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Two-Edged Faces in Arrangements

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Abstract

Given an arrangement of n not all coincident lines on the Riemann Sphere we show that there can be no more than $\lfloor 4n/3 \rfloor$ two-edged faces and give explicit examples to show that this bound is tight. We describe the connection this problem has to the problem of obtaining lower bounds on the number of ordinary points in arrangements of not all coincident, not all parallel lines (the sharpened dual to Sylvester's Problem).

1. Introduction

As an extension of our investigation of line arrangements where not all lines are coincident and not all lines are parallel [7], we have been led to consider arrangements on the Riemann sphere and count those faces (cells) that contain just two edges.

Given a collection of points, a line which passes through precisely two of the points is usually called an **ordinary line**. We shall analogously refer to a point which lies at the intersection of precisely two lines in an arrangement of lines as an **ordinary point**. The classical Theorem of Sylvester states that given a collection of not all collinear points there must exist an ordinary line. In [7] we prove the following sharp dual to Sylvester's Theorem:

Theorem 1. *Given any finite set of lines in the Euclidean plane, not all coincident and not all parallel, then there is a point where precisely two lines intersect.*

The proof relies on results of Kelly and Moser [6] and Csima and Sawyer [2] which give lower bounds on the number of ordinary lines in the dual collection of not all collinear points. In [7] we give analogous bounds on the number of ordinary points in arrangements of lines satisfying the hypotheses of Theorem 1. Bounds on the number of two-edged faces imply bounds on the number of ordinary points through Euler's relation for the Riemann sphere.

2. Euler's Relation on the Riemann Sphere: Connection to Two-Edged Faces

To obtain an arrangement of lines on the Riemann Sphere from an arrangement of n lines in the plane one simply adds a "point at infinity" and collects all unbounded edges at this point. If V denotes the number of vertices, E the number of edges, and F the number of faces in an arrangement on the Riemann Sphere, then

$$V - E + F = 2. \tag{1}$$

We shall heretofore talk about line arrangements simply on the *sphere*, but in so doing, in all cases we mean to say line arrangements on the *Riemann Sphere*.

Now, putting

$$\begin{aligned} t_j &= \textit{number of vertices where } j \textit{ lines cross} \\ p_k &= \textit{number of faces surrounded by } k \textit{ edges} \end{aligned}$$

write

$$Y = \sum_{j \geq 2} (3-j)t_j + \sum_{k \geq 2} (3-k)p_k \tag{2}$$

With this notation we have

$$\sum_{j \geq 2} t_j = V, \ \sum_{k \geq 2} p_k = F. \tag{3}$$

Furthermore, since every edge is shared by two faces,

$$\sum_{k \geq 2} kp_k = 2E \tag{4}$$

and every edge is incident to two vertices,

$$\sum_{j \geq 2} jt_j = E \tag{5}$$

Plugging relations (3), (4), (5) together with Euler's relation (1) into equation (2) gives

$$Y = 3(V - E + F) = 6 \tag{6}$$

But now, the point at infinity has n lines crossing, so if we exclude this point from the vertex set, we have

$$\sum_{j \geq 2} (3-j)t_j + \sum_{k \geq 2} (3-k)p_k = n + 3 \tag{7}$$

Now in equation (7) only the t_2 and p_2 terms are positive and they both have coefficient 1. t_2 is the number of vertices where two lines cross, and p_2 is the number of two-edged faces. Upper bounds on the number of two-edged faces therefore immediately imply lower bounds on the number of ordinary points. In particular, if we knew that there could be no more than $n + 2$ two-edged faces, relation (7) would immediately imply Theorem 1.

However, there are examples of a line arrangement with $n + 3$ two-edged faces. Figure 1 gives an arrangement of 9 lines with 12 two-edged faces. This arrangement has a family of "essentially equivalent variations" with respect to the property of having $n + 3$ two-edged faces as the examples of Figure 2 illustrate. The essential ingredients are two oppositely oriented similar triangles, with coinciding medians which each cut pairs of adjacent two-edged faces.

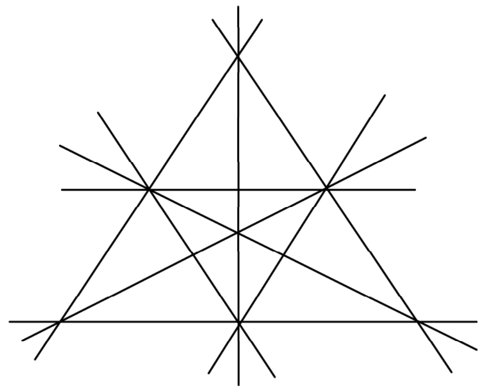

FIGURE 1. Example with $n+3$ two-edged faces.

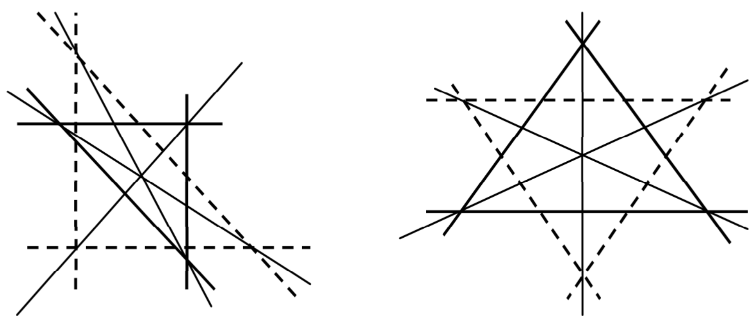

FIGURE 2. Essentially equivalent variations.

One might be tempted to conjecture that these are the only line arrangements with $n+3$ two-edged faces. However, this conjecture is false. Figure 3 gives an example with 16 lines and 12 two-edged faces and another with 20 two-edged faces and 15 lines. The examples of Figure 3 are obtained by taking two identically

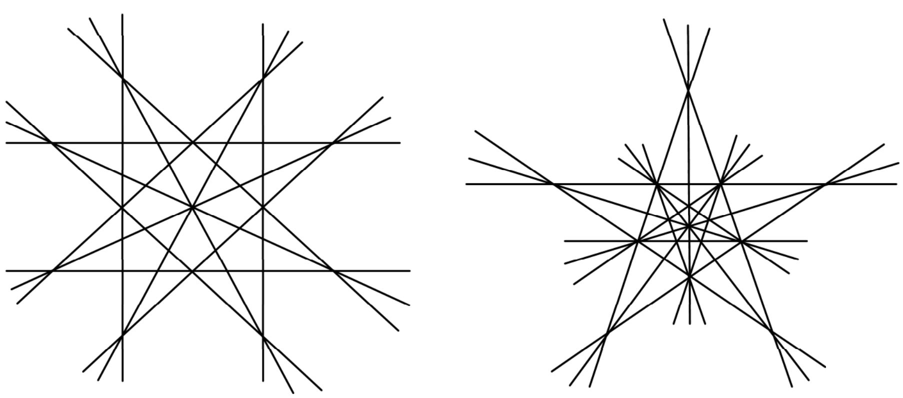

FIGURE 3. $4n/3$ examples.

oriented, nested regular n-gons, rotating one by π/n degrees, extending all edges, and then adding n lines to bisect all of the thus-formed two-edged faces. We prove that this process gives $4n/3$ two-edged faces for any $n \geq 3$ in Theorem 3.

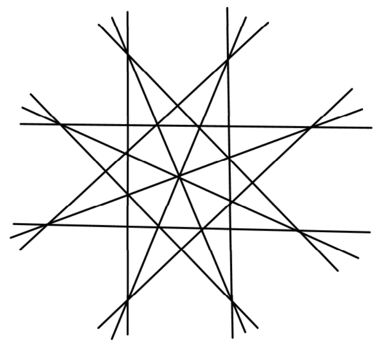

FIGURE 4. An essentially equivalent variation in the $n+4$ case.

3. The Theory of Two-Edged Faces

Lemma 1. *In any arrangement of n lines, not all of which are coincident and no two of which are parallel, there are at most n two-edged faces.*

Proof. Order the lines by increasing slope $l_1, ..., l_n$. We obtain n pairs (l_1, l_2), ...,(l_{n-1}, l_n), (l_n, l_1). A two-edged face must be formed between some such pair, on one unbounded end or the other (what we shall refer to as either the "top" or "bottom," where the choice of "top" is arbitrary). Since no two lines are parallel, each line contains at least two points of intersection. Hence, if (l_i, l_{i+1}) corresponds to a two-edged face on "top" then it cannot correspond to a two-edged face on "bottom" (and vice versa). The lemma follows. □

It is very easy to achieve an arrangement of n lines, no two parallel, with n two-edged faces, as the example in Figure 5 shows. Another example is provided by extending the edges of a regular n-gon, for n odd. In fact, extending the edges of a regular n-gon, for any $n \geq 3$, gives rise to an arrangement with n two-edged faces[1]. If we label the lines which extend the edges in counter-clockwise order according to how they are attached in the n-gon, starting with an arbitrary l_1, as $l_1, ..., l_n$, then $(l_1, l_{\lceil n/2 \rceil})$ is a two-edged face. Continuing around cyclically we obtain n such faces. This observation is important for the $4n/3$ construction in Theorem 3.

Lemma 2. *An arrangement of n lines with $n+k$ two-edged faces must contain k pairs of parallel lines, no two pairs of which are parallel to one another.*

Proof. Adding a line to an arrangement can add at most two two-edged faces: one at the "top" of the line and one at the "bottom." To see this, suppose that adding a line l_k could add more than one two-edged face at the "top." Far from all points of intersection, in the direction of "top," l_k lies between two lines, l_1 and l_2 say. For l_k to create more than one two-edged face at the "top" there must not have been a

[1] One need not stop there; the same reasoning shows that extending the edges of an arbitrary convex n-gon gives rise to an arrangement of n lines with n two-edged faces.

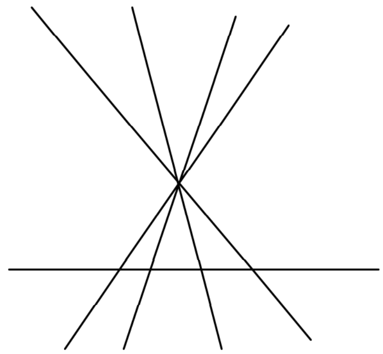

FIGURE 5. n two-edged faces with no two lines parallel.

two-edged face between l_1 and l_2 before placing l_k, and there must be a pair of such faces after placing l_k. But for there to have been no two-edged face earlier means that there had to be a line l_j intersecting both l_1 and l_2 in the direction of "top" after the point where l_1 and l_2 intersect (i.e closer to "top"). But then clearly the newly placed line l_k can create a two-edged face with l_1 or l_2 but not both.

Adding a third parallel line between two existing parallel lines clearly cannot contribute any two-edged faces. The lemma follows by first placing a maximal subset of the n lines, no two of which are parallel, and then placing the remaining lines and recording the maximum number of two-edged faces that can be thereby obtained. □

The observation that one cannot add more than 2 two-edged faces by placing a single line will be used later on.

Figure 6 is an example of 4 lines with 3 two-edged faces, and is thus an example of n lines with less than n two-edged faces. The two-edged faces are each marked with an "x". In fact one can produce examples with 3 two-edged faces and arbitrarily many lines. In Figure 6, we may either add many lines parallel to the line l, all placed between points p and r, or if we prefer an example with no two lines parallel, add lines through p which pass into the regions U and B, but do not pass through the point q.

Lemma 3. *Let $\mathcal{L}$ be a line arrangement and let $\mathcal{C}$ denote the extreme points of the convex hull of the intersection points of the members of $\mathcal{L}$. Then $\mathcal{L}$ must contain at least one two-edged face for each element of $\mathcal{C}$.*

Proof. $\mathcal{C}$ is formed by first taking intersection points of the members of $\mathcal{L}$, finding those points in the collection of intersection points which lie in the convex hull, and then removing non-extreme points. Hence the elements of $\mathcal{C}$ are all themselves intersection points. Any two edges e_i, e_j, which emanate from a point $p \in \mathcal{C}$ and extend from p to infinity, each form an edge of a two-edged face. There may be an edge between e_i and e_j but nonetheless we can associate at least one two-edged face with the point $p \in \mathcal{C}$. The same is true for any other point $q \in \mathcal{C}$, and clearly any edge extending from p to infinity is distinct from any edge extending from q to infinity, and hence their associated two-edged faces are distinct. The lemma follows. □

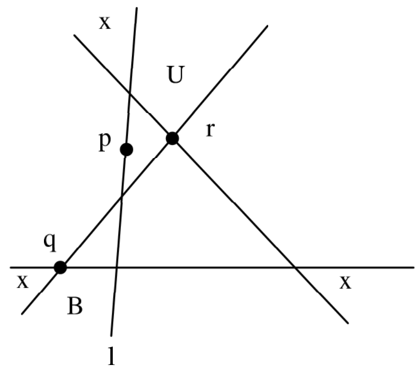

FIGURE 6. 4 or more lines with just 3 two-edged faces.

Corollary 1. *In any arrangement of n lines, not all of which are parallel, there must be at least 3 two-edged faces.*

Proof. If all the intersection points of the collection of lines $\mathcal{L}$ lie on one line then we have a family of parallel lines cutting a single line. In this case there are 4 two-edged faces. Otherwise some three intersection points form a triangle and Lemma 3 applies. □

The following was proved by Atallah for lines in "general position," i.e. no two lines parallel, and no three lines intersecting in a common point, in [1].

Corollary 2. *In any arrangement of n lines it is possible to find the convex hull of the set of intersection points in time $\mathcal{O}(n \log n)$.*

This corollary does not immediately follow from the classical convex hull algorithms for sets of points, because here the given data is the set of lines, and we do not have time to generate the $\mathcal{O}(n^2)$ points of intersection.

Proof. Without loss of generality, assume no line has infinite slope. Sort the lines by increasing slope, and in case of ties, by increasing y-intercept, labelling the sorted sequence $l_1, ..., l_n$. Follow this with a sequence $l'_1, ..., l'_n$ which is the same sequence, but this time sorted by increasing slope, and in case of ties, by *decreasing* y-intercept. We consider in turn the intersection of pairs

$$(l_1, l_2), ..., (l_{n-1}, l_n), (l_n, l'_1), (l'_1, l'_2), ..., (l'_{n-1}, l'_n), (l'_n, l_1).$$

Each two-edged face appears in this list. Lemma 3 implies that the set of intersection points of line pairs in this list includes the extreme points of the convex hull of intersection points of all pairs. Compute the convex hull of this reduced set using any of the known $\mathcal{O}(n \log n)$ algorithms to complete the proof. □

Once we have found the convex hull it is easy enough to generate a list of all intersection points on the hull, still in $\mathcal{O}(n \log n)$ time. For each point found in the final convex hull above, keep track of the "first" and "last" lines intersecting at that point in the cyclical ordering $l_1, .., l_n, l'_1, ..., l'_n, l_1$. If l, k are the first and last lines crossing at a point p_i and m, n are the first and last lines crossing at the next point p_{i+1} of the convex hull, where points are ordered in clockwise order, then we must just gather the intersection points of each line between k, m (in the cyclical ordering), and the line connecting p_i, p_{i+1}. The collection of all such points, together with the original hull points, is a complete hull point enumeration.

It is also possible to enumerate all two-edged faces in $\mathcal{O}(n \log n)$ time, however, not simply by enumerating the hull points. There may be "inner" two-edged faces - two-edged faces that emanate from a point inside the convex hull of the intersection points. The left hand diagram in Figure 7 provides an example.

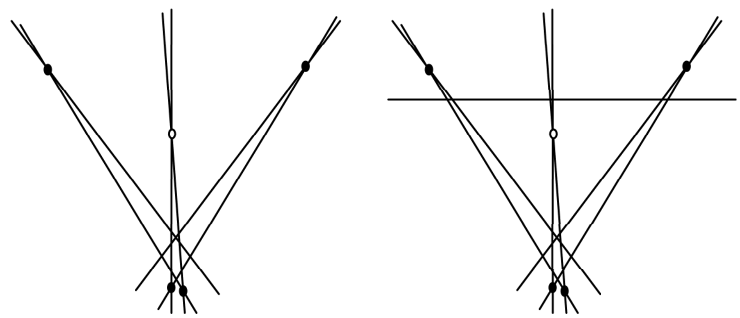

FIGURE 7. Inner two-edged faces and difficulty detecting them.

The convex hull points are marked with solid dots. The vertex of the inner two-edged face is marked with a hollow dot. The difficulty is detecting the difference between the diagram on the left, which has an inner two-edged face, and the diagram on the right, which does not. Note that we can fit arbitrarily many inner two-edged faces next to the two-edged face on the left, thus showing that there can be $\mathcal{O}(n)$ inner two-edged faces. In fact it is not clear that one can tell whether an arbitrary face, given by (l_i, l_{i+1}) in the cyclical ordering, is a two-edged face in less than $\mathcal{O}(n)$ time. Nonetheless, one has:

Theorem 2. *In any arrangement of n lines in the plane, it is possible to completely describe all unbounded faces in time $\mathcal{O}(n \log n)$.*

Proof. First determine the convex hull of the intersection points. Use the hull to draw two horizontal lines in addition to the n lines of the arrangement, one above all intersection points, and one below all intersection points. Now apply the classical Zone Theorem[2]. All two-edged faces are in the zone of one (or both) of the added horizontal lines. Let us focus on the zone of the top line. By the Zone Theorem the complexity of the zone is $\mathcal{O}(n)$. Create a cyclical list of unbounded edges which includes unbounded edges for the two new lines, and do a sweep beginning from the right edge of the top horizontal line and tracing out the lower zone corresponding to the top horizontal line. See Figure 8.

Determining each vertex takes constant time since we are just taking the intersection of the line of an edge with its right or left neighbor in the cyclical list. When we cross a vertex we reverse the order of the two contributing edges. Since there are at most $\mathcal{O}(n)$ vertices the delineation of the zone can be done in $\mathcal{O}(n)$ time. The same is obviously true for the bottom horizontal line and zone. Hence the the most

[2]See for example [8], p.169.

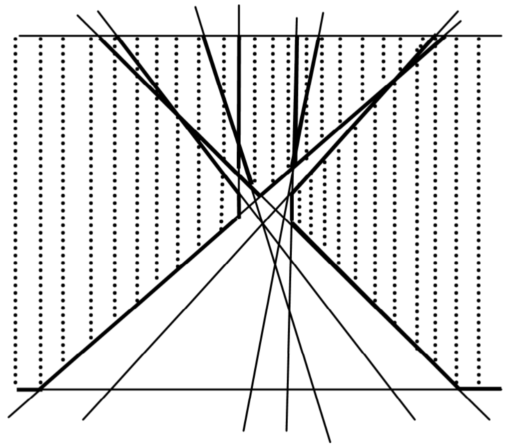

FIGURE 8. Top zone of the unbounded faces.

costly operations are obtaining the initial convex hull and the cyclical unbounded edge list, both of which take time $\mathcal{O}(n \log n)$. Thus the theorem is established. □

Theorem 3. *In an arrangement of n lines, not all of which are coincident, there are at most $\lfloor 4n/3 \rfloor$ two-edged faces. Furthermore, this bound is tight.*

Proof. We begin by establishing the $\lfloor 4n/3 \rfloor$ bound. Given n lines, in order for there to be $n+k$ two-edged faces, by Lemma 2, requires k independent pairs of parallel lines. But between the k pairs of parallel lines there can be no two-edged face. In other words, of the $2n$ pairs

$$(l_1, l_2), \dots, (l_{n-1}, l_n), (l_n, l'_1), \dots, (l'_{n-1}, l'_n), (l'_n, l_1),$$

$2k$ of the pairs cannot be two-edged faces. It follows that $n+k \leq 2n-2k$, i.e. $k \leq n/3$. So the number of two-edged faces is at most $\lfloor 4n/3 \rfloor$.

It remains to establish tightness of the bound. To this end we give explicit examples showing the bound being attained. We first give examples for the case when $n \geq 9$ is a multiple of 3 which we described informally earlier. Start with two identically oriented regular $n/3$-gons with a common center point and rotate one of the $n/3$-gons by $\pi/(n/3)$ degrees and extend all edges. This gives a set of lines with the same set of slopes as a regular $2n/3$-gon. We introduce the following labelling which is convenient in this situation. Label the edges (lines) of the unrotated $n/3$-gon in counter-clockwise order according to how they are attached: l_1,l_3, ...,$l_{2n/3-1}$ and the lines on the corresponding edges (prior to rotation) of the rotated $n/3$-gon l_2, ...,$l_{2n/3}$. Then the two-edged faces are formed by $(l_1, l_{n/3})$, $(l_2, l_{n/3+1})$, etc. It follows that the line that bisects the two-edged face $(l_1, l_{n/3})$ also bisects the two-edged face formed by the pair of corresponding parallel lines, i.e. $(l_{n/3+1}, l_{2n/3})$. There is one bisector line for each such pair of two-edged faces. Since there were $2n/3$ original lines with $2n/3$ two-edged faces, adding $n/3$ bisector lines adds $2n/3$ two-edged faces yielding a total of n lines with $4n/3$ two-edged faces.

The case $n \geq 9$ where n is a multiple of 3 is thus established. Figure 9 establishes the cases $n = 3, ..., 8$.

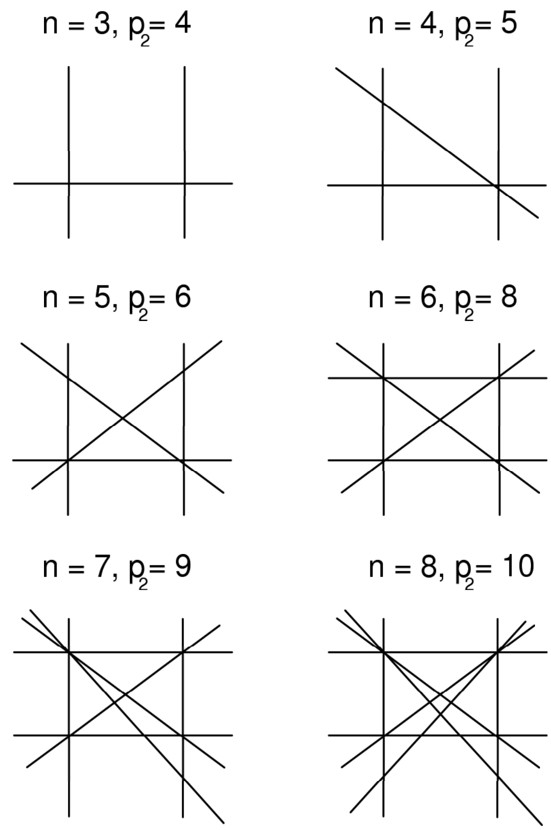

FIGURE 9. $\lfloor 4n/3 \rfloor$ cases $n = 3, \ldots, 8$.

It remains to consider the cases of $n+1$ and $n+2$ where $n \geq 9$ with n a multiple of 3. Note that $\lfloor 4(n+1)/3 \rfloor = 4n/3 + 1$ and $\lfloor 4(n+2)/3 \rfloor = 4n/3 + 2$ so we need to show how to add a line at a time to an example like those of Figures 1 through 3 while also adding a single two-edged face. The last two diagrams in Figure 9 actually provide a model for this. To see this formally, find a circle big enough so that it contains all intersection points. Traverse the circle counter-clockwise beginning at its intersection with any pair of parallel lines. Label the lines as they are encountered l_1, l_2,...,l_n,l_2,l_1,l_3,l_5,l_4,l_6,...,l_n. We encounter a pair of parallel lines, followed by a splitting line, a pair of parallel lines, followed by a splitting line, and so forth, until we run through all lines. After reaching l_n we encounter the parallel lines in reverse order, but otherwise the overall order is unchanged. Now suppose we add a new line l_{n+1} that splits the two-edged face (l_2, l_3). Since this new line has no parallel, the new ordering will be l_1,l_2,l_{n+1},l_3...,l_n,l_2,l_1,l_{n+1},l_3,l_5,l_4,l_6,...,l_n. Previously (l_1, l_3) was a two-edged face. Now (l_1, l_{n+1}) is a two edged since l_{n+1} plainly intersects l_1 after l_3 does, which was the previous last point of intersection. Of course, (l_{n+1}, l_3) is not a two-edged face, since it is also bounded by l_1. In any case, the net result of adding the splitter l_{n+1} is that we have added a two-edged face.

Similarly we can split the two-edged face (l_3, l_4) with a line l_{n+2} yielding the sequence l_1,l_2,l_{n+1},l_3,l_{n+2},l_4,...,l_n,l_2,l_1, l_{n+1},l_3,l_{n+2},l_5,l_4,l_6,...,l_n, and thereby adding a two-edged face.

We have thus established the tightness of the $\lfloor 4n/3 \rfloor$ bound and hence the theorem follows. □

In [7] we give a lower bound of $\lceil 5n/39 \rceil$ on the number of ordinary points in an arrangement of n not all coincident, not all parallel lines. If a famous conjecture of Dirac and Motzkin is true, this bound could be improved to $\lceil n/6 \rceil$, except for the cases $n = 7, 13$. We can use the theory of two-edged faces to get a precise statement in the $n = 7$ case.

Lemma 4. *An arrangement of $n = 7$ not all coincident, not all parallel lines must contain at least 2 ordinary points.*

Proof. Let $\mathcal{L}$ be an arrangement of lines satisfying the conditions of the lemma. If $p_2 \leq 8$ then relation (7) implies that there must be at least 2 ordinary points. Hence assume $p_2 = 9$. By Lemma 2, $\mathcal{L}$ must contain at least two pairs of parallel lines. See Figure 10.

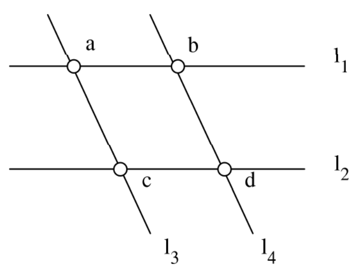

FIGURE 10. Initial placement of the 2 mandatory sets of parallel lines.

Now, for there to be only one ordinary point, some two of the remaining three lines must intersect some two adjacent points $\{a, b, c, d\}$ in the given (cyclical) ordering. Without loss of generality assume that the line l_5 intersects the point a and line l_6 intersects the point b. If l_5 does not pass through d and l_6 does not pass through c, then the 6 lines $\{l_1, ..., l_6\}$ form just 6 two-edged faces. Laying the last line can add at most 2 two-edged faces, contradicting $p_2 = 9$. Hence assume that l_5 intersects both points a, d. If now l_6 does not pass through point c, then prior to the placement of l_7, l_6 intersects l_5, l_2, and l_3 at ordinary points. Laying just one line can only take care of one of these points - so at least two will remain ordinary.

We are thus left with one final case, where l_6 passes through both points b and c. Now line l_7 must create one new two-edged face so that $p_2 = 9$. Without loss of generality, suppose l_7 passes through the point a. We are then in one of the situations of Figure 11 and so clearly have more than one ordinary point. The lemma follows. □

A configuration of 7 lines meeting the hypotheses of Lemma 4 with 2 ordinary points can be obtained by adding a third parallel line through the single ordinary point in the $n = 6$ example of Figure 9.

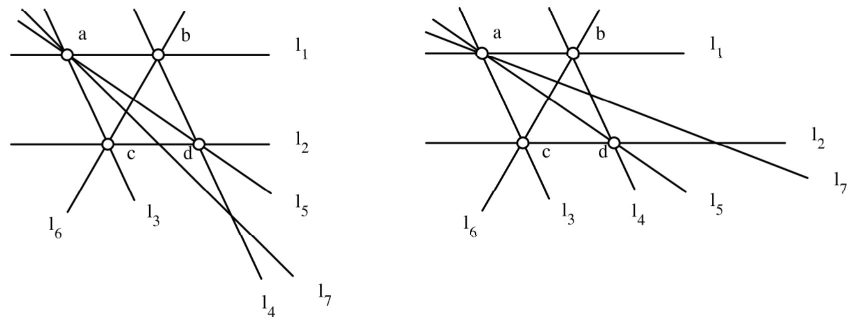

FIGURE 11. Possible placements of l_7.

The following additional best-possible bounds are easily obtained:

# lines	Minimum # of ordinary points
3	2
4	2
5	2
6	1

Finally, given a suitable random distribution on the space of lines, it is interesting to consider the *expected* number of two-edged faces in an arrangement. If we could ignore the possibility of there being inner two-edged faces, this problem would reduce to the problem of determining the expected number of points residing on the convex of hull of the intersection points. To see this, note that:

(i) We could reduce to the case of Lemma 1, because almost surely no two lines are parallel, and no three lines are coincident.
(ii) We can assume every point of the convex hull is either an extreme point or the site of a line crossing which is not the vertex of a two-edged face, since almost surely the vertex of a two-edged face does not happen to lie somewhere in the middle of a supporting line.
(iii) We can assume there is no more than one two-edged face at an extreme point, because again, almost surely, no three lines are coincident.

Lemma 1 would then imply that given a set of n lines the expected number of two-edged faces would be the same as the expected number of points on the convex hull of the intersection points.

It is an interesting fact that the expected number of points in the convex hull of the points of intersection of lines chosen randomly as duals from a uniform distribution of points in a square [5] or disk [4] is actually bounded by a constant for all n, where n is the number of lines. In the case of the square, the constant, although not estimated precisely by the authors, is thought to be relatively small (< 10).

As $n \to \infty$ it clearly becomes more and more unlikely that a randomly chosen internal unbounded face is two-edged, but this does not necessarily imply that the number of internal two-edged faces goes to zero, or is even unbounded. In any case, we hazard the following:

Conjecture. *The expected number of internal two-edged faces tends to zero as the number of lines in a suitably chosen random arrangement of lines tends to infinity.*

4. Concluding Remarks

Doubtless, the methods of Lemma 4 can be pushed to obtain best possible lower bounds in several further cases. However, it would be nice to find a way to push these results to obtain best possible bounds for arbitrarily large n, and thus see if the bounds obtained in [7] are asymptotically tight for any $n \to \infty$.

Finally, it would be interesting to extend our analysis of two-edged faces to the 3 dimensional case where the unbounded cells that take the place of two-edged faces are the "three-faced cells." In exceptional cases there can also be two-faced cells. Such an analysis could inform the "ordinary plane problem" akin to Sylvester's ordinary line problem.

The ordinary plane problem, however, can get a bit messy. One would certainly like to call a plane that passes through precisely three non-collinear points ordinary. However, there are point configurations in $\mathbb{R}^3$, with not all points coplanar, such that no such plane exists. For example, consider a configuration with three or more points on each of two skew lines. No plane exists which passes through precisely three non-collinear points, since such a plane would have to include two points from one of the skew lines, and hence all of the three or more points. Allowing for such a circumstance, given a configuration $\mathcal{S}$ of points in $\mathbb{R}^3$, Devillers and Mukhopadhyay [3] call a plane *ordinary* if all but one of the points of $\mathcal{S}$ that lie on it are collinear. If the points of $\mathcal{S}$ are not all coplanar, they show that such an ordinary plane always exists, and then using point/plane duality and Gaussian diagrams they show that this plane can be found in $\mathcal{O}(n \log n)$ time. The authors note that their ideas extend to higher dimensions.

À la Devillers and Mukhopadhyay, one could try to extend our theory of two-edged faces to dimensions $n > 3$, where it would become a theory of n-faced cells.

Acknowledgements

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