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Ergodic Type Theorems for Actions of Finitely Generated Semigroups on von Neumann Algebras II

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ERGODIC TYPE THEOREMS FOR ACTIONS OF FINITELY GENERATED SEMIGROUPS ON VON NEUMANN ALGEBRAS II

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ABSTRACT. In the sequel, we continue the study initiated in: Grabarnik, G.Ya., Katz, A.A., Shwartz, L., "Ergodic Theorems for Actions of Finitely-Generated Semi-Groups on von Neumann Algebras, I", Proceedings of the 3rd Annual Hawaii International Conference on Statistics, Mathematics and Related Fields, Honolulu, Hawaii, USA. We obtain a non-commutative version of Bufetov's Ergodic Theorem for Skew Product.

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1. INTRODUCTION

First Ergodic Theorems for actions of arbitrary countable groups were obtained by Oseledets [24], who followed an idea of Kakutani [15]. For actions of free groups Guivarc'h [12] considered uniform averages over spheres of increasing radii in a group and proved the related mean ergodic theorem. Grigorchuk [10] announced the Pointwise Ergodic Theorem for Česaro averages of the spherical averages. Nevo [22] and Nevo and Stein [23] published a proof of the Pointwise Ergodic Theorem. In [11] Grigorchuk announced an Ergodic Theorem for Actions of Free Semigroups. In [3] Bufetov generalized classical and recent Ergodic Theorems of Kakutani, Oseledets, Guivarc'h, Grigorchuk, Nevo and Nevo and Stein for measure-preserving actions of free semigroups and groups.

The first results in the field of non-commutative Ergodic Theorems were obtained by Sinai and Anshelevich [26] and Lance [20]. Developments of the subject are reflected in the monographs of Jajte [13] and Krengel [19].

Majorant ergodic theorem for the operators affiliated to tracial von Neumann algebras was proved in [6].

In [7] the authors initiated study with the aim to generalize Bufetov's results from [3] to the non-commutative case to obtain non-commutative Ergodic Theorems for the actions of finitely generated semigroups on von Neumann algebras with faithful normal finite tracial state. In the sequel a non-commutative version of Bufetov's Ergodic Theorems for Skew Product is obtained.

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2. Preliminaries

Let us recall some notions ans results from [7].

Let the pair (A, τ) be a **non-commutative probability space**, where A is a von Neumann algebra with a faithful, normal tracial state τ.

Let

$$\alpha_1, \alpha_2, ..., \alpha_m : A \longmapsto A$$

be **positive kernels**

$$(\alpha_i \mathbf{1} \leq \mathbf{1}; \tau \circ \alpha_i \leq \tau). \tag{2.1}$$

All $\{\alpha_i\}$ could be extended to the operators

$$L_1(A, \tau) \longmapsto L_1(A, \tau),$$

which we will also call without the loss of generality $\{\alpha_i\}$.

Let

$$\Omega_m = \{\omega = \omega_1 \omega_2 ... \omega_n ... : \omega_i = 1, ..., m\} \tag{2.2}$$

be the space of all one-sided infinite sequences in the symbols $1, ..., m$.

We denote by σ_m the shift on Ω_m, defined by the formula

$$(\sigma_m \omega)_i = \omega_{i+1}. \tag{2.3}$$

Consider the set

$$W_m = \{w = w_1 w_2 ... w_n : w_i = 1, ..., m\} \tag{2.4}$$

of all finite words in the symbols $1, ..., m$.

Denote by $|w|$ the length of the word w. For each

$$w \in W_m,$$

let

$$C(w) \subset \Omega_m$$

be the set of all sequences starting from the word w. For an arbitrary Borel measure μ on Ω_m, set

$$\mu(w) = \mu(C(w)). \tag{2.5}$$

For each

$$w \in W_m,$$

introduce the operator

$$\alpha_w = \alpha_{w_n} \alpha_{w_{n-1}} ... \alpha_{w_1}. \tag{2.6}$$

Let μ be a Borel σ_m-invariant probability measure on Ω_m. Consider the words w with

$$|w| = l \tag{2.7}$$

and the sum of the corresponding operators α_w with the weights $\mu(w)$,

$$s_l^\mu(\alpha) = \sum_{|w|=l} \mu(w) \alpha_w. \tag{2.8}$$

Average $s_l^\mu(\alpha)$ over $l = 0, ..., n-1$,

$$c_n^\mu(\alpha) = \frac{1}{n} \sum_{l=0}^{n-1} s_l^\mu(\alpha). \tag{2.9}$$

Suppose μ is a σ_m-invariant Markov measure on Ω_m. We will show that the averages $c_n^\mu(\alpha)\varphi$ converge both double side almost everywhere and in $L_1(A,\tau)$ for any operator

$$\varphi \in L_1(A,\tau).$$

Definition 1. *A matrix Q with non-negative entries is said to be irreducible if, for some $n > 0$, all entries of the matrix*

$$Q + Q^2 + ... + Q^n \tag{2.10}$$

are positive (if Q is stochastic, then this is equivalent to saying that in the corresponding Markov chain any state is attainable from any other state).

Definition 2. *A matrix P with non-negative entries is said to be strictly irreducible if P and PP^T are irreducible (here P^T stands for the transpose of the matrix P).*

Definition 3. *A Markov chain is said to be strictly irreducible if the corresponding chain is strictly irreducible.*

Theorem 1. *Let (A,τ) be a non-commutative probability space,*

$$\alpha_1, ..., \alpha_m : A \longmapsto A$$

are positive kernels, and

$$\alpha_1, ..., \alpha_m : L_1(A,\tau) \longmapsto L_1(A,\tau)$$

are their corresponding extensions. Let μ be a σ_m-invariant Markov measure on Ω_m. Then, for any operator

$$\varphi \in L_1(A,\tau),$$

there exists a function

$$\overline{\varphi} \in L_1(A,\tau),$$

such that

$$c_n^\mu(\alpha)\varphi \to \overline{\varphi} \tag{2.11}$$

both double side almost everywhere and in $L_1(A,\tau)$ as $n \to \infty$. We have

$$\tau(\varphi) = \tau(\overline{\varphi}). \tag{2.12}$$

If the measure μ is strictly irreducible, then

$$\alpha_j \overline{\varphi} = \overline{\varphi}$$

for $j = 1, ..., m$. If

$$\varphi \in L_p(A,\tau),$$

$p \geq 1$, then

$$c_n^\mu(\alpha)\varphi \to \overline{\varphi} \tag{2.13}$$

in $L_p(A,\tau)$ as well.

Theorem 1 generalizes Ergodic Theorems of Grogorchuk [11], Nevo [22], Nevo and Stein [23], and Bufetov [3] to the non-commutative case.

In this paper we proof Skew Ergodic Theorem (originated by S.Kakutani). Next, we also apply Theorem 1 to the proof of Ergodic Theorem for finitely generated locally free semigroup acting as contraction on tracial von Neumann algebra, introduced in the [27].

3. Ergodic Type Theorem and Skew Product Transformations

For each $\omega \in W_m$ denote by T_ω operator defined as follows

$$T_\omega : A \longmapsto A, T_\omega = \alpha_{\omega_n} \circ ... \circ \alpha_{\omega_1} \tag{3.1}$$

For any operator φ in $L_1(A, \tau)$ we denote

$$S_j^\mu(T)(\varphi) = \sum_{|\omega|=j} \mu(\omega) T_\omega(\varphi) \tag{3.2}$$

Česaro average of the $S_j^\mu(T)(\varphi)$ over j we denote by

$$C_n^\mu(T)(\varphi) = \frac{1}{n} \sum_{j=0}^{n-1} S_j^\mu(T)(\varphi) \tag{3.3}$$

Theorem 2. *Let φ be an operator affiliated to tracial von Neumann algebra A with separable predual, and integrated with modules, $A \in L_1(A, \tau)$. Then there exists an operator $\overline{\varphi} \in L_1(A, \tau)$ such that*

$$C_n^\mu(T)(\varphi) \longmapsto \overline{\varphi} \text{ in norm of the } L_1(A, \tau), \tag{3.4}$$

and

$$\tau(\varphi) = \tau(\overline{\varphi}). \tag{3.5}$$

If, in addition, $\varphi \in L_{1+\epsilon}(A, \tau)$, then

$$C_n^\mu(T)(\varphi) \longmapsto \overline{\varphi} \text{ double side almost everywhere }. \tag{3.6}$$

For any $\omega \in W_m$, $\omega = \omega_1 ... \omega_n$ denote by ω^* expression $\omega = \omega_n ... \omega_1$. If μ is σ_m invariant Borel probability on Ω_m, than formula $\mu^*(\omega) = \mu(\omega^*)$ defines σ_m invariant Borel probability with property $\mu^{**} = \mu$. Later implies that

$$C_n^\mu(T) = c_n^{\mu^*}(T) \tag{3.7}$$

Let $\Psi = L_\infty(\Omega_m, \mu) \otimes A$ be a tensor product of von Neumann algebras $L_\infty(\Omega_m, \mu)$ and A. Ψ may be considered as a von Neumann algebra $L_\infty(\Omega_m, \mu, A)$ of almost everywhere bounded ultra-weakly bounded functions mapping $\psi : \Omega_m \mapsto A$, with the norm $||\psi|| = \limsup_{\omega \in \Omega_m} ||\psi(\omega)||_\infty$. Trace for the algebra Ψ is given by formula

$$\nu(\psi) = \int_{\Omega_m} \tau(\psi(\omega)) d\mu(\omega), \text{ for } \psi \in \Psi \tag{3.8}$$

Consider an operator Φ define as

$$\Phi(\psi) = \Phi(\psi(\omega)) = \alpha_{\omega_1}(\psi(\sigma(\omega))) \text{ for } \psi \in \Psi \tag{3.9}$$

It is easy to see that $\Phi : \Psi \mapsto \Psi$, and Φ is a positive kernel on Ψ.

Extension of the operator Φ to the $L_1(\Omega_m, \mu, A)$, pre-conjugate of the algebra $L_\infty(\Omega_m, \mu, A) = (L_1(\Omega_m, \mu, A))^*$ by norm $||.||_1$ we denote also by Φ.

For every $x \in A$ denote by x^e embedding of x into Ψ as an identical function. This embedding extends by norm $||.||_1$ onto embedding of $L_1(A, \tau)$ into $L_1(\Omega_m, \mu, A)$. We denote the images under imbedding by A^e and $L_1(A^e)$.

Lemma 1. *For the $\omega \in W_m$ with $|\omega| = j$ and $\varphi \in L_1(A, \tau)$, n-th iteration of Φ satisfy*

$$\Phi^{(n)}(\psi) = \Phi^{(n)}(\psi(\omega)) = \alpha_{\omega_n} \cdots \alpha_{\omega_1}(\psi(\sigma^{(n)}\omega)) \text{ for } \psi \in \Psi \tag{3.10}$$

and

$$\Phi^{(j)}(\varphi^e) = T_\omega(\varphi) \text{ for } \varphi \in L_1(A,\tau) \tag{3.11}$$

Lemma 2. *For $\varphi \in L_1(A,\tau)$ the following equality holds:*

$$\int_{\Omega_m} \Phi^j(\varphi^e)d\mu(\omega) = \sum_{|\omega|=j} \mu(\omega)T_\omega(\varphi) \tag{3.12}$$

Denote by

$$\varphi_n^e = \frac{1}{n}\sum_{j=0}^{n-1}\Phi^j(\varphi^e) \tag{3.13}$$

the n-th Česaro average of φ^e.

Lemma 3. *For $\varphi \in L_1(A,\tau)$ the following equality holds:*

$$\int_{\Omega_m} \varphi_n^e d\mu(\omega) = \frac{1}{n}\sum_{j=0}^{n-1}\sum_{|\omega|=j} \mu(\omega)T_\omega(\varphi) \tag{3.14}$$

and

$$C_n^\mu(T)(\varphi) = \int_{\Omega_m} \varphi_n^e d\mu(\omega) \tag{3.15}$$

Denote by $Proj(A)$ set of all orthogonal projection operators in A. Let E_e be a conditional expectation of the Ψ onto subalgebra A^e defined as

$$E_e(\psi(\omega)) = \int_{\Omega_m} \psi(\omega)d\mu(\omega) \tag{3.16}$$

Lemma 4. *Sequence φ_n^e convergence in norm of $||.||_1$.*

Proof. Follows from the Ergodic Theorem, see for example [29]. □

Lemma 5. *Suppose that $\varphi \in L_{1+\epsilon}(A,\tau)$, than there exists decreasing to 0 sequence of positive operators B_n from $L_1(\Omega_m,\mu,A)$ such that for positive φ*

$$-B_n \leq \varphi_n^e - \overline{\varphi} \leq B_n, \text{ for all } n \tag{3.17}$$

Proof. Follows from the theorem about majorant convergence, see [6]. □

Since conditional expectation is contraction in the norm $||.||_1$, latest lemma and (3.13)-(3.16) imply $||.||_1$ convergence in the Theorem 2.

Lemma 6. *Let E be an conditional expectation of von Neumann algebra $\mathfrak{A}$ onto subalgebra $\mathfrak{B} \subset \mathfrak{A}$. If sequence $\{X_n\}_{n=1}^\infty$ of self-adjoint operators from the algebra $\mathfrak{A}$ such that it majorant converges to 0, than $\{E(X_n)\}_{n=1}^\infty$ majorant converges to 0.*

Proof. Follows from application of expectation E to the majorant convergence inequality. □

Proof of the theorem follows from the lemmas 1-5 and the fact that majorant convergence imply double side almost everywhere convergence.

4. Ergodic Type Theorem for Action of Finitely Generated Locally Free Semigroups

Definition 4. *Locally free semigroup (see* [28] *and references there)* $\mathcal{LFS}_{m+1}$ *with* m *generators is define as semigroup determined by generators satisfying following relations:*

$$\mathcal{LFS}_{m+1} = \{g_1, ..., g_m \mid g_i g_j = g_j g_i;\ i, j \in \{1, ..., m\}, |i - j| > 1\} \tag{4.1}$$

Semigroup $\mathcal{LFS}_{m+1}$ is associated with to topological Markov chain with states $\{1, ..., m\}$ and transition matrix

$$M = (m_{i,j}),\ \ m_{i,j} = \begin{cases} 1, & \text{if } |i - j| \le 1 \text{ or } i \le j; \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

The set of admissible words in the chain corresponds to the W_M, set of admissible one-sided sequences corresponds to Ω_M and left shift σ_M corresponds to shift on Ω_M. Each word $\omega_1 ... \omega_n$ corresponds to $g_\omega = g_{\omega_1} ... g_{\omega_n}$.

Correspondence $\omega \mapsto g_\omega$ defines a bijection between W_M and $\mathcal{LFS}_{m+1}$, and from (4.2) it follows that system (Ω_M, σ_M) mixes topologically, hence it has ergodic measure non-generative on words W_M.

Now we assume that semigroup $\mathcal{LFS}_{m+1}$ acts as semigroup with generators g_i mapped to the kernels α_i acting on tracial von Neumann Algebra (A, τ). Applying Theorem 1, we obtain ergodic theorem for action of $\mathcal{LFS}_{m+1}$.

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