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NEUTRAL ONE–DIMENSIONAL ATTRACTOR OF A TWO–DIMENSIONAL SYSTEM DERIVED FROM NEWTON'S MEANS

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Abstract. We investigate a special case of Newton's means as an example of a two dimensional rational dynamical system with an observed neutral behavior. We provide the reason for such a behavior and state a program for further investigations.

1. **Newton's means.** The most known case of means as dynamical systems is the case of arithmetic-geometric means, which form a 2-to-2 iterative relation in $\mathbb{C}^2$, $\hat{g}^2 = ag$, $2\hat{a} = a + g$, and its limit value (in case $a, g > 0$) can be used to calculate elliptic integrals. There are no obvious generalization of this system.

Newton's means are the generalization of a system of arithmetic-harmonic means $(a, h) \mapsto ((a + h)/2, 2/(a^{-1} + h^{-1}))$, with two advantages: the system is given by a map (not a relation) and the map is rational. As it will be explained below Newton's means are related to the zeros of the polynomials which, by itself, makes them an interesting object of investigation.

The object of the present study is a two dimensional dynamical system derived from the Newton's means in case of three variables. There are not many specific examples of two dimensional relatively simple maps with interesting dynamics. Our map shows some resemblance to the two dimensional map of a triangle considered in [5] (a problem by Sharkovsky). There, a segment was invariant and the dynamic on it was that of a full quadratic map. It was proven that this invariant set attracts a dense subset of a triangle, and that there exists a dense subset of a triangle consisting of points not being attracted by the invariant set, but whose ω–limits are not disjoined with the invariant set. This map was defined by two quadratic polynomial. Our map is defined by rational functions and it's degree is higher, consequently the dynamics are much more complicated.

1.1. **Definition and known properties.** Fix $n \geq 2$. Let $x = (x_1, \dots x_n) \in \mathcal{C}^n$. Denote $c_k = \binom{n}{k}$, then for $k = 0, \dots, n$ we define standard symmetric polynomials $s_k(x)$, normalized symmetric polynomials $p_k(x)$ and for $k > 0$ the k-th Newton mean $\hat{x}_k$ by

$$s_k(x) = \sum_{\substack{A \subset \{1,\dots,n\} \\ \#A = k}} \prod_{m \in A} x_m \, , \quad p_k(x) = \frac{s_k(x)}{c_k} \, , \quad \hat{x}_k = \frac{p_k(x)}{p_{k-1}(x)} \; .$$

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Clearly by Vieta's formulas x are zeros of the polynomial $W_p(z) = \sum(-1)^k c_k p_k(x) z^{n-k}$. It follows in particular that the discriminant $\mathcal{D}(x)$ of this polynomial is of constant sign. A discriminant of a polynomial is an algebraic expression of its coefficients which is zero when the polynomial has at least one multiple root.

According to [1] already Newton [3] knew that for any real x we have:

$$p_{k-1}(x)p_{k+1}(x) \leq p_k(x)^2 \,, \tag{1.1}$$

which shows that for $x > 0$ (all coordinates) one has

$$\min(x_m) \leq \hat{x}_n \leq \cdots \leq \hat{x}_1 \leq \max(x_m)$$

and explains the name.

Let $F(x) = \hat{x}$. It was proven previously that:

Theorem I (**Same signs** [4])**.** *Let $H(x)$ denote the convex hull of the points $x_1, x_2, \ldots, x_n \in \mathcal{C}$. If $H(x) \subset \mathcal{C} \setminus \{0\}$ then $H(\hat{x}) \subset H(x)$ has the same property and $F^N(x) \to x_0(1, 1, \ldots, 1)$ with $x_0^n = \prod_{m=1}^n x_m$.*

Theorem II (**Different signs** [2])**.** *For any $n \geq 2$ there exists an invariant planar curve Γ which in the coordinates $a_k = x_{k+1}/(x_{k+1} - x_k)$, $k = 1, \ldots, n-1$ is an ellipse, where the dynamics of F is conjugated to the dynamics of $z \mapsto z^n$ on the unit circle. The conjugacy is given by the natural parametrization of the ellipse by the argument.*

One can prove [2] that for real x different signs is an invariant property under F. Numerical observations show that the invariant curve attracts a large set of trajectories. For $n > 3$ and starting points in a large set the attracting curve is visible after a few (below 100) iterates, which suggests that it is a hyperbolic attractor. In the case $n = 2$ all the dynamic is (2-to-1 semi-) conjugate to the $z \to z^2$ on S^1, as it will be explained shortly.

In the case $n = 3$ the high iterates of any starting point seem to form a double line around the invariant curve, approaching very slowly (after thousands of iterates the distinction is still visible) cf. Figure 4. As usual in simulations this maybe due to the numerical errors, but clearly is not compatible with a hyperbolic attraction. We shall provide a partial explanation of such behavior. The dynamics reduced to its natural dimension are complicated as the map contains unavoidable singularities.

1.2. **Dimension of the problem.** Before we go any further we remark that for any n as $F(\lambda x) = \lambda F(x)$ the map is homogeneous of degree 1 and that the product $\prod_{k=1}^n x_k$ is invariant. Therefore one can reduce the dimension of the problem as the dynamics on the surfaces of constant (non-zero) product are all conjugated by the rays. The map of the implied $n-1$ dimensional system will be denoted G. It depends on the way we reduce the number of variables, but usually conserves some symmetry and is still rational.

On the other hand one can investigate a polynomial map on variables p:

$$\hat{p}_k = \frac{1}{c_k} \sum_{\substack{A \subset \{1,\ldots,n\} \\ \sharp A = k}} \prod_{m \in A} p_m \prod_{m \notin A} p_{m-1} \quad k = 0, \ldots n \,,$$

getting rid of the denominators by the price of working in one dimension higher [2]. This map is polynomial and homogeneous of degree n, however it is not projective as non-zero arguments can have zero values.

We shall reduce the original n dimensional system to two $n-1$ dimensional equivalent ones. Firstly we can get rid of the homogeneity using the variables from Theorem II. Secondly we can use the variables p, with the provision $p_0 = 1 = p_n$. This is possible as the equality $p_0 = p_n$ can be achieved by the rescaling of the variable z and is invariant. The equality to 1 can be done by dividing the polynomial by the coefficient at the highest power of z, which does not change the roots.

1.3. **Case $n = 2$.** In this case the choice of the variable $a = x_2/(x_2 - x_1)$ yields 1–dimensional dynamics of the full quadratic map $a \mapsto \hat{a} = G(a) = 4a(1-a)$. This map is derived from the 2-to-1 projection of $(x, y) = z \to z^2 = (x^2 - y^2, 2xy)$ from the unit circle $|z|^2 = x^2 + y^2 = 1$ onto the diameter $y = 0, -1 \le x \le 1$, where $x \mapsto \hat{x} = 2x^2 - 1$, which becomes $\hat{a} = 4a(1-a)$ if $a = (1-x)/2$. The map is 2-to-1 on the invariant interval $[0, 1]$, which is its Julia set, and for any $w \in \mathbb{C} \setminus [0, 1]$ the iterations $G^N(w) \to \infty$, which in this coordinates represent the super sink of the iterations in other words the condition $x_1 = x_2$ (equal to the geometric mean of the two).

Now we are ready to investigate the case $n = 3$.

$$F(x_1, x_2, x_3) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{x_1x_2 + x_2x_3 + x_3x_1}{x_1 + x_2 + x_3}, \frac{3x_1x_2x_3}{x_1x_2 + x_2x_3 + x_3x_1} \right).$$

As said above we shall use two equivalent sets of coordinates, and we shall use the same letter G to denote the map in both systems. Although, from the formal point of view, these are two different maps, this notation is intuitive, and it will always follow clearly from the context which of the two maps we are considering. We shall denote by G_1 (G_2) the first (second) coordinate of G respectively.

2. Dynamics around the ellipse – coordinates (a, b).

We first describe the dynamical system expressed in coordinates:

$$a = \frac{x_2}{x_2 - x_1}, \qquad b = \frac{x_3}{x_3 - x_2},$$

which is defined by $G(a, b) = (\hat{a}, \hat{b})$:

$$\begin{aligned} \hat{a} &= \frac{3a(1-b)(1-a-2b+3ab)}{(a-b)^2 + 1 - a - 2b + 3ab}, \\ \hat{b} &= \frac{3(1-a)b(1-2a-b+3ab)}{(a-b)^2 + 1 - 2a - b + 3ab}. \end{aligned}$$

2.1. **Invariant sets.** The invariant ellipse $\mathcal{E}$ is given by the equation:

$$a + b - a^2 - ab - b^2 = 0,$$

and with its parametrization:

$$\begin{aligned} a(t) &= \frac{1 - \cos(t) + \sqrt{3}\sin(t)}{3} \\ b(t) &= \frac{1 - \cos(t) - \sqrt{3}\sin(t)}{3}, \end{aligned}$$

we have $G(a(t), b(t)) = (a(3t), b(3t))$ on $\mathcal{E}$, conforming with Theorem II. All the dynamics happen inside the parabola:

$$\mathcal{D}(a, b) = (a-b)^2 + 2(a+b) - 3 \le 0,$$

which is the discriminant of $p_0z^3 - 3p_1z^2 + 3p_2z + p_3$ in the coordinates (a, b). This is stronger than $a \le 1, b \le 1$ which follow from the Newton's inequalities (1.1). The

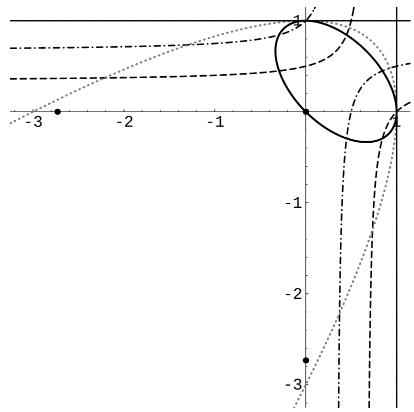

FIGURE 1. Invariant sets in coordinates (a, b).

latter with same signs condition implies $a \leq 0$ and $b \leq 0$, hence the different signs condition is satisfied in the region bounded by the discriminant parabola and the (invariant) axis $a = 0$ and $b = 0$. Remark that the axis have no good interpretation in variables x_k, however the behavior of the iterations of G in the neighborhood of the axis relates to the behavior of the iterates of F. In particular from the six fixed points of G: $(0, 0)$, $\left(\frac{2}{3}, \frac{2}{3}\right)$, $(0, -1-\sqrt{3})$, $(0, -1+\sqrt{3})$, $(-1-\sqrt{3}, 0)$ and $(0, -1+\sqrt{3})$ only the second one produces a fixed point of F, but the invariant manifolds of the others produce invariant sets for F. On the other hand the points $x_1 = x_2 = x_3$ which are fixed for F produce a super sink at infinity for G. The point (0,0) is a repelling linear node with diagonal linearization matrix whose both eigenvalues are equal to 3. The points $(-1 - \sqrt{3}, 0)$ and $(0, -1 - \sqrt{3})$ are hyperbolic saddles with stable manifold along the axis (resp. $b = 0$ and $a = 0$) and a transversal unstable manifold. The map on each of the axis is a two-fold, conjugated to the full quadratic map $z \mapsto 4z(1 - z)$.

The unstable invariant manifold of $\left(\frac{2}{3}, \frac{2}{3}\right)$ is the ellipse while the line $a = b$ is invariant with a neutral (order reversing) attractor at this point with dynamics governed by the map $z \mapsto 3z(1 - z)$.

2.2. **Discontinuities.** The map G is not continuous only at two points: $(1, 0)$ and $(0, 1)$ which lie on the intersection of the ellipse, the parabola and the border line $a = 1$ (resp. $b = 1$). The images of discontinuity points depend on the angle of approach and with the approach tangent to the border line they are mapped into $(0, 0)$ with the angle of approach to the image depending on the second order term. The preimages of the discontinuity points are hyperbolas, if we approach a point on the hyperbola we obtain a specific tangential approach to the discontinuity point and in the next step a specific angular approach to $(0, 0)$. We describe this in details as the invariant ellipse crosses the hyperbolas and so would any bassin of its attraction. Hence any invariant domain would be pinched at the discontinuity points and thus also at the origin.

The map G is singular (locally irreversible) on the hyperbolas and on the line $a+b = 1$ which is mapped onto the discriminant parabola. This is the two-fold. On the other hands the hyperbolas, the axis and the diagonal $a = b$ form a partition which is mapped onto itself in a three-fold way.

3. **Dynamics around the line – coordinates** (d, s)**.** In order to rectify the invariant ellipse we use the second set of coordinates: $u = p_1(x_1, x_2, x_3) = \frac{x_1+x_2+x_3}{3}$, $v = p_1(x_1, x_2, x_3) = \frac{x_1x_2+x_2x_3+x_3x_1}{3}$. It is easy to check that:

$$\begin{aligned} \hat{u} &= \frac{1}{3}\left(u + \frac{1}{v} + \frac{v}{u}\right), \\ \hat{v} &= \frac{1}{3}\left(\frac{1}{u} + \frac{u}{v} + v\right). \end{aligned}$$

Remark 1. Observe that (u, v), $\left(\frac{1}{v}, \frac{1}{u}\right)$, $\left(\frac{v}{u}, \frac{1}{u}\right)$, $\left(u, \frac{u}{v}\right)$, $\left(\frac{1}{v}, \frac{u}{v}\right)$, $\left(\frac{v}{u}, \frac{1}{u}\right)$ form a group of substitutions isomorphic to the group of permutations of 3 elements, the roots of the polynomial of degree 3. Hence the preimages of a given $(\hat{u}, \hat{v})$ form a group.

A calculation shows that $u = v$ and $u + v = -1$ are invariant, the former representing the diagonal $a = b$ and the latter the invariant ellipse. In order to make the axis invariant we rotate the coordinates $d = u - v$, $s = u + v + 1$, losing the symmetry. Now system takes the form $G(d, s) = (\hat{d}, \hat{s})$,

$$\begin{aligned} \hat{d} &= \frac{d(d^2 - (s-3)^2)}{3(d^2 - (s-1)^2)}, \\ \hat{s} &= \frac{s(3 + d^2 - 2s - s^2)}{3(d^2 - (s-1)^2)}. \end{aligned}$$

The line $s = 0$ corresponds to the ellipse $\mathcal{E}$, we denote it by $\mathcal{P}$.

Remark 2. In this coordinates Theorem II applies to the dynamics of $\mathcal{P}$. Indeed:

$$G(\sqrt{3}\cot(\varphi), 0) = \sqrt{3}\cot(3\varphi).$$

3.1. **Invariant regions.** The zeros of the discriminant $\mathcal{D}$ of the polynomial $W(z) = z^3 - 3uz^2 + 3vz - 1$ defines an algebraic curve γ:

$$\mathcal{D}(u, v) = 1 + 4u^3 - 6uv - 3u^2v^2 + 4v^3$$

In the coordinates (d, s) we get the expression $\mathcal{D}(d, s) = -27 - 18d^2 - 3d^4 + 108s + 36d^2s - 90s^2 + 6d^2s^2 + 28s^3 - 3s^4$ (up to multiplicative constant).

Remark 3. The curve $\mathcal{D}(d, s) = 0$ divides the real plane into three regions Δ_-, Σ and Δ^+. The image $G(\mathbb{R}^2) = \Delta_- \cup \Delta^+$. Moreover, both regions Δ_- and Δ^+ are forward–invariant, more precisely $G(\Delta_-) \subsetneq \Delta_-$ and $G(\Delta_+) \subsetneq \Delta_+$.

3.2. **Behavior near the line at infinity.** We shall use standard projective coordinates

$$w = 1/d, \qquad z = s/d.$$

We get

$$\begin{aligned} \hat{w} &= \frac{3w((w-z)^2 - 1)}{(3w - z)^2 - 1}, \\ \hat{z} &= \frac{z(z^2 + 2wz - 3w^2 - 1)}{(3w - z)^2 - 1}. \end{aligned}$$

Note that $\hat{z}|_{\{w=0,z\}} = z$, so for every curve γ intersecting the line at infinity its image has the same asymptotic as the original curve.

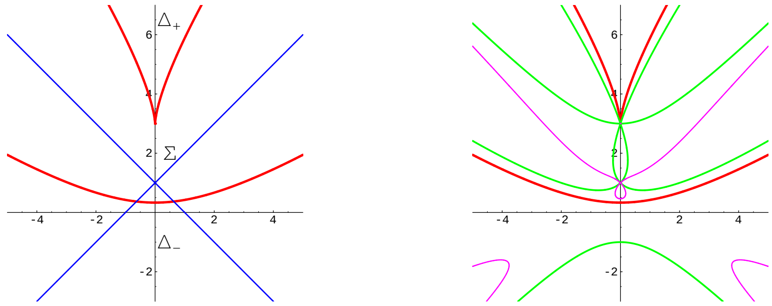

FIGURE 2. Invariant regions for the map G, preimages of ∞ and preimages of the curve $D(d,s)=0$.

The derivative at the line $w=0$ is equal to:

$$\begin{pmatrix} 3 & 0 \\ \frac{8z^2}{z^2-1} & 1 \end{pmatrix}.$$

That means that infinity is a hyperbolic repellor. This corresponds to the dynamics near the origin in the (a,b) coordinates (where locally the directions are also invariant).

4. **Invariant vector field.** On the axis $s=0$ the map is strongly mixing, hence the typical behavior of a trajectory is commanded by an average properties of the points. The very slow approach of the trajectories to the axis $s=0$ suggests that the map is in the average neutral in the transversal direction. in fact a stronger fact holds.

Theorem . *There exists a vector field $v(d)$, $v:\mathbb{R}\to\mathbb{R}^2$ such that*

$$dG(d,0)(v(d)) = v(\hat{d}(d,0))\,. \tag{4.1}$$

The field $v(d)$, where $d=\sqrt{3}\cot\varphi$ can be written explicitly

$$v(d) = \frac{1}{\sin^2(\varphi)}\begin{pmatrix} -2\sum_{n=0}^{\infty}\frac{\sin(3^n\varphi)\sin(2\cdot 3^n\varphi)}{3^n} \\ \sqrt{3}\sin(\varphi)\end{pmatrix}. \tag{4.2}$$

Therefore in the average the first approximation of the map neither moves away nor approaches the axis in the transversal direction.

Remark 4. Note that formula ((4.2)) defines actually two vectors attached to each point $(0,d)$. Indeed, when we replace φ with $\varphi+\pi$, $d=\sqrt{3}\cot\varphi$ remains unchanged, but the vector $v(d)$ changes sign. This is why we present v using parametric representation, instead of striving to express v using coordinates (d,s). This explains also the double winding of the iterates around the invariant curve.

Figure 3 shows a curve defined by $(d(\varphi),0)+0.55v(d(\varphi))$.

Remark 5. Vector field v given by (4.2) is continuous, but not differentiable. Indeed, it's first coordinate looks like a famous nowhere–differentiable Weierstraß function.

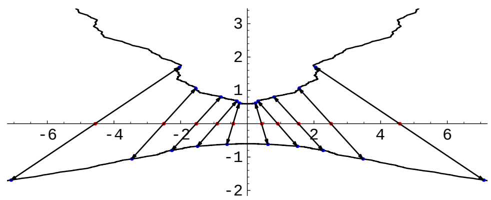

FIGURE 3. Invariant vector field.

Proof of Theorem . We start with the formula for dG on the d axis. It is natural to use coordinate $d = \sqrt{3}\cot\varphi$ (see Remark 2). We have:

$$dG\Big|_{(\sqrt{3}\cot\varphi,0} = \begin{pmatrix} \frac{3}{(1+2\cos(2\alpha))^2} & \frac{2\sqrt{3}\sin(2\alpha)}{(1+2\cos(2\alpha))^2} \\ 0 & \frac{1}{1+2\cos(2\alpha)} \end{pmatrix} = \begin{pmatrix} 3 & 2\sqrt{3}\sin 2\varphi \\ 0 & \frac{\sin 3\varphi}{\sin\varphi} \end{pmatrix} \cdot \left(\frac{\sin\varphi}{\sin 3\varphi}\right)^2 .$$

Denoting $w(\varphi) = \sin^2\varphi v(\varphi)$ we obtain reformulation of Equation (4.1)

$$w(3\varphi) = \begin{pmatrix} 3 & 2\sqrt{3}\sin 2\varphi \\ 0 & \frac{\sin 3\varphi}{\sin\varphi} \end{pmatrix} \cdot w(\varphi)\,. \tag{4.3}$$

Let w_1, w_2 denote the first and the second coordinate of w respectively. From (4.3) we get $w(3\varphi) = \frac{\sin(3\varphi)}{\sin(\varphi)} = w(\varphi)$, so we can chose $w_2(\varphi) = \sqrt{3}\sin(\varphi)$. Now we get the equation $w_1(3\varphi) = 3[w_1(\varphi) - (\cos(3\varphi) - \cos(\varphi))]$ and thus inductively:

$$w_1(\varphi) = \sum_{n=0}^{\infty} \frac{1}{3^n}\left[\cos\left(3^{n+1}\varphi\right) - \cos\left(3^n\varphi\right)\right] .$$

Using $\cos(3\varphi) - \cos(\varphi) = -2\sin(2\varphi)\sin(\varphi)$ we get:

$$w_1(\varphi) = -2\sum_{n=0}^{\infty} \frac{\sin\left(3^n\varphi\right)\sin\left(2\cdot 3^n\varphi\right)}{3^n} ,$$

and:

$$v(\varphi) = \frac{1}{\sin^2(\varphi)} \begin{pmatrix} -2\sum_{n=0}^{\infty} \frac{\sin(3^n\varphi)\sin(2\cdot 3^n\varphi)}{3^n} \\ \sqrt{3}\sin(\varphi) \end{pmatrix} .$$

□

The ragged character of the invariant field makes it impossible to construct a smooth family of smooth curves that descends toward the invariant curve and are mapped monotonously into itself.

5. **Simulations and numeric approximations.** Numerical simulations clearly indicate, that the ellipse $\mathcal{E}$ in coordinates (a, b) (and – consequently – the line $\mathcal{P}$ in coordinates (d, s)) is an attractor. For a random choice of starting points their images converge to $\mathcal{E}$ and $\mathcal{P}$ respectively, see Figure 4.

Next Figure 5 shows unstable manifold of the point $(0, -1-\sqrt{3})$. It is "winding up" onto the invariant ellipse $\mathcal{E}$ in a complicated, irregular way. A small portion of the unstable manifold extends toward the hyperbola, then under one iteration reaches the discontinuity point, and under the next one runs around the ellipse. All

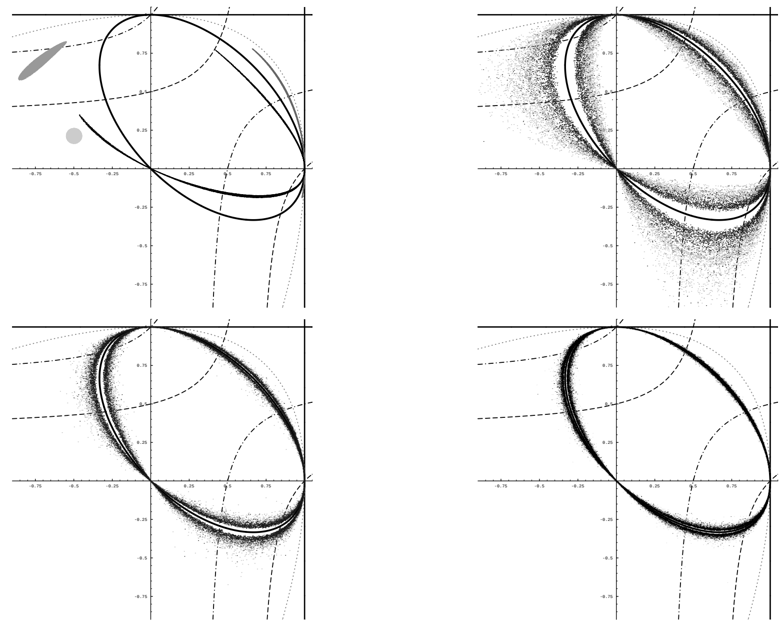

FIGURE 4. Subsequent images of a small circle under the action of G. Images of iterates number 0 (the original set), 1, 2, 3; then 10, 50; next 150, 350; and finally 650, 1150, 2150.

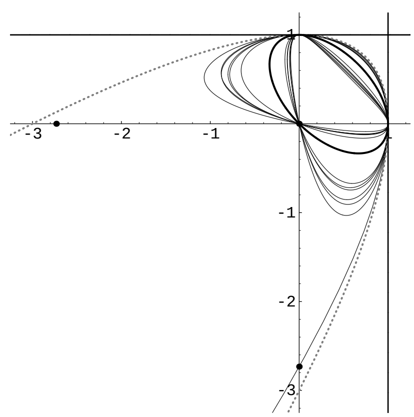

FIGURE 5. Unstable manifold of the fixed point $(0, -1-\sqrt{3})$.

subsequent iterates repeat the process while keeping the old portion invariant, and we obtain an invariant curve which runs around the ellipse infinitely many times.

Figure 6 below shows 9 generations of preimages of the point $(0, -1)$ in the coordinates (d, s). They seem to slowly approach the line $\mathcal{P}$, which might suggest that it is not an attractor.

However, the approaching is very slow, and the observed phenomenon is specific only for this particular system of coordinates. Indeed, an analogous picture in coordinates (a, b) does not show any such behavior. One might argue that there is

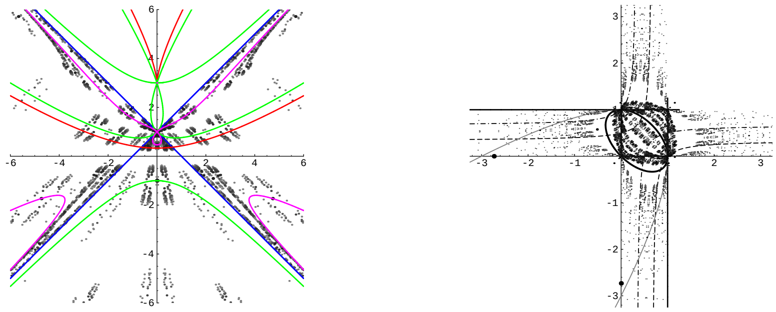

FIGURE 6. Preimages of the point $(0,-1)$ in coordinates (d,s). and of the point $(-.515\dots,.576\dots)$ in coordinates (a,b).

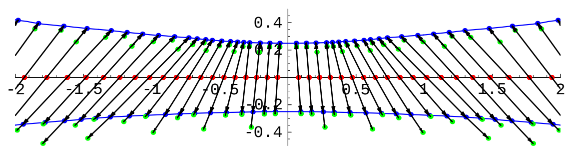

FIGURE 7. "True" and "linear" images of the invariant vector field v.

a visible gap between the preimages and $\mathcal{P}$ which might suggest the existence of the bassin of attraction.

Figures 4 and 5 strongly suggest the existence of an invariant open set in the shape of a "triple sausage" containing the ellipse. It is worth noting that we obtain very similar figures for any choice of starting point, as long as it is in "general position", i.e. not on any invariant line etc.

5.1. **Image of the invariant vector field v.** The vector field v is invariant under the action of G more precisely, it is invariant with respect to the map dG acting on the tangent bundle $\mathrm{T}\mathbb{R}^2$. It is not true in general that $G\left((d,0)+v(d)\right)=G(d,0)+v(G_1(d,0))$. The next picture shows difference between the "linear" and "true" images of the vector v. More rigorously speaking, the long arrows at the picture are the vectors $G(d,0)+dG|_{(d,0)}v(d)$ and the short arrows are the vectors $v(G_1(d,0))-dG|_{(d,0)}v(d)$ attached at points $G(d,0)+dG|_{(d,0)}v(d)$.

Strikingly, the "true" images of vectors v are almost colinear with their "linear" images. This means, that although the lamination by straight segments $(d,0)+tv(d)$ is not invariant, the map G perturbs it very slightly from small t. Of course, under further iterations the perturbation increases rapidly due to the strong expansion in the direction tangent to the invariant curve.

The next interesting observation is, that the "true" images of v lying above the line $\mathcal{P}$ are all shorter then their "linear" images, while the "true" images of v lying below the line $\mathcal{P}$ are all longer then their "linear" images.

We made many attempts to find a region Ξ bounded by two graphs of functions, such that the line $\mathcal{P}$ would be contained in Ξ and that the image of Ξ would contained in it. This seemed to be a natural way to prove formally that $\mathcal{P}$ (and consequently $\mathcal{E}$) is an attractor. Also, many simulations seemed to suggest that such construction should be possible. We found a smooth perturbation of a curve $\gamma=(d(\varphi),0)+\varepsilon v(d(\varphi))$, namely $\tilde{\gamma}=(d(\varphi),0)+0.06(1-0.2\sin^2(2\varphi))v(d(\varphi))$ such that the image of $\tilde{\gamma}$ is contained in the stripe Ξ containing $\mathcal{P}$ bounded by $\tilde{\gamma}$. However such region Ξ is not invariant under G, a boundary of it's image is not the image of it's boundary. In fact, already the second iterate of $\tilde{\gamma}$ is no longer contained in Ξ.

The reason we believe that the ellipse is an attractor after all is the following: any trajectory close enough to $\mathcal{E}$ (or $\mathcal{P}$) winds around it as it follows the strong eigen direction along the invariant curve and for a long time mimics the trajectory on the curve, therefore (by ergodicity) visits often the neighborhood of the fixed point $(\frac{2}{3}, \frac{2}{3})$ where it is attracted to the curve.

6. **Conjectures and open questions.**

Conjecture . Basing on the numeric simulations and theoretical investigations we conjecture that the following assertions hold

1. There does not exist any open, proper subset A of Δ_- containing the line $s = 0$ with the property $G(A) \subset A$.
2. For any P–nowhere dense subset of the axis $s = 0$ there is no open, proper subset B of Δ_- containing the set $\{s = 0\} \setminus P$ with the property $G(B) \subset B$.
3. The set $C = \{(d, s) : \omega(d, s) \subset \{d = 0\}\}$ of points being attracted by $d = 0$ is a fractal set with positive measure.
4. The set $D = \{(d, s) : \omega(d, s) \cap \{d = 0\} \neq \emptyset\}$ is a fractal set whose complement in Δ_- has measure 0.
5. For every open subset E of Δ_- there holds $\bigcup_{n=0}^{\infty} G(E) = \Delta_-$.
6. For almost every $(d, s) \in \Delta_-$ preimages of (d, s) form a dense subset of Δ_-.
7. There exist an open set F containing the axis $s = 0$ and a continuous $\mathcal{F}$ foliation of F with smooth leaves which is invariant under the action of G.

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