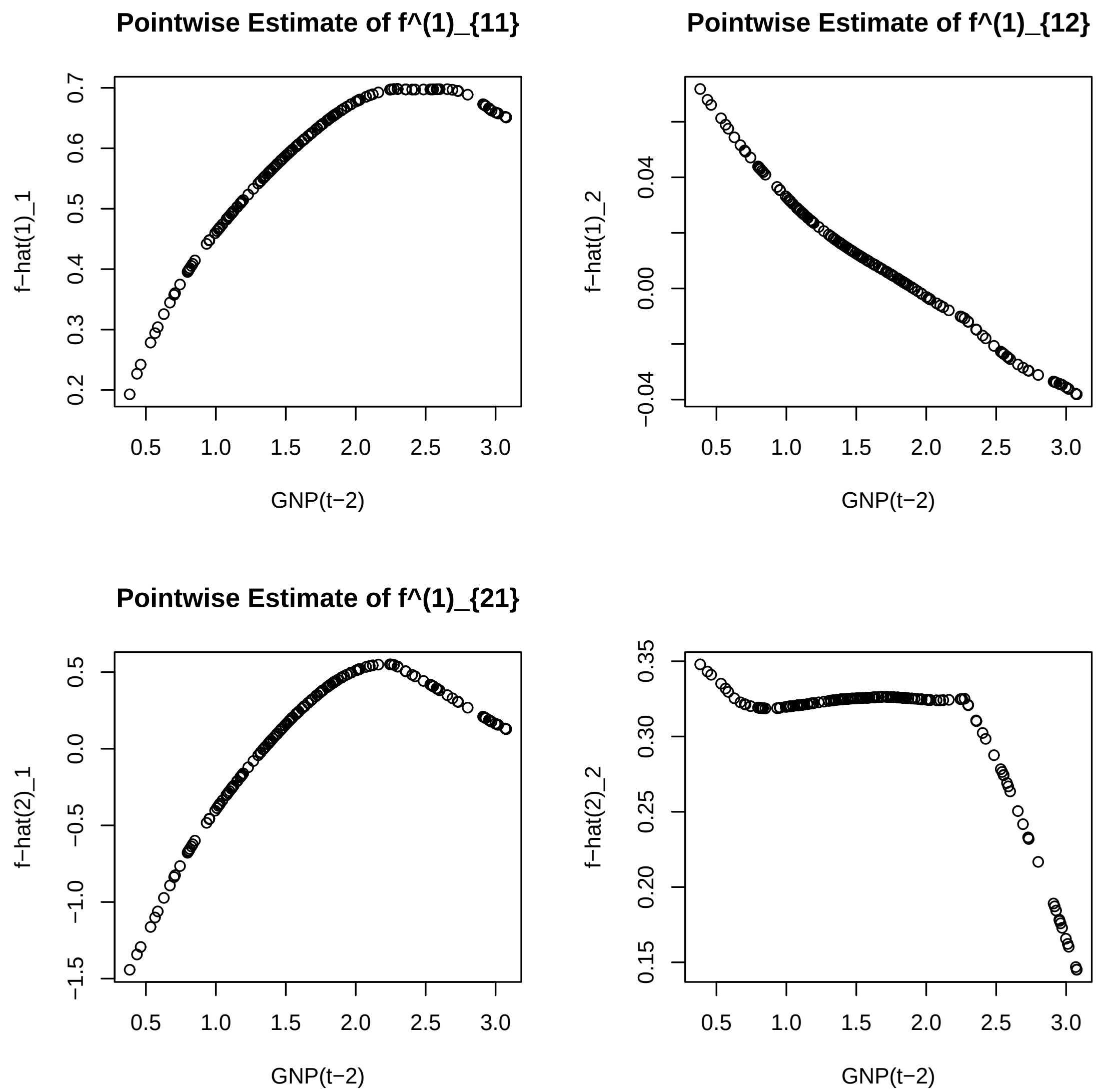


Figure 5: Bootsrap 90% Prediction Intervals for U.S. Unemployment and GNP (Q1, 2000 - Q3, 2003).