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On Weak Convergence of Iterates in Quantum L_p-Spaces (p >= 1)

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ON WEAK CONVERGENCE OF ITERATES IN QUANTUM L_p-SPACES $(p \geq 1)$

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Abstract. Equivalent conditions are obtained for weak convergence of iterates of positive contractions in the L_1 spaces for general von Neumann algebra and general JBW-algebras, as well as for Segal-Dixmier L_p-spaces $(1 \leq p < \infty)$ affiliated to semifinite von Neumann algebras and semifinite JBW-algebras without direct summands of type I_2.

1. Introduction and Preliminaries

This paper is devoted to a presentation of some results concerning ergodic type properties of weak convergence of iterates of operators acting in L_1 space for general von Neumann algebras and JBW-algebras, as well as Segal-Dixmier L_p-spaces $(1 \leq p < \infty)$ of operators affiliated with semifinite von Neumann algebras and semifinite JBW-algebras.

The first results in the field of non-commutative ergodic theory were obtained independently by Sinai and Anshelevich [21] and Lance [15]. Developments of the subject are reflected in the monographs of Jajte [13] and Krengel [14] (see also [8],[9],[10],[18]).

We will use facts and the terminology from the general theory of von Neumann algebras ([5],[7],[17],[19],[22]), the general theory of Jordan and Real operator algebras ([2],[3],[11],[16]), and the theory of non-commutative integration ([20],[24],[23]).

Let M be a von Neumann algebra, acting on a separable Hilbert space H, M_* is a pre-dual space of M , which always exists according to the Sakai theorem [19]. It is well known that M_* could be identified with L_1-space for M.

Spaces L_1 and L_2 of the operators affiliated with the semifinite von Neumann algebra M with semifinite faithful trace τ were introduced by Segal (see [20]). This result was extended to L_p space of operators affiliated with von Neumann algebra M, τ and integrated with p-th power by Dixmier (see [6]). For an alternative exposition of building L_p based on Grothendieck's idea of using rearrangements of functions see also [24]. The theory of L_p spaces was extended further to the von

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Neumann algebras with faithful normal weight ρ. However, these spaces luck some of the properties, for example, in general, these spaces do not intersect.

Recall some standard terminology ([8],[9],[10],[14]).

Definition 1. *A linear mapping T from M_* in itself is called a **contraction** if its norm is not greater then one.*

Definition 2. *A contraction T is said to be **positive** if*

$$TM_{*+} \subset M_{*+}. \tag{1.1}$$

We will consider the two topologies on the space M_*: the *weak topology*, or the $\sigma(M_*, M)$ topology, and the *strong topology* of the M_*-space norm convergence.

Definition 3. *A matrix $(a_{n,i})$, $i, n = 1, 2, ...$ of real numbers is called **uniformly regular**, if:*

$$\sup_n \sum_{i=1}^{\infty} |a_{n,i}| \le C < \infty; \tag{1.2}$$

$$\lim_{n\to\infty} \sup_i |a_{n,i}| = 0; \tag{1.3}$$

$$\lim_{n\to\infty} \sum_i a_{n,i} = 1. \tag{1.4}$$

2. Main Result- the case of quantum L_1-spaces

2.1. **The case of non-commutative L_1-spaces.** The following theorem is valid:

Theorem 1. *The following conditions for a positive contraction T in the pre-dual space of a Complex von Neumann algebras M are equivalent:*

i). *The sequence $\{T^i\}_{i=1,2,...}$ converges weakly,*
ii). *For each strictly increasing sequence of natural numbers $\{k_i\}_{i=1,2,...}$,*

$$n^{-1} \sum_{i<n} T^{k_i}, \tag{2.1}$$

converges strongly,

iii). *For any uniformly regular matrix $(a_{n,i})$, the sequence $\{A_n(T)\}_{n=1,2,...}$,*

$$A_n(T) = \sum_i a_{n,i} T^i, \tag{2.2}$$

converges strongly.

Proof of the Theorem 1. We first prove the following lemma:

Lemma 1. *Let there exists a uniformly regular matrix $(a_{n,i})$ such that for each strictly increasing sequence $\{k_i\}_{i=1,2,...}$ of natural numbers,*

$$B_n = \sum_i a_{n,i} T^{k_i}, \tag{2.3}$$

converges strongly. Then the sequence $\{T^i\}_{i=1,2,...}$ converges weakly.

Proof. Let $(a_{n,i})$ be a matrix with the aforementioned properties. Then the limit B_n is not dependant upon the choice of the sequence $\{k_i\}_{i=1,2,...}$. In fact, let $\{k_i\}_{i=1,2,...}$ and $\{l_i\}_{i=1,2,...}$ be the sequences for which the limits B_n are different. This means that for some $x \in M_*$,

$$\sum_i a_{n,i} T^{k_i} x \to x_1, \tag{2.4}$$

and

$$\sum_i a_{n,i} T^{l_i} x \to x_2, \tag{2.5}$$

for $n \to \infty$. For a matrix $(a_{n,i})$ let us build increasing sequences $\{i_j\}_{j=1,2,...}$ and $\{n_j\}_{j=1,2,...}$, such that

$$\lim_{j\to\infty}\Big(\sum_{i<i_{j-1}} |a_{n_j,i}| + \sum_{i>i_j} |a_{n_j,i}|\Big) = 0. \tag{2.6}$$

Let

$$m_i = k_i \text{ for } i \in [i_{2j-1}, i_{2j}) \text{ and } m_i = l_i \text{ for } i \in [i_{2j}, i_{2j+1}), j = 1, 2, ... \tag{2.7}$$

Then

$$\lim_j \left\| \sum_i a_{n_{2j+1},i} T^{m_i} x - x_1 \right\| = 0; \tag{2.8}$$

$$\lim_j \left\| \sum_i a_{n_{2j},i} T^{m_i} x - x_2 \right\| = 0, \tag{2.9}$$

which contradicts (2.3), and therefore $x_1 = x_2$. Let now $y \in M$ is such that

$$(T^n x - x_1, y) \to 0, \tag{2.10}$$

when $n \to \infty$. Let us choose a subsequence $\{k_i\}$ such that

$$(T^{k_i} x - x_1, y) \to \gamma \neq 0, \tag{2.11}$$

where γ is a real number. Then, from the uniform regularity of the matrix $(a_{n,i})$ it follows that

$$\lim_n \Big(\sum_i a_{n,i} T^{k_i} x - x_1, y\Big) = \gamma, \tag{2.12}$$

which contradicts the choice of the matrix $(a_{n,i})$. □

Proof of the Theorem 1 (cont.) The implication $iii) \Longrightarrow ii)$ is trivial, because the matrix $(a_{n,i})$,

$$a_{n,i} = \frac{1}{n} \sum_{i<n} \delta_{j,k_i}, \tag{2.13}$$

is uniformly regular. Applying the above Lemma 1 to the matrix

$$a_{n,i} = \frac{1}{n}, \tag{2.14}$$

$i \le n$ and $a_{n,i} = 0$ for $i > n$, we get the implication $ii) \Longrightarrow i)$. □

To prove the implication $i) \Longrightarrow iii)$, we would need the following lemma:

Lemma 2. *Let Q be a contraction in the Hilbert space H. Then the weak convergence of $Q^n x$ in H,where $x \in H$, implies the strong convergence of*

$$\sum_i a_{n,i} Q^i x \tag{2.15}$$

for any uniformly regular matrix $(a_{n,i})$.

Proof. If the weak limit $Q^n x$ exists and equal to x_1, then

$$Q x_1 = Q(\lim_{n\to\infty} Q^n x) = x_1, \tag{2.16}$$

where the limit is considered in the weak topology, i.e. x_1 is Q- invariant. Replacing x on $x - x_1$ (if necessary), we may suppose that $Q^n x$ converges weakly to $\mathbf{0}$, and hence

$$(Q^n x, x) \to 0. \tag{2.17}$$

We are going to show that

$$\sum_n a_{i,n} Q^n x \xrightarrow{\|\cdot\|} \mathbf{0}, \tag{2.18}$$

where $(a_{i,n})$ uniformly regular matrix. One can see that

$$\left\| \sum_i a_{N,i} Q^i x \right\|^2 \leq \sum_i \sum_j a_{N,i} a_{N,j} (Q^i x, Q^j x) \leq \sum_i \sum_j \left| a_{N,i} a_{N,j} (Q^i x, Q^j x) \right|. \tag{2.19}$$

Let us fix $\varepsilon > 0$. Because Q is a contraction, the limit $\|Q^n x\|$ does exist. Now, we can find $K > 0$, such that for $k > K$ and $j \geq 0$,

$$\left\| Q^k x \right\| - \left\| Q^{k+j} x \right\| \leq \varepsilon^2 \tag{2.20}$$

and

$$\left| (Q^k x, x) \right| \leq \varepsilon. \tag{2.21}$$

Then,

$$\begin{aligned} \left| (Q^k x, x) - (Q^{k+j} x, Q^j x) \right| &= \left| (Q^k x, x) - (Q^{*j} Q^{k+j} x, x) \right| \leq \\ &\leq \left\| Q^k x - Q^{*j} Q^{k+j} x \right\| \cdot \|x\| = (\left\| Q^k x - Q^{*j} Q^{k+j} \right\|^2)^{\frac{1}{2}} \cdot \|x\| = \\ &= (\left\| Q^k x \right\|^2 - 2 \left\| Q^{k+j} x \right\|^2 + \left\| Q^{*j} Q^{k+j} x \right\|^2)^{\frac{1}{2}} \cdot \|x\| \leq \\ &\leq (\left\| Q^k x \right\|^2 - \left\| Q^{k+j} x \right\|^2) \cdot \|x\| \leq \varepsilon \cdot \|x\|, \end{aligned} \tag{2.22}$$

and therefore

$$\left| (Q^{k+j} x, Q^j x) \right| \leq \varepsilon \cdot (1 + \|x\|) \tag{2.23}$$

for all $k > K$ and $j \geq 0$, or for $|i - j| \geq k$, the inequality

$$\left| (Q^i x, Q^j x) \right| \leq \varepsilon \cdot (1 + \|x\|), \tag{2.24}$$

is valid. We will fix $\eta > 0$, and let N be such a natural number that

$$\max_i |a_{n,i}| < \eta, \tag{2.25}$$

for $n \geq N$. Then the expression (1) for $n \geq N$ could be estimated the following way:

$$\sum_i \sum_j \left| a_{N,i} a_{N,j} (Q^i x, Q^j x) \right| =$$
$$= \sum_{|i-j| \leq k} \left| a_{n,i} a_{n,j} (Q^i x, Q^j x) \right| + \sum_{|i-j|>k} \left| a_{n,i} a_{n,j} (Q^i x, Q^j x) \right| \leq$$
$$\leq \sum_i |a_{n,i}| \cdot \eta \cdot \|x\|^2 \cdot (2k-1) + \sum_i \sum_j |a_{n,i} a_{n,j}| \cdot \varepsilon \cdot (1 + \|x\|) \leq$$
$$\leq C \cdot \eta \cdot \|x\|^2 \cdot (2k-1) + C^2 \cdot \varepsilon \cdot (1 + \|x\|). \tag{2.26}$$

From the arbitrarity of the values of ε and η it follows that the strong convergence is present and the lemma is proven. □

Proof of the Theorem 1 (cont.) Let us prove the implication $i) \Longrightarrow iii)$. Let $x \in M_{*+}$ and the sequence $\{T^i x\}_{i=1,2,...}$ converges weakly. Without the loss of generality we can consider $\|x\| \leq 1$, and let

$$\overline{x} = \lim_{n \to \infty} T^n x, \tag{2.27}$$

where the limit is understood in the weak sense. Let us consider

$$y = \sum_{n=0}^{\infty} 2^{-n} T^n x. \tag{2.28}$$

The series that defines y is convergent in the norm of the space M_*. From the positivity of x and the properties of the operator T it follows that

$$Ty \leq 2y, \tag{2.29}$$

and, therefore, for all $k = 1, 2, ...,$

$$s(T^k y) \leq s(y), \tag{2.30}$$

where by $s(z)$ we denote the support of the normal functional z. □

Lemma 3. *Let $u \in M_{*+}$ and $s(u) \leq s(y)$. Then $s(\overline{u}) \leq s(\overline{x})$, where*

$$\overline{u} = \lim_{n \to \infty} T^n u. \tag{2.31}$$

Proof. In fact, let us fix $\varepsilon > 0$. From the density of the set

$$\mathfrak{L}_y = \{w \in M_{*+}, w \leq \lambda y, \text{ for some } \lambda > 0\}, \tag{2.32}$$

in the set

$$\mathfrak{S} = \{w \in M_{*+}, s(w) \leq s(y)\}, \tag{2.33}$$

in the norm of the space M_* it follows that there are $\lambda > 0$ and $w \in \mathfrak{L}_y$ such that

$$\|w - u\| \leq \varepsilon \text{ and } w \leq \lambda y. \tag{2.34}$$

Let

$$\overline{w} = \lim_{n \to \infty} T^n w. \tag{2.35}$$

Then

$$\overline{w}(\mathbf{1}-s(\overline{x})) = \\ = \lim_{n\to\infty}(T^n(w))(\mathbf{1}-s(\overline{x})) \le \\ \le \lambda\cdot \lim_{n\to\infty}(T^n y)(\mathbf{1}-s(\overline{x})) \le \\ \le \lambda\cdot \lim_{n\to\infty}(\sum_{k=0}^{\infty} 2^{-k}\cdot (T^{n+k}x)(\mathbf{1}-s(\overline{x}))) = \\ = \lambda\cdot \sum_{k=0}^{\infty} 2^{-k} \lim_{n\to\infty}(T^{n+k}x)(\mathbf{1}-s(\overline{x})) = 0. \tag{2.36}$$

Because the operator T does not increase the norm of the functionals from M_*, we get that

$$\overline{u}(\mathbf{1}-s(\overline{x})) = \lim_{n\to\infty}(T^n u)(\mathbf{1}-s(\overline{x})) \le \\ \le \lim_{n\to\infty}(T^n w)(\mathbf{1}-s(\overline{x})) + \lim_{n\to\infty}\|T^n(w-u)\| \le \varepsilon. \tag{2.37}$$

The needed inequality follows from the arbitrarity of ε. □

Proof of the Theorem 1 (cont.). Let us introduce the following notion. For $\mu\in M_*$, we will denote by $\mu.E$, where E is a projection from the algebra M, the functional

$$(\mu.E)(A) = \mu(EAE), \tag{2.38}$$

where $A\in M$.

Let us fix $\varepsilon>0$. We will find a number N, such that

$$(T^n x)(\mathbf{1}-s(\overline{x})) < \varepsilon^2 \tag{2.39}$$

for $n>N$.

Than

$$\left\|T^N x.s(\overline{x}) - T^N x\right\| = \\ \sup_{\substack{A\in M\\ \|A\|_\infty\le 1}} \left|(T^N x)((\mathbf{1}-s(\overline{x}))A(\mathbf{1}-s(\overline{x})))+ \right. \\ \left. +(T^N x)((s(\overline{x}))A(\mathbf{1}-s(\overline{x}))) + (T^N x)((\mathbf{1}-s(\overline{x}))A(s(\overline{x})))\right| \le \\ \le \varepsilon\cdot(\varepsilon + 2\|x\|^{\frac{1}{2}}), \tag{2.40}$$

because

$$|\mu(AB)|^2 \le \mu(A^*A)\cdot\mu(B^*B), \tag{2.41}$$

where $\mu\in M_{*+}$ and $A,B\in M$.

Let $w\in\mathfrak{L}_{\overline{y}}$ is such that

$$w\le\lambda\overline{x} \tag{2.42}$$

for some $\lambda>0$ and

$$\left\|T^N x.s(\overline{x}) - w\right\| \le \varepsilon. \tag{2.43}$$

Then, for $n>N$, the following is valid:

$$\|T^n x - T^{n-N} w\| \le \|T^{n-N}(T^N x - T^N x.s(\overline{x}))\| +$$
$$+ \|T^{n-N}(T^N x.s(\overline{x}) - w)\| \le 4 \cdot \varepsilon. \tag{2.44}$$

By taking the weak limit in the inequality (2.40) and because the unit ball of M_* is closed weakly, we will get

$$\|\overline{x} - \overline{w}\| \le 4 \cdot \varepsilon, \tag{2.45}$$

where

$$\overline{w} = \lim_{n\to\infty} T^n w. \tag{2.46}$$

Let us now consider the algebra $M_{s(x)}$. The functional $\overline{x}$ is faithful on the algebra $M_{s(x)}$. We will consider the representation $\pi_{\overline{x}}$ of the algebra $M_{s(x)}$ constructed using the functional x [7]. Because the functional $\overline{x}$ is faithful, we can conclude that the representation $\pi_{\overline{x}}$ is faithful on the algebra $M_{s(\overline{x})}$, and therefore $\pi_{\overline{x}}$ is an isomorphism of the algebra $M_{s(\overline{x})}$ and some algebra $\mathfrak{A}$. The algebra $\mathfrak{A}$ is a von Neumann algebra, and its pre-conjugate space $\mathfrak{A}_*$ is isomorphic to the space $M_*.s(\overline{x})$ ([19]). Let us note now that

$$TM_*.s(\overline{x}) \subset M_*.s(\overline{x}). \tag{2.47}$$

In fact,

$$T\mathfrak{L}_y \subset \mathfrak{L}_y, \tag{2.48}$$

and therefore, by taking the norm closure, we get

$$TS \subset S; \tag{2.49}$$

by taking now the linear span, we get

$$TM_*.s(\overline{x}) \subset M_*.s(\overline{x}). \tag{2.50}$$

Let denote by $\overline{T}$ the isomorphic image of the operator T, acting on the space $\mathfrak{A}_*$. Let

$$u \in \mathfrak{A}_{*+} \text{ and } u \le \lambda\overline{x} \tag{2.51}$$

for some $\lambda > 0$. Then there exists the operator $B \in \mathfrak{A}'$, where $\mathfrak{A}'$ is a commutant of $\mathfrak{A}$, such that

$$(AB\Omega, \Omega) = u(A) \tag{2.52}$$

for all $A \in \mathfrak{A}$. Note, that from the lemma 2

$$(\overline{T}u)(A) = u((\overline{T})^* A) = (((\overline{T})^* A)B\Omega, \Omega) = (A((\overline{T}^*)' B)\Omega, \Omega). \tag{2.53}$$

Also, from

$$\overline{T}\mathfrak{A}_{*+} \subset \mathfrak{A}_{*+}, \|\overline{T}u\| \le \|u\| \text{ and } \overline{T}\overline{x} = \overline{x} \tag{2.54}$$

it follows that

$$(\overline{T})^*\mathfrak{A}_+; (\overline{T}^*)\mathbf{1} \le \mathbf{1} \text{ and } \|(\overline{T})^* A\|_\infty \le \|A\|_\infty \tag{2.55}$$

for all $A \in \mathfrak{A}$. Based on the lemma we now conclude that

$$\left\|(\overline{T}^{*}B)\right\|_{\infty} \leq \|B\|_{\infty}; \overline{T}^{*\prime}\mathfrak{A}'_{+} \subset \mathfrak{A}'_{+}; \overline{T}^{*\prime}\mathbf{1} \leq \mathbf{1} \tag{2.56}$$

for all $B \in \mathfrak{A}'$.

The space $\mathfrak{A}'_{sa}$ is a pre-Hilbert space of the self adjoint operators from $\mathfrak{A}'$ with the scalar product

$$(B, C)_{\overline{x}} = (CB\Omega, \Omega), \tag{2.57}$$

and, using the Kadison inequality [5] we have

$$((\overline{T}^{*\prime}B)(\overline{T}^{*\prime}B)\Omega, \Omega) \leq (\overline{T}^{*\prime}(B^2)\Omega, \Omega) \leq (B\Omega, B\Omega), \tag{2.58}$$

i.e. the operator $\overline{T}^{*\prime}$ is a contraction in the pre-Hilbert space $(\mathfrak{A}'_{sa}, (.,.)_{\overline{x}})$.

We will identify $M_{*}.s(\overline{x})$ and $\mathfrak{A}_{*}$. Because $w \in \mathfrak{L}$, i.e.

$$w \leq \lambda\overline{x} \tag{2.59}$$

for some $\lambda > 0$, then

$$\overline{w} \leq \lambda\overline{x} \tag{2.60}$$

as well. Let

$$w(A) = (BA\Omega, \Omega) \text{ and } \overline{w}(A) = (\overline{B}A\Omega, \Omega) \tag{2.61}$$

for all $A \in \mathfrak{A}$, where B, $\overline{B} \in \mathfrak{A}'$.

Let now $(a_{n,i})$ be a uniformly regular matrix. Using lemma 2 we will find $k \in \mathbb{N}$ so that

$$\left\|\sum_i a'_{k,i}T^i w - \overline{w}\right\| = \sup_{\substack{A\in\mathfrak{A}\\ \|A\|_{\infty}=1}} \left|(\sum_{i=1}^{\infty} a'_{k,i}((\overline{T}^{*\prime})^i(B-\overline{B})A\Omega, \Omega)\right| \leq$$

$$\leq (\sum_{i=1}^{\infty} a'_{k,i}(\overline{T}^{*\prime})^i(B-\overline{B})\Omega, \sum_{i=1}^{\infty} a'_{k,i}(\overline{T}^{*\prime})^i(B-\overline{B})\Omega)^{\frac{1}{2}} \cdot \sup_{\substack{A\in\mathfrak{A}\\ \|A\|_{\infty}\leq 1}} (A\Omega, A\Omega)^{\frac{1}{2}} \leq$$

$$\leq (\overline{x}(\mathbf{1}))^{\frac{1}{2}} \cdot \left\|\sum_{i=1}^{\infty} a'_{k,i}(\overline{T}^{*\prime})^i(B-\overline{B})\right\|_{(.,.)_{\overline{x}}} < \varepsilon \tag{2.62}$$

for $k > K$, where by $(a'_{n,i})$ we will denote a matrix with the elements

$$a'_{n,i} = (\sum_{i>N} a_{n,j})^{-1} a_{n,j+N}. \tag{2.63}$$

It is easy to see that the matrix $(a'_{n,i})$ will be uniformly regular as well.

Then, for a big enough $k > K$ we will have

$$\left\|\sum_i a_{k,i}T^i x - \overline{x}\right\| \leq \sum_{i\leq N} |a_{k,i}| \left\|T^i x - \overline{x}\right\| + \sum_{i>N} |a_{k,i}| \left\|T^i x - T^{i-N}w\right\| +$$

$$+ \sum_{i>N} |a_{k,i}| \left|1 - (\sum_{i>N} a_{k,i})^{-1}\right| \left\|T^{i-N}w\right\| + \left\|\sum_{j=1}^{\infty} a_{k,j+N} \cdot (\sum_{i>N} a_{k,i})^{-1} T^j w - \overline{w}\right\| +$$

$$+\left\|\left(\sum_{i\le N} a_{k,i}\right)\cdot\overline{w}\right\| + \left|\sum_{i>N} a_{k,i}\right| \|\overline{w}-\overline{x}\| \le$$

$$\le \sum_{i\le N} 2\cdot\frac{\varepsilon}{N} + \sum_{i>N}|a_{k,i}|\cdot 4\varepsilon + \sum_{i>N}|a_{k,i}|\,(1-(1+\varepsilon)^{-1})\cdot 2 + \sum_{i\le N} 2\cdot\frac{\varepsilon}{N} + (1+\varepsilon)\cdot 4\varepsilon \le$$

$$\le 2\varepsilon + (1+\varepsilon)\cdot 4\varepsilon + \varepsilon\cdot 2\cdot(1+\varepsilon) + \varepsilon + 2\varepsilon + (1+\varepsilon)\cdot 4\varepsilon \le 25\varepsilon. \tag{2.64}$$

The arbitrarity of ε proves the needed statement. The proof of the theorem is now completed. □

2.1.1. *The case of L_1-spaces for JBW-algebras.* The L_1-spaces for semifinite JBW-algebras were considered by [4] (see also [1],[12]), were it has been proven that they do coincide with predual spaces. A semifinite JBW-algebra A always represented as

$$A = A_{sp} \dot{+} A_{ex}, \tag{2.65}$$

where A_{sp} is isometrically isomorphic to operator JW-algebra, and A_{ex} is isometrically isomorphic to the space $C(X, M_3^8)$ of all continuous mappings from a Hyperstoanean compact topological space X onto the exceptional Jordan algebra M_3^8 ([11]). In th case when A does not have direct summands of type I_2, it is going to be a self-adjoint part of a Real von Neumann algebra $R(A_{sp})$, whose complexification

$$R(A_{sp}) \dot{+} iR(A_{sp}) = M, \tag{2.66}$$

where M is the enveloping von Neumann algebra of A_{sp}, and the predual space of A, the space

$$A_* = (A_{sp})_* \dot{+} (A_{ex})_*, \tag{2.67}$$

where $(A_{sp})_*$ is the predual space of A_{sp}, and $(A_{ex})_*$ is the predual space of A_{ex} (see, for example [11] and [2]). The main result for the summand A_{ex} follows immediately from the result for $C(X)$, and the fact that the algebra M_3^8 is finite-dimentional. So, without the loss of generality we are interested in the operator case only. But in the operator case, the space $(A_{sp})_*$ is a self-adjoint part of $R_* = (R(A_{sp}))_*$, and

$$M_* = R_* \dot{+} iR_*, \tag{2.68}$$

(see [2] and [16] for details). So, the main result for R_* is thus follows from the complex case by restriction of scalars, and we obtain the main result for L_1-spaces affiliated to semifinite JBW-algebras without direct type I_2 summand.

3. The case of quantum L_p-spaces, $(1 < p < \infty)$

In the case of a non-commutative L_p-space for a semifinite von Neumann algebra, the main result is disscussed in [25].

We will disscuss here the non-associative case.

In this section A denotes a semifinite JBW-algebra without direct summands of type I_2, with a faithful normal trace τ. By L_p we denote the space of operators affiliated to A, and integrated with p-th power ($p > 1$, see for example [1],[2],[12]). Space L_q (here $q = \frac{p}{p-1}$) is a dual as Banach space to L_p ([1],[12]). The following theorem is valid:

Theorem 2. *The following conditions for a positive contraction T in the L_p are equivalent:*

i). *The sequence $\{T^i\}_{i=1,2,\dots}$ converges in $\sigma(L_p, L_q)$ topology,*

ii). *For each strictly increasing sequence of natural numbers $\{k_i\}_{i=1,2,\dots}$,*

$$n^{-1}\sum_{i<n} T^{k_i}, \tag{3.1}$$

converges in norm of L_p,

iii). *For any uniformly regular matrix $(a_{n,i})$, the sequence $\{A_n(T)\}_{n=1,2,\dots}$,*

$$A_n(T) = \sum_i a_{n,i}T^i, \tag{3.2}$$

converges in norm of L_p.

For the sake of completeness we give the following definitions (see, for example [25]), and sketch of the proof. Let ϕ be a gauge function

$$\phi : \mathbb{R}^+ \mapsto \mathbb{R}^+, \tag{3.3}$$

with

$$\phi(0) = 0, \tag{3.4}$$

and

$$\lim_{t\to\infty} \phi(t) = \infty. \tag{3.5}$$

Hahn-Banach theorem implies for strictly convex Banach spaces E with conjugate E' that there exists a duality map

$$\Phi : E \mapsto E', \tag{3.6}$$

associated with ϕ such that

$$\langle x, \Phi(x)\rangle = \|x\|\|\Phi(x)\|, \tag{3.7}$$

and

$$\|\Phi(x)\| = \phi(x). \tag{3.8}$$

Definition 4. *Map Φ is said to satisfy property (S) uniformly if, for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$, such that for any $x, y \in E$,*

$$|\langle x, \Phi(y)\rangle| < \delta(\epsilon), \tag{3.9}$$

implies

$$|\langle y, \Phi(x)\rangle| < \epsilon. \tag{3.10}$$

Proof. From section 4 in [12], it follows that the duality map defined as

$$\Phi(a) = s|a|^{p-1}, \tag{3.11}$$

for

$$a = s|a| \in A, \tag{3.12}$$

where $a = s|a|$ is a polar decomposition of element a, satisfies the property (S) uniformly. Hence, the statement of the theorem follows from Theorem 3.1 in [25]. □

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