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On the Dual and Sharpened Dual of Sylvester's Theorem in the Plane

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Abstract

Given a collection of n lines in the Euclidean plane, not all coincident and not all parallel, we prove that there must be a point where precisely two lines intersect. We show that this result is a sharpened version of the dual of the classical Theorem of Sylvester. We consider Sylvester's Theorem as well as its various duals in hyperbolic space. Finally we obtain lower bounds on the number of ordinary points in arrangements satisfying the hypotheses of our sharpened dual theorem.

1. Introduction

In 1893 J. J. Sylvester posed the following celebrated problem [12]: Given a collection of points in the plane, not all lying on a line, prove that there exists a line which passes through precisely two of the points. Sylvester's problem which today is usually referred to as Sylvester's Theorem or the Theorem of Sylvester and Gallai, was reposed by Erdos in 1944 [3] and then solved the same year by by T. Gallai [5]. It is our observation that authors that discuss Sylvester's Theorem and deal with point-line duality in one way or another either

(i) Incorrectly assert the following to be Sylvester's dual: "In a collection of n lines in the plane, not all of which are parallel and not all of which are coincident, there must exist a point of intersection of precisely two lines [Sharp Dual Sylvester]," or
(ii) State the correct dual version of the theorem, namely: "In a collection of n lines in the plane, no *two* of which are parallel and not all of which are coincident, there must exist a point of intersection of precisely two lines [Dual Sylvester]," without mentioning the associated problem for which not all lines are parallel, or
(iii) State only the projective dual, namely: "In a collection of n lines in the real projective plane, $\mathbb{RP}^2$, not all of which are coincident, there exists a point of intersection of precisely two lines. [Projective or *Full* Dual Sylvester]"

In this paper we point out that the proper dual to Sylvester's Theorem, stated for the Euclidean plane, is (ii) and give a proof that in fact the sharpened version (i) is also true. We also make observations about Sylvester's Theorem and its various duals in other geometries.

Given a collection of points, a line which passes through precisely two of the points is usually called an **ordinary line**. Much work has gone into obtaining lower bounds on the number of ordinary lines in a collection of points satisfying the hypothesis of Sylvester's Theorem. We shall analogously refer to a point which lies at the intersection of precisely two lines in an arrangement of lines as an **ordinary point**. We obtain corresponding lower bounds on the number of ordinary points in an arrangement of lines, not all of which are coincident and not all of which are parallel.

2. Sylvester and Dual Sylvester Theorems

Theorem 1. ***Sylvester's Theorem.*** *Given any finite set of points in the plane, not all of which are collinear, there is some line containing precisely two of the points.*

The following proof is due to Melchior[8].

Proof. Clearly if Sylvester's Theorem is true in the Projective plane it is true in the Euclidean plane - just consider the planar model of $\mathbb{RP}^2$ and collections of points not on the line at infinity. By projective duality Sylvester's Theorem in the Projective plane is equivalent to the statement that given a finite set of Projective lines, not all coincident, there exists a point which lies at the intersection of precisely two lines. It is this result which we now establish.

If V denotes the number of vertices, E the number of edges, and F the number of faces in a projective arrangement of lines, then Euler's formula says that

$$V - E + F = 1. \tag{1}$$

Under the assumption that Sylvester's Theorem is false, every vertex lies at the intersection of 3 or more lines. Hence,

$$6V \leq 2E. \tag{2}$$

Also, each face contains at least 3 edges, and $\sum_{i=1}^{F}(edges\ per\ face_i)$ counts each edge twice, once for each face on either side of it. Hence

$$3F \leq 2E \tag{3}$$

Putting (2) and (3) together gives $6(V - E + F) \leq 0$ which contradicts (1). Sylvester's Theorem follows. ☐

As an immediate consequence of the preceding proof we have

Theorem 2. ***Sylvester's Theorem for the Projective plane.*** *Given any finite set of points in the Projective plane, not all of which are collinear, there is some line containing precisely two of the points.* ☐

Theorem 3. ***Dual Sylvester Theorem for the Projective plane.*** *Given any finite collection of lines in the projective plane, not all of which are coincident, there is some point of intersection of precisely two lines.* ☐

Since 2-dimensional elliptical space[1] is a model of the real projective plane, the above results imply that the Sylvester and Dual Sylvester Theorems hold in this geometry. Since the Dual Sylvester Theorem holds in 2-dimensional elliptical geometry it clearly also holds in 2-dimensional spherical geometry. Now consider the usual Sylvester Theorem, and suppose we are given n points, not all of which lie on a great circle. If some two of the points are antipodal, then these two points are

[1] Elliptical geometry is spherical geometry with antipodal points identified. To see that two-dimensional elliptical geometry is a model of the real projective plane start with the model of $\mathbb{RP}^2$ where points are lines through the origin and lines are planes through the origin and take intersections with a sphere centered at the origin.

contained in infinitely many great circles, some of which clearly miss the remaining $n-2$ points. If there are no antipodal points then we use the result from elliptical geometry. Hence both Sylvester and its dual hold in 2-dimensional spherical geometry, $\mathbb{S}^2$, as well as 2-dimensional elliptical geometry, $\mathbb{S}^2/\ \mathbb{Z}^2$.

The "full" Dual Sylvester Theorem in $\mathbb{R}^2$ (finite collection of lines in the Euclidean plane, not all coincident) is false; a counterexample is provided by any finite set of parallel lines. If we start with a set of not all coincident lines in $\mathbb{R}^2$, dualize the lines to points, say using the map $(u,v)\mapsto\{(x,y)\in\mathbb{R}^2 : y=ux-v\}$, and obtain an ordinary line of the points, then this line, when dualized back to a point, will be a point of intersection of precisely two of the original lines so long as the two original lines were not parallel. (Otherwise, we would not be able to dualize back, since we would end up with a vertical line.) If we insist that no two lines in our arrangement be parallel, then we are certainly assured of there being an ordinary point. This seemingly "weak" dual actually implies Sylvester's Theorem. Start with a collection of not all collinear points. Since the problem of finding an ordinary line is rotation invariant, rotate so that no two points lie on a line orthogonal to the x-axis. Dualize using $(u,v)\mapsto y=ux-v$ to get a line arrangement where no two lines are parallel. Use the seemingly "weak" dual to obtain an ordinary point. Dualize back to get a line through precisely two points in the original configuration. Thus:

Theorem 4. ***Dual Sylvester Theorem for the Euclidean plane.*** *Given a finite collection of lines in the Euclidean plane, not all coincident and no two of which are parallel, there exists at least one point which lies at the intersection of precisely two lines.* □

We shall come back to what we call the *Sharpened Dual of Sylvester's Theorem for the Euclidean plane*, where not *all* lines are parallel in section 4. In the interim, we make some observations about the situation in the hyperbolic plane.

3. Sylvester and Dual Sylvester Theorems in the Hyperbolic Plane

We use the so-called **Klein** or **Projective Disk model** of the hyperbolic plane, $\mathbb{H}^2$, where points are points in the open unit disk, and lines (geodesics) are open chords. See [11], section 6.1, for details on the Klein model. The one unpleasant feature of this model is that it is not conformal, in other words the angle between geodesic segments is not the Euclidean angle between the segments. However, since we are only interested in incidence properties, this feature will not concern us. In this model, two lines are parallel if the chords do not intersect, or equivalently the lines supporting the chords intersect on or outside the unit disk, or are parallel in the Euclidean plane.

Theorem 5. ***Sylvester's Theorem for the Hyperbolic plane.*** *Given any finite set of points in the hyperbolic plane, not all of which are collinear, there is some geodesic containing precisely two of the points.*

Proof. The proof follows immediately from Sylvester's Theorem for the Euclidean plane plus the fact that in the Projective Disk model of $\mathbb{H}^2$, geodesics are straight line segments. □

The full Dual Sylvester Theorem is clearly false in $\mathbb{H}^2$ for the same reason it is false in $\mathbb{R}^2$ (the case of parallel lines). In addition it is easy to see that the Sharpened Dual of Sylvester's Theorem (not all lines coincident and not *all* lines parallel) is false for $\mathbb{H}^2$, as the example in Figure 1 illustrates.

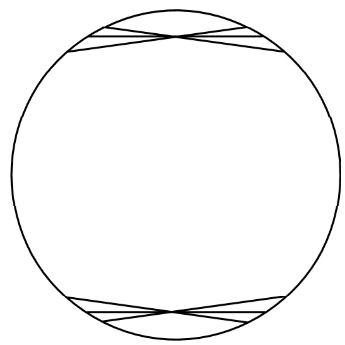

FIGURE 1. The Sharpened Dual of Sylvester's Theorem is false in $\mathbb{H}^2$.

However, we do have the following:

Theorem 6. ***Dual Sylvester Theorem for the Hyperbolic plane.*** *Given any finite collection of geodesics in the hyperbolic plane, not all coincident and no two parallel, then there exists at least one point where exactly two geodesics intersect.*

Proof. An immediate consequence of the fact that Dual Sylvester holds in the Euclidean plane (Theorem 4) coupled with the fact that no two geodesics are parallel so all points of intersection lie in the unit disk. □

Although Theorems 5 and 6 have some appeal, it would be more satisfying if we could produce an incidence-preserving duality map in $\mathbb{H}^2$. However, such a map is not easy to come by. Since $\mathbb{H}^2$ is not a vector space, one is led to consider the hyperboloid model (see [11], chapter 3, for details) of $\mathbb{H}^2$, which sits inside the purely time-like vector subspace of the Lorentzian space $\mathbb{R}^{1,2}$.[2] The most natural dual of a point in $p \in \mathbb{H}^2$ would then be the set of vectors Lorentz orthogonal to p, in other words a plane in $\mathbb{R}^{1,2}$ containing just space-like vectors and an object that does not properly reside in $\mathbb{H}^2$. This map *is* incidence preserving for the same reason point-line duality is incidence preserving in $\mathbb{RP}^2$. Despite the fact that this map takes one in and out of $\mathbb{H}^2$, it may warrant further study.

There are also point-horocycle[3] duality maps that one could consider, though in general these maps would *not* be incidence preserving.

[2]Lorentzian space, $\mathbb{R}^{1,2}$ is just $\mathbb{R}^3$ together with the indefinite inner product $x \circ y = -x_1y_1 + x_2y_2 + x_3y_3$. x and y are then Lorentz orthogonal iff $x \circ y = 0$. Analogously define the Lorentzian norm of a vector x to be $||x|| = (x \circ x)^{1/2}$. A vector x is *time-like* if $||x||$ is imaginary, and *space-like* if $||x|| > 0$.

[3]A horocycle in the Projective Disk model is a circle which is tangent to the circle at infinity.

4. Counting Ordinary Lines and the Sharpened Dual to Sylvester's Theorem

Many people have worked on the problem of determining a lower bound on the number of ordinary lines in an arrangement of n points. The proof of Sylvester's Theorem given earlier can actually be used to show that there must be at least 3 ordinary lines. If not, then all but at most two vertices lie at the intersection of 3 lines, so in place of equation (2) we have

$$6V \leq 2E + 4. \tag{4}$$

Combining this with equation (3) gives $6(V - E + F) \leq 4$, contradicting the projective Euler relation (1). This was first noted by Dirac in [2]. Motzkin [9] showed that the number of ordinary lines must be at least $\sqrt{2n} - 2$. Dirac and Motzkin each conjectured that the number of such lines is at least $\lceil n/2 \rceil$. There are exceptional cases to the $\lceil n/2 \rceil$ conjecture, however, when $n = 7$ and $n = 13$.

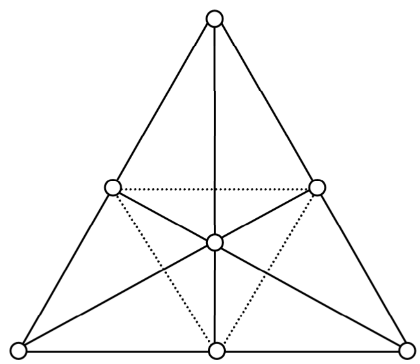

Figure 2. The exceptional case $n = 7$ with 3 ordinary lines.

The exceptional case $n = 7$, which has just 3 ordinary lines, is illustrated in Figure 2. The ordinary lines are the dotted lines that connect the midpoints of the edges of the containing triangle.

Asymptotically, the conjectured bound of $\lceil n/2 \rceil$, with the exceptions $n = 7, 13$, is known to be tight, at least for even n, by virtue of the examples of Böröczky (as cited in [1]). Böröczky's examples for $n = 10, 12$, which reside in $\mathbb{RP}^2$, are given in Figure 3. The examples are obtained by extending the edges of a regular $(n/2)$-gon and adding the $n/2$ lines of reflective symmetry. Note that in the $n = 12$ example any line which has a parallel has two parallels, so there are no ordinary points at infinity.

A bound of $\lceil 3n/7 \rceil$ was proved by Kelly and Moser in 1958 [6], and then refined to $\lceil 6n/13 \rceil$ except for the exceptional case of $n = 7$ by Csima and Sawyer in 1993 [1]. With these results we can prove

Theorem 7. ***Sharp Dual Sylvester Theorem for the Euclidean plane.*** *Given any finite set of lines in the Euclidean plane, not all coincident and not all parallel, then there is a point where precisely two lines intersect.*

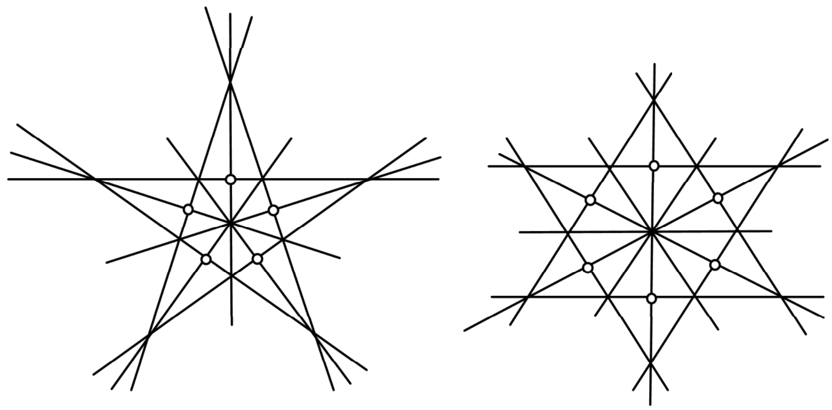

FIGURE 3. Böröczky's examples for $n = 10$ and $n = 12$ with n/2 ordinary points in $\mathbb{RP}^2$.

Proof. Suppose we are given a collection $\mathcal{L}$ of lines in the plane, not all coincident and not all parallel. The property of having an ordinary point is translation invariant, so we may assume that no line in $\mathcal{L}$ passes through the origin. We use the duality map

$$(u, v) \mapsto \{(x, y) : vy = -ux - 1\} \tag{5}$$

which takes points in $\mathbb{R}^2 \setminus \{0\}$ to lines omitting the origin in $\mathbb{R}^2$. Further, points on a line through the origin in the primal plane are taken to parallel lines in the dual plane, and vice versa. Apply the inverse of the map (5) to the collection of lines $\mathcal{L}$. This gives a set of points $\mathcal{L}^*$, not all of which are collinear.

By the result of Kelly and Moser, there are at least $\lceil 3n/7 \rceil$ ordinary lines associated with the set of points $\mathcal{L}^*$. If any of these lines did not pass through the origin, we could use (5) to conclude that the collection $\mathcal{L}$ contained an ordinary point. Hence, assume that all $\lceil 3n/7 \rceil$ ordinary lines pass through the origin. This utilizes all but at most $\lfloor n/7 \rfloor$ of the points. To the collection of points $\mathcal{L}^*$ we now add a point at the origin. None of the original $\lceil 3n/7 \rceil$ ordinary lines are still ordinary and we have created at most $\lfloor n/7 \rfloor$ new ordinary lines. But in this augmented configuration, the assumption of at least $\lceil 3n/7 \rceil$ ordinary lines also applies. $\lceil 2n/7 \rceil$ of these newly found ordinary lines must avoid the origin. Hence these $\lceil 2n/7 \rceil$ lines are ordinary in the original configuration, and so when dualized lead us to ordinary points. □

Collecting our results for the various geometries, we have:

	Sylvester	Full Dual Sylvester	Sharp Dual Sylvester	Dual Sylvester
$\mathbb{S}^2, \mathbb{S}^2/\ \mathbb{Z}^2$	TRUE	TRUE	TRUE	TRUE
$\mathbb{R}^2$	TRUE	FALSE	TRUE	TRUE
$\mathbb{H}^2$	TRUE	FALSE	FALSE	TRUE

Analogous to bounding the number of ordinary lines from below as a function of the number of (not all collinear) points, one can think about bounding the number of ordinary points from below as a function of the number of (not all concurrent, not all parallel) lines. A little experimentation shows that the lower bound on the number of ordinary points in an arrangement is generally considerably lower than the corresponding lower bound on the number of ordinary lines. Figure 4 shows an arrangement of 6 lines with just 1 ordinary point.

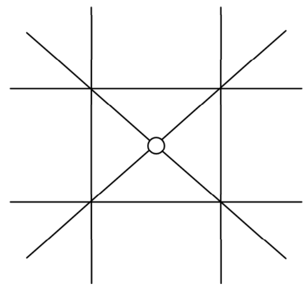

FIGURE 4. An arrangement of 6 lines with 1 ordinary point.

A close examination of the proof of the Sharp Dual Sylvester Theorem yields the following:

Theorem 8. *In any arrangement of n lines in the Euclidean plane, not all concurrent and not all parallel, there must exist at least* $\lceil 5n/39 \rceil$ *ordinary points.*

Proof. Begin with an arrangement $\mathcal{L}$ of n lines in the plane as in the statement of the theorem, none of which we assume passes through the origin. Since if $n = 7$, $\lceil 5n/39 \rceil = 1$ and the result holds, we may assume $n \neq 7$. Dualize the lines to points using (5). We seek to show there are $\lceil 5n/39 \rceil$ ordinary lines through these points, none of which go through the origin. By the result of Csima and Sawyer there exist at least $\lceil 6n/13 \rceil$ ordinary lines. If the statement of the Theorem is false, then at least $\lceil n/3 \rceil$ of the ordinary lines from the Csima and Sawyer bound must lie on lines through the origin. This leaves at most $\lfloor n/3 \rfloor$ points no two of which lie on lines through the origin. Add the point at the origin. In the new configuration, at most $\lfloor n/3 \rfloor$ of the $\lceil 6n/13 \rceil$ ordinary lines go through the origin, again leaving $\lceil 5n/39 \rceil$ which must have been ordinary in the original configuration. □

Thus if $\lceil pn \rceil$ is the bound on the number of ordinary lines in the Sylvester problem, then $\lceil qn \rceil$ is a bound on the number of ordinary points in the sharpened dual problem where

$$pn - (n - 2(p - q)n) = qn,$$

so $q = (3p - 1)/3$. If the $\lceil n/2 \rceil$ conjecture is true then there must be at least $\lceil n/6 \rceil$ ordinary points for $n \neq 7, 13$.[4]

Actually we have not been as careful as possible in the above calculation. After adding the point at the origin we actually have $n + 1$ points so there are $\lceil p(n+1) \rceil$ ordinary lines. Thus the above equation can be improved to

$$p(n+1) - (n - 2(p - q)n) = qn \tag{6}$$

whence

$$q = \frac{p(3n+1) - n}{3n}$$

[4]In [7] we show that there must be at least 2 ordinary points in an arrangement of 7 lines meeting the hypotheses of Sharp Dual Sylvester.

with a corresponding dual Sylvester bound, $DS(n)$, of

$$DS(n) = \left\lceil \frac{p(3n+1)-n}{3} \right\rceil. \tag{7}$$

Strictly speaking, if we are using the Csima-Sawyer bound of $\lceil 6n/13 \rceil$ the bound (7) is valid for $n \neq 6, 7$ since in (6) we are using Sylvester bounds for both n and $n+1$ points. Similarly, with respect to the $\lceil n/2 \rceil$ conjecture, the bound applies only for $n \neq 6, 7, 12, 13$.

5. Concluding Remarks

There is a well known algorithm for finding an ordinary line in a collection of points in time $\mathcal{O}(n \log n)$ due to Mukhopadhyay, Agrawal and Hosabettu [10]. Analogously, it would be interesting to know if one could find an ordinary point in an arrangement of not all coincident, not all parallel lines in time less than $\mathcal{O}(n^2)$. Since our existence proof boils down to dualizing lines to points, adding a point, and then finding several ordinary lines in the augmented configuration, we would be interested in an algorithm that could find just a *second* ordinary line in a collection of points in time $\mathcal{O}(n \log n)$. If the $\lceil n/2 \rceil$ conjecture turns out to be true, then finding one point suffices, and we can just adapt the Mukhopadhyay *et al.* algorithm.

Finally, while Böröczky's examples show that $\lceil n/2 \rceil$ would be asymptotically tight for even n, we do not have evidence that our bounds are asymptotically tight for any n as $n \to \infty$, even under the assumption of the $\lceil n/2 \rceil$ conjecture.

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