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On Two Consensus Problems in Groups of Interacting Agents with Linear Dynamics

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Abstract

We study two recent consensus problems in multi-agent coordination with linear dynamics. In Saber and Murray an agreement problem was studied with linear continuous-time state equations and a sufficient condition was given for the given protocol to solve the agreement problem; namely that the underlying graph is strongly connected. We give sufficient and necessary conditions which include graphs that are not strongly connected. In addition, Saber and Murray show that the protocol solves the average consensus problem if and only if the graph is strongly connected and balanced. We show how multi-rate integrators can solve the average consensus problem even if the graph is not balanced. We give lower bounds on the rate of convergence of these systems which are related to the coupling topology. Saber and Murray also considered the case where the coupling topology changes with time but remain a balanced graph at all times. We relate this case of switching topology to synchronization of nonlinear dynamical systems with time-varying coupling and give conditions for solving the consensus problem even when the graphs are not balanced.

Jadbabaie et al. study a model of leaderless and follow-the-leader coordination of autonomous agents using a discrete-time model with time-varying linear dynamics and show coordination if the underlying undirected graph is connected across intervals. Mureau extended this to directed graphs which are strongly connected across intervals. We prove that coordination is possible even if the graph is not strongly connected by utilizing recent results on the weak ergodicity of inhomogeneous Markov chains.

I. Introduction

Recently, there has been some activity in studying consensus problems of a group of interacting agents where the underlying dynamics is linear and possibly nonautonomous [1], [2]. The goal is for the agents to reach a consensus, expressed as some state variables in each agent agreeing to each other. The goals of this paper are to study two such problems and to present results improving upon previously reported results. As both these problems focus on the coupling topology expressed as a graph, we review some notation in graph theory in Section II. In Section III we study the agreement problem considered by Saber and Murray [2] and in Section IV we study the coordination problem studied by Jadbabaie et al. [1]. A preliminary version of this paper appears in Proceedings of IEEE International Symposium on Circuits and Systems, 2005.

II. Graphs and networks

A directed graph (digraph) or a network $\mathcal{G}$ is defined as (V, E), where V is the set of vertices and $E \subset V \times V$ is the set of edges. A digraph can be weighted, i.e. there is a positive weight associated with each edge. The cardinality of V is called the order of $\mathcal{G}$ and generally denoted as n. We assume that $V = \{1, \dots, n\}$. The *reversal* of a graph is the graph obtained by reversing the orientation of all the edges, i.e. if a graph has adjacency matrix A, then its reversal has adjacency matrix A^T. The outdegree of a vertex is defined as the sum of the weights of all edges emanating from it. The indegree is defined similarly. The Laplacian matrix of a graph is defined as $L = D - A$, where A is the adjacency matrix and D is the diagonal matrix of vertex outdegrees. This means that L is a zero row sums matrix. A digraph is a directed tree if it has n vertices and $n-1$ edges and there exists a root vertex with directed paths to all other vertices. A directed tree $\mathcal{H}$ is a spanning directed tree of a graph $\mathcal{G}$ if $\mathcal{H}$ has the same vertex set as $\mathcal{G}$. A forest is a collection of trees. A graph is balanced if the outdegree of each vertex is equal to its indegree. Unless the graph is balanced, the Laplacian matrix of the reversal of a graph is generally not equal to L^T. A digraph is strongly connected if there exists a directed path between every ordered pair of distinct vertices. A digraph is weakly connected if ignoring the orientation of the edges, the resulting undirected graph is connected. In [3] it was shown that for balanced graphs, these two notions of connectivity are equivalent. We denote by $\mathbf{1}$ the vector $(1, \dots, 1)^T$. For a Hermitian matrix A, we order the eigenvalues as as $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$.

III. An agreement problem with linear dynamics

In [2] an agreement problem among n agents is modeled by the following continuous-time autonomous linear dynamical system:

$$\dot{x} = -Lx \tag{1}$$

where $x \in \mathbb{R}^n$ and L is the Laplacian matrix of some graph of order n. The coupling topology can be expressed by a weighted digraph $\mathcal{G}$ (with no loops) as follows: If x_i depends on x_j for $i \neq j$, then there is a directed edge of weight $-L_{ij}$ from vertex j to vertex i. In other words if $L_{ij} \neq 0$, then there is an edge in $\mathcal{G}$ from vertex j to vertex i. We call $\mathcal{G}$ the *interaction* graph of L. It is clear that L is the

Laplacian matrix of the reversal of $\mathcal{G}$. We believe that the interaction graph is the proper way to define the underlying coupling topology of the dynamical system in Eq. (1), in contrast to [2] where the graph of L is used, i.e. the reversal of the interaction graph.

Since L is a Laplacian matrix of a digraph, its eigenvalues have nonnegative real parts and $L\mathbf{1} = \mathbf{0}$. If the zero eigenvalue is simple, then the kernel of L is spanned by the vector $\mathbf{1}$ and all other eigenvalues of L have positive real parts due to Gershgorin's circle criterion. In this case, x in Eq. (1) approach the kernel of L, and $x \to x^*$ where $x_i^* = x_j^*$, for all $1 \leq i, j, \leq n$. This is equivalent to saying that the system solves an *agreement* or *consensus* problem. In [2] it was shown that the zero eigenvalue is simple if the graph of L is strongly connected. Since $\mathcal{G}$ is strongly connected if and only if L is irreducible, this is also a consequence of Perron-Frobenius theory [4].

Theorem 1: The multiplicity of the zero eigenvalue of L is equal to the minimum number of directed trees which forms a spanning forest in the interaction graph.

Proof: See [5].

Corollary 1: The zero eigenvalue of L is simple if and only if the interaction graph has a spanning directed tree.

Strong connectedness of the underlying graph is sufficient but not necessary to solve the agreement problem. The next result gives a sufficient and necessary condition for Eq. (1) to solve the agreement problem.

Theorem 2: The state x in Eq. (1) approaches $span(\mathbf{1})$ and thus solves an agreement problem for all initial x if and only if the interaction graph of L contains a spanning directed tree.

Proof: Follows from Corollary 1 and the above discussion. □

This result is intuitive since the existence of a spanning directed tree in the interaction graph implies that there is a root vertex which influences directly or indirectly all other vertices. If no such spanning directed tree exists, then there exists two groups of vertices which do not influence each other [6] and thus cannot possibly reach an agreement for arbitrary initial disagreement. This is one reason why the interaction graph rather than the graph of L is a useful tool to study the consensus criteria of Eq. (1). Since the agent at the root vertex influences all other agents, it can be considered a leader, which might not be unique if there are more than one spanning directed tree. In section IV we look at a case where the root agent and thus the leader is unique. The following theorem extends the corresponding result in [2]:

Theorem 3: Let L be the Laplacian matrix whose interaction graph contains a spanning directed tree. Let w_r and w_l^T be the right and left eigenvectors of L corresponding to the zero eigenvalue. Then

$\lim_{t\to\infty} e^{-Lt} = \frac{w_r w_l^T}{w_r^T w_l}$.

Proof: The proof is essentially the same as in [2]. □

If $x \to x^*$ with $x_i^* = x_j^* = \frac{1}{n}\sum_i x_i(0)$, then Eq. (1) is said to solve the *average consensus* problem. In [2] it was shown that if $\mathcal{G}$ is a strongly connected balanced graph, then Eq. (1) solves the average-consensus problem. The following result shows that if $\mathcal{G}$ is strongly connected, but not balanced, then by using multi-rate integrators we can still solve the average consensus problem.

Theorem 4: Let L be the Laplacian matrix of a strongly connected graph and let w be a positive vector such that $w^T L = 0$. Let $W = \text{diag}(w_1, \ldots w_n)$. Then $W^{-1}\dot{x} = -Lx$ solves the average consensus problem.

Proof: Note that since L is irreducible, a positive vector w exists by Perron-Frobenius theory. Next note that $\mathbf{1}^T WL = w^T L = 0$ and thus WL has zero column sums, i.e. WL is the Laplacian matrix of a balanced graph. Since $\dot{x} = -WLx$, the result follows from the balanced graph case. □

A. Rate of exponential convergence

We say that $x(t)$ converges exponentially towards $x^*(t)$ with rate k if $\|x(t) - x^*(t)\| \leq O(e^{-kt})$. Since Eq. (1) is linear, clearly x converges towards x^* with rate at least[1] $\gamma(L) = \min_i\{\text{Re}(\lambda_i) : \lambda_i \in \text{Spec}(L), \lambda_i \neq 0\}$ which is positive for interaction graphs with a spanning directed tree, with rate equal to $\gamma(L)$ for some initial conditions.

When the graph is balanced, in [6], [7] it was shown that $\kappa = \lambda_2(\frac{1}{2}(L + L^T)) \leq \gamma(L)$, and thus x converges towards x^* with rate at least κ, a result which was proved directly in [2] using a quadratic Lyapunov function. This can be generalized to strongly connected graphs as follows.

Theorem 5: Suppose the graph of L is strongly connected, with a positive vector w such that $w^T L = 0$ and $\max_i w_i = 1$, and $W = \text{diag}(w_1, \ldots, w_n)$. Then in Eq. (1), x converges towards x^* with rate $\beta > 0$ where

$$\beta = \min_{x\perp\mathbf{1}, x\neq 0} \frac{x^T W L x}{x^T\left(W - \frac{ww^T}{w^T\mathbf{1}}\right)x} \geq \lambda_2\left(\frac{1}{2}(WL + L^T W)\right)$$

Proof: Follows from that fact that $\beta \leq Re(\lambda)$ for all nonzero eigenvalue λ of L [3]. □

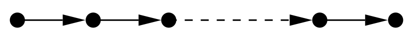
Note that for balanced graphs, $W = I$ and $\beta = \lambda_2(\frac{1}{2}(L + L^T))$. For the case where the interaction graph of L contains a spanning directed tree, a lower bound on $\gamma(L)$ can be found in [5]. See also the quantities a_4 and μ in [8] for other lower bounds. When the graph is undirected, adding extra undirected

[1]Or at least arbitrarily close to $\gamma(L)$, if L has nontrivial Jordan blocks.

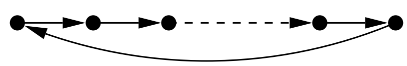
edges cannot decrease γ [9], [10]. However, this is not true for digraphs, as illustrated by the following example.

First note that the Laplacian matrix of an acyclic digraph can always have its rows and columns be simultaneously permuted to a upper-triangular matrix, say $\tilde{L}$. Since $\tilde{L}$ has zero row sums, $\tilde{L}_{nn} = 0$ and thus $\gamma(L) = \gamma(\tilde{L}) = \min_{i<n} \tilde{L}_{ii}$. Thus for an acyclic graph $\gamma(L) = \min_{i \neq j} L_{ii}$ where j is an index such that $L_{jj} = 0$. Since L_{ii} are the indegrees of the interaction graph, this in particular implies that $\gamma(L) = 1$ if the interaction graph of L is a tree.

For the directed path graph with Laplacian matrix L, $\gamma(L) = 1$ since it is a tree and is isomorphic to its interaction graph:



By adding one edge we get the directed cycle graph with a circulant Laplacian matrix L and $\gamma(L) = 1 - \cos\left(\frac{2\pi}{n}\right)$ (See e.g. [11]):



Thus by adding a single edge, γ changes from $\gamma = 1$ to $\gamma = 1 - \cos\left(\frac{2\pi}{n}\right)$ which decreases to 0 as $O(\frac{1}{n^2})$. One way this behavior can be explained is by studying the strongly connected components (SCC) of these graphs. In the directed path graph, the SCC's are single vertices. In other words, each vertex influences directly the next vertex and thus the agreement problem is solved with the first 2 vertices, and then consensus is reached between vertex 2 and 3, etc. In other words, consensus can be reached in stages, each time considering a subgraph of two vertices. On the other hand, for the directed cycle graph, there is a single SCC with each vertex influencing and influenced by vertices which are far apart, i.e. the diameter of the SCC is large for large n. This long path of communication between agents is what causes the consensus to take significantly more effort. In [12] it was shown that for undirected graphs with bounded vertex degrees, if the diameter grows faster than $\ln(n)$ then $\gamma(L) \to 0$ as $n \to \infty$. In addition, we can obtain bounds on γ that depend on the algebraic connectivities of the SCC's and the number of edges between the SCC's [5], [8].

B. Time-varying topology

Ref. [2] also considers the case where $L(t)$ is a time-varying matrix. For this case, the following result is proved:

Theorem 6: If at each time t, $L(t)$ is the Laplacian matrix of a strongly connected balanced digraph $\mathcal{G}(t) \in \Gamma$ where Γ is a finite set, then Eq. (1) satisfies the agreement problem with rate $\kappa^* = \min_{\mathcal{G}\in\Gamma} \lambda_2(\frac{1}{2}(L(\mathcal{G}) + L(\mathcal{G})^T))$.

We extend this result to graphs which are not necessarily balanced nor strongly connected by using results in the synchronization of nonlinear dynamical systems. In the last decade or so, there has been much activity in studying the synchronization in networks of coupled systems [10], [11], [13]–[16]. A network of coupled systems synchronizes if the states of the individual systems approach each other. The consensus problem can thus be considered as a synchronization problem.

Definition 1: A function $f(y,t)$ is V-uniformly decreasing if $(y-z)^T V(f(y,t)-f(z,t)) \leq -\mu\|y-z\|^2$ for some $\mu > 0$ and all y, z, t.

Theorem 7 ([10]): Let $Y(t)$ be a matrix and V be a symmetric positive definite matrix such that $f(x,t) + Y(t)x$ is V-uniformly decreasing. Then the state equation $\dot{x} = (f(x_1,t),\ldots,f(x_n,t))^T + (C(t)\otimes D(t))x$ synchronizes in the sense that $x_i \to x_j$ as $t \to \infty$, if there exists a symmetric irreducible zero row sums matrix U with nonpositive off-diagonal elements such that $(U\otimes V)(C(t)\otimes D(t) - I\otimes Y(t))$ is negative semidefinite for all t. Here $x = (x_1,\ldots,x_n)^T$ and $G \otimes D$ is the Kronecker product of the matrices G and D.

In the case of the agreement problem, we set $y = xe^{kt}$, and obtain $\dot{y} = \dot{x}e^{kt} + ky = ky - L(t)y$. Applying Theorem 7 with $f(x_i,t) = kx_i$ and $V = D(t) = 1$ and choosing $-w > k$, we obtain $y_i \to y_j$ if there exists U such that $U(-L(t) + kI)$ is negative definite for all t. As $x = ye^{-kt}$ this implies that $x_i \to x_j$ with rate k. By choosing $U = I - \frac{1}{n}\mathbf{1}\mathbf{1}^T$, it was shown in [7] that $U(-L(t)+kI)$ is negative definite if $\min_{x\perp\mathbf{1},\|x\|=1} x^T L(t)x > k$ for all t. Thus we have proved the following Corollary to Theorem 7:

Corollary 2: Eq. (1) solves the agreement problem with $x_i \to x_j$ and rate at least k, where $k = \inf_{t,x\perp\mathbf{1},\|x\|=1} x^T L(t)x$.

Since $\min_{x\perp\mathbf{1},\|x\|=1} x^T L(t)x = \lambda_2(\frac{1}{2}(L+L^T))$ for balanced graphs, Corollary 2 generalizes Theorem 6.

C. *Slowly-varying coupling topology*

In contrast to the case of constant L, there might not be agreement even if $\gamma(L(t)) > 0$ for all t. However, agreement can be achieved if $\gamma(L(t)) \geq \alpha > 0$ for all t and $L(t)$ varies slow enough or the variation of $L(t)$ is small enough.

Theorem 8: Let $L(t)$ be piecewise continuous as a function of time t and there exists $\alpha > 0$ such that all nonzero eigenvalues of $L(t)$ has real part larger than α for all $t \geq 0$. Suppose there exists $m > 0$ such that $\|L(t)\| \leq m$ for all $t \geq 0$. Then Eq. (1) solves the agreement problem if one of the following conditions is satisfied:

1) $\alpha > 8m(n-1)$;
2) $L(\cdot)$ is piecewise differentiable and
$$\|\dot{L}(t)\| \leq \delta < \frac{\alpha^{4n-2}}{(2n-1)(2n-2)^{4n-3}m^{4n-4}}$$
3) For some $k \geq 0$, $0 < \eta < 1$, $\alpha > 4m(n-1)\eta - \frac{n-1}{k}\log\eta$ and
$$\sup_{0\leq\tau\leq k}\|L(t+\tau) - L(t)\| \leq \delta < \frac{\eta^{n-1}}{2(n-1)}\left(\alpha - 4m(n-1)\eta + \frac{n-1}{k}\log\eta\right)$$
4) $\alpha > n-1$ and for some $0 < \eta < 1$
$$\sup_{h>0}\left\|\frac{L(t+h) - L(t)}{h}\right\| \leq \delta < \frac{\eta^{n-1}}{n-1}(\alpha - 4m(n-1)\eta + (n-1)\log\eta)$$

Proof: Let
$$C = \begin{pmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & & \cdots & & \\ & & & & 1 & -1 \end{pmatrix}$$
$$D = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ & & \ddots & & 1 \\ & & & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
and $y_i = x_i - x_{i+1}$ for $i = 1, \ldots n-1$. Therefore $y = Cx$ and thus $\dot{y} = -CL(t)x$. In [11], [13] it was shown that $CL(t) = A(t)C$ where $A(t) = CL(t)D$ and $\gamma(L(t)) = \min_i\{\text{Re}(\lambda_i) : \lambda_i \in \text{Spec}(A(t))\}$. Thus $\dot{y} = -A(t)y$, and agreement is achieved if $y \to 0$. The result is then proved by applying Theorem 3.2 in [17] and noting that $\|A(t)\| \leq \|L(t)\|\|C\|\|D\| \leq 2(n-1)\|L(t)\|$. □

D. Nonlinear coupling

Consider the following state equations which extend Eq. (1) to the case where the coupling is nonlinear.
$$\dot{x}_i = \sum_j \phi_{ij}(x_j - x_i) \tag{2}$$

Suppose each ϕ_{ij} satisfies:

1) $\phi(x)$ is continuous and locally Lipschitz,
2) $\phi(x) = 0 \Leftrightarrow x = 0$,
3) $\phi(-x) = -\phi(x)$ for all x,
4) $(x - y)(\phi(x) - \phi(y)) > 0$ for all $x \neq y$,
5) $\phi_{ij} = \phi_{ji}$.

then it was shown in [18] that Eq. (2) solves a consensus problem when the underlying graph is connected for all x. We generalize this result by considering less restrictive conditions on ϕ_{ij}.

Theorem 9: Consider ϕ_{ij} such that

1) $\phi_{ij}(0) = 0$,
2) $x\phi_{ij}(x) > 0$ for $x \neq 0$.

Let $G(x)$ be the matrix such that for $i \neq j$

$$G_{ij}(x) = \begin{cases} \frac{\phi_{ij}(x_j - x_i)}{x_j - x_i} & x_j \neq x_i \\ 1 & \text{otherwise} \end{cases}$$

and $G_{ii}(x) = -\sum_{j \neq i} G_{ij}(x)$. If $\inf_{x, y \perp \mathbf{1}, \|y\|=1} y^T G(x) y > 0$, then Eq. (2) solves a consensus problem.

Proof: First note that the state equation can be rewritten as $\dot{x}_i = \sum_{ij} G_{ij}(x)(x_j - x_i)$ which is equivalent to $\dot{x} = G(x)x$. The result then follows from Corollary 2. □

Note that $(x - y)(\phi(x) - \phi(y)) > 0$ for all $x \neq y$ implies that $x\phi(x) > 0$ for all $x \neq 0$. Also note that ϕ_{ij} needs not be equal to ϕ_{ji} in Theorem 9.

Corollary 3: Consider ϕ_{ij} such that

1) $\phi_{ij} = \phi_{ji}$,
2) $\phi_{ij}(0) = 0$,
3) $x\phi_{ij}(x) > 0$ for $x \neq 0$.

If x lies in a bounded closed set B for all t and the graph of $G(x)$ is connected for all $x \in B$, then Eq. (2) solves a consensus problem.

Proof: If $\phi_{ij} = \phi_{ji}$, then $G(x)$ is symmetric and $\inf_{y \perp \mathbf{1}, \|y\|=1} y^T G(x) y$ is the algebraic connectivity of the graph of $G(x)$ which is positive if the graph of $G(x)$ is connected. Compactness of B implies that $\inf_{x, y \perp \mathbf{1}, \|y\|=1} y^T G(x) y > 0$ and thus Eq. (2) solves a consensus problem by Theorem 9. □

IV. COORDINATION OF AUTONOMOUS AGENTS

In [1] a set of interacting agents is modeled as a nonautonomous linear discrete time system[2]. It was shown that the agents' states converge to each other if the agents are connected over a bounded time period.

A. Leaderless coordination

The state equation is given by:

$$x(k+1) = M(k)x(k) \tag{3}$$

where $M(k)$ is a stochastic matrix (i.e. a nonnegative matrix with each row sum equal to 1). To each matrix $M(k)$ we can again associate an interaction digraph $\mathcal{G}(k)$ with an edge (i,j) if $M_{ji} \neq 0$. In [1] the nonzero elements within each row of $M(k)$ are equal. This form of $M(k)$ corresponds to dynamics where each component $x_i(k+1)$ is determined as the average of $x_i(k)$ plus its neighbors $x_j(k)$ in the graph $\mathcal{G}(k)$. In [1] the graphs $\mathcal{G}(k)$ are undirected and belong to a finite set.

In [1] it was shown that if there is a fixed N such that the union of the graphs $\mathcal{G}(kN+1), \mathcal{G}(kN+2), \ldots, \mathcal{G}((k+1)N)$ is connected for each k, then Eq. (3) solves a consensus problem. in the sense that $x \rightarrow x^*$ where $x_i^* = x_j^*$ for all i,j. The way this is proved is by showing that $M(n)M(n-1)\cdots M(1)$ converges to a rank one matrix of the form $\mathbf{1}c^T$ as $n \rightarrow \infty$. In [19] this is extended to the case where $\mathcal{G}(k)$ are digraphs, and the requirement is that the union of $\mathcal{G}(k), \ldots, \mathcal{G}(k+N)$ are strongly connected and in addition, for each edge (i,j) in $\mathcal{G}(k)$, there is a directed path from j to i in the union of $\mathcal{G}(k+1), \ldots, \mathcal{G}(k+N)$.

We extend these results in several ways. First we consider dynamics where, instead of setting $x_i(k+1)$ to be the average of $x_i(k)$ with its neighbors, we consider a weighted average, i.e. the nonzero entries in each row need not be equal. Second, we consider digraphs which are not necessarily strongly connected. Third, $\mathcal{G}(k)$ belong to a set of graphs which can be infinite. We use results from the theory of inhomogeneous Markov chains which we will summarize next.

Definition 2: A matrix A is *scrambling* if A is stochastic and for each pair of indices i,j there exist a column of A such that the i and j-th entries are both nonzero.

[2] In [1] these agents are termed "autonomous" since they can act on their own without centralized control. We refrain from using this term to avoid confusion with its use in systems theory to denote systems that do not receive external (or equivalently) time-varying stimuli.

Definition 3: A matrix A is stochastic, indecomposable and aperiodic (SIA) if A is stochastic and $\lim_{n\rightarrow\infty} A^n = \mathbf{1}c^T$ for some vector c [20].

Let $\mu(A) = \min_{j,k}\sum_i \min(A_{ji}, A_{ki})$ be the ergodicity coefficient of a matrix A [21]. Note that $0 \leq \mu(A) \leq 1$ for stochastic matrices with $\mu(A) > 0$ if and only if A is scrambling. For a set of matrices S, let S^m denote the set of products of matrices from S of length m. We start with a Lemma which slightly generalizes the results in [20], [22]. Although not explicitly stated in [20], it was discussed in the concluding remarks.

Lemma 1: Let S be a set of matrices such that products of matrices in S are SIA. If $\inf_{A\in S^m}\mu(A) > 0$ for some $m > 0$, then $A_nA_{n-1}\cdots A_1$ with $A_i \in S$ converges to a rank-one matrix of the form $\mathbf{1}c^T$ as $n \rightarrow \infty$.

Theorem 10: Let S be a compact set of matrices such that product of matrices in S are SIA, then $A_nA_{n-1}\cdots A_1$ with $A_i \in S$ converges to a rank-one matrix of the form $\mathbf{1}c^T$ as $n \rightarrow \infty$.

Proof: Since S^m is compact, the argument in [20] shows that $\inf_{A\in S^m}\mu(A) > 0$ for large enough m and thus the result follows from Lemma 1. □

In fact, it is not necessary for every matrix A_i to be in S. It suffices that there are infinitely many long stretches of A_i in S:

Theorem 11: Let S be a compact set of matrices such that product of matrices in S are SIA. Let A_i be a set of stochastic matrices. Let s_i, t_i be two sets of increasing integers such that $s_i \leq t_i < s_{i+1} \leq t_{i+1}$ for each i. If for each i, $A_j \in S$ for all $s_i \leq j \leq t_i$ and $t_i - s_i + 1 \geq \frac{1}{2}(3^n - 2^{n+1} + 1)$ then $A_nA_{n-1}\cdots A_1$ with $A_i \in S$ converges to a rank-one matrix of the form $\mathbf{1}c^T$ as $n \rightarrow \infty$

Proof: The proof is essentially the same as in [20], [22] and we use the fact that products of at least $\frac{1}{2}(3^n - 2^{n+1} + 1)$ matrices in S is scrambling. □

Definition 4: S_d is defined as the set of stochastic matrices with positive diagonal elements.

Theorem 12: For a matrix $A \in S_d$, A is SIA if and only if the interaction graph of A contains a spanning directed tree. If A, B are SIA matrices in S_d, then AB is SIA.

Proof: see [5]. □

The next result shows that the bound of $\frac{1}{2}(3^n - 2^{n+1} + 1)$ in Theorem 11 can be reduced to $O(n^2)$ for matrices in S_d.

Theorem 13: For $k = n(n-2)+1$, let $A_1, \ldots, A_k$ be SIA matrices in S_d. Then the matrix product $A_1A_2\cdots A_k$ is a scrambling matrix.

Proof: Let $\mathcal{G}_i$ be the interaction graph of A_i. Since each $\mathcal{G}_i$ contains a root vertex of a spanning directed tree, it is clear that there are $n-1$ graphs among $\mathcal{G}_i$ with the same root vertex r. Let us denote the corresponding matrices as $A_{m_1}, \ldots, A_{m_{n-1}}$. Since $A_i \in S_d$, it suffices to show that the matrix product $A_{m_1} \cdots A_{m_{n-1}}$ is a scrambling matrix.

Let us denote the children of r (in which we include r itself) in the interaction graph of $A_{m_1} \cdots A_{m_i}$ as C_i. Note that C_1 has at least two elements. Since the diagonal elements of A_i are positive, C_{i+1} is equal to C_i plus the children of C_i in $\mathcal{G}_{m_{i+1}}$. Since $r \in C_i$ and is the root of a spanning directed tree in $\mathcal{G}_{m_{i+1}}$, the children of C_i in $\mathcal{G}_{m_{i+1}}$, must include some vertex not in C_i (unless $C_i = V$). This implies that $C_{n-1} = V$ which implies that the r-th column of $A_{m_1} \ldots A_{m_{n-1}}$ is positive[3], which implies that it is a scrambling matrix. □

Definition 5: $S_d(v)$ is defined as the set of stochastic matrices with positive diagonal elements where each nonzero element is larger than v.

The next results extends the result in [1], [19]:

Theorem 14: Let $\mathcal{G}(k)$ be the weighted interaction digraph of $M(k)$. Suppose there exists $v > 0$, $N > 0$ and an infinite sequence $k_1 \leq k_2 \leq \cdots$ such that

1) $M(k) \in S_d(v)$ for all k,
2) $k_{i+1} - k_i \leq N$,
3) For each i, the union of the graphs $\mathcal{G}(k_i), \mathcal{G}(k_i + 1), \cdots, \mathcal{G}(k_{i+1} - 1)$ contains a spanning directed tree,

then $x \to x^*$ as $t \to \infty$ in Eq. (3), where $x_i^* = x_j^*$ for all i, j.

Proof: Without loss of generality, assume $v < 1$. The product $P_i = M_{k_{i+1}-1} \cdots M_{k_i+1} M_{k_i}$ is in $S_d(v^N)$ and P_i are SIA matrices whose products are SIA [5]. Theorems 11 and 13 show that $B_i = P_{m(i+1)} \cdots P_{mi+2} P_{mi+1}$ is a scrambling matrix for some integer m. Since $\mu(B_i) > 0$ and $B_i \in S_d(v^{Nm})$, this means that $\mu(B_i) \geq v^{Nm} > 0$. By Lemma 1 $\lim_{n\to\infty} B_n \cdots B_0 = \mathbf{1}c^T$ and the result then follows from Lemma 3 in [23]. □

The constant $N < \infty$ is important in Theorem 14. The example in [19] shows that if such an N does not exist, then there could be no consensus among agents. In other words, it is not sufficient (although it is easy to see that it is necessary) in order to reach consensus to have two sequences k_i, n_i such that the union of $\mathcal{G}(k_i)$,$\mathcal{G}(k_i + 1), \ldots, \mathcal{G}(k_i + n_i)$ contains a spanning directed tree for all i. Furthermore, a modification of the example in [19] shows that the hypothesis in Theorem 14 is sufficient, but not necessary for consensus. On the other hand, if each digraph $\mathcal{G}(k)$ is a disjoint union of strongly connected components, then the constant N is not necessary in Theorem 14, i.e. $k_{i+1} - k_i$ can be arbitrarily large

[3]A stochastic matrix with a positive column is called *Markov*.

[24]. This occurs, for example, if $\mathcal{G}(k)$ (after ignoring the weights on the edges) are undirected graphs. In particular, there is no need for a uniform bound N in the results in [1].

Suppose that some of the matrices $M(k)$ are stochastic matrices that are not SIA, while the rest satisfies Theorem 14, would we still have consensus? The answer is no, as the following example indicates. Consider the stochastic matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix $A \in S_d$ is SIA and system (3) with $M(k) = A$ reaches consensus. However, setting $M(k) = A$ when k is even and $M(k) = B$ when k is odd will not result in consensus since BA is a decomposable matrix and decouples the agents from interacting with each other.

On the other hand, Theorems 11 and 13 shows that we can still have consensus if the matrices that are not SIA are sparse enough among $M(k)$.

B. *Follow the leader dynamics and leadership in coordinated agents*

Ref. [1] also considered a follow-the-leader configuration, where n agents are connected via an undirected graph. An additional agent, the leader, influences some of these n agents, but is itself not influenced by other agents. In other words, the state of the leader is constant.

This case is a special case of the discussion in Section IV-A since the state equation can still be written as $M(x) = I - \frac{1}{\alpha}L$ where L is the Laplacian matrix of the underlying interaction graph. Since the leader vertex has indegree 0, every spanning directed tree must have the leader vertex as root. Spanning directed trees exist since the subgraph of the n agents is strongly connected.

This unifies both leaderless and the leader following dynamics considered in [1]. In fact, we can generalize the concept of a leader as follows. A digraph can be partitioned into strongly connected components (SCC) using linear time algorithms [25]. From this structure we create a condensation digraph [26] by associating the SCC to vertices of the condensation digraph with an edge from i to j if and only if there are some edges from the i-th SCC to the j-th SCC. The condensation digraph does not contain directed cycles and satisfies:

Lemma 2: The condensation digraph $\mathcal{H}$ of $\mathcal{G}$ contains a spanning directed tree if and only if $\mathcal{G}$ contains a spanning directed tree.

Proof: Suppose that $\mathcal{H}$ contains a spanning directed tree. For each edge (i, j) in $\mathcal{H}$, we pick an edge from the i-th SCC to the j-th SCC. We also pick a spanning directed tree inside each of the SCC of $\mathcal{G}$

such that the edges above end at the roots of these trees. It is clear that such a choice is also possible. Adding these edges to the spanning directed trees we obtain a spanning directed tree for $\mathcal{G}$. If $\mathcal{H}$ does not contain a spanning directed tree, then there exists at least 2 vertices in $\mathcal{H}$ with indegree zero. This means that there are no edges into at least two SCC's and thus no spanning directed tree can exist in $\mathcal{G}$. □

Thus when $\mathcal{G}$ contains a spanning directed tree, the unique SCC which corresponds to the root of the condensation digraph can be considered a "leading" strongly connected component (LSCC), with the property that agents in the LSCC influencing all other agents outside the component, but not vice versa. When $\mathcal{G}$ changes with time, the LSCC also changes with time. It is clear that the roots of spanning directed trees are equal to the vertices in LSCC.

When $\mathcal{G}$ does not change with time, an alternative way of viewing the dynamics is the following. First the agents in the LSCC reach a consensus. Their states are then "collapsed" into a single "leader" state. The agents that the LSCC influences then reach a consensus following the "leader" state and are absorbed into the "leader" state etc, until finally all agents reach a consensus. This reduces the problem to the case of a single leader.

In addition, we can consider a range of "leadership" in the collection of agents, with the set of root vertices of spanning directed trees as the leaders in the system. The system can be considered leaderless if the size of this set (which is equal to LSCC) approaches the number of agents.

V. Higher dimensional states

In [1], [2] each agent has a scalar state x_i. This can be extended to multidimensional states x_i which are vectors. In this case, the full state x is a concatenation of the state vectors x_i. Suppose that the state equations are written as $\dot{x} = -(L(t) \otimes A(t))x$ and that all eigenvalues of $L(t)$ are nonnegative. Since the eigenvalues of $L(t) \otimes A(t)$ are obtained by multiplying the eigenvalues of $L(t)$ with the eigenvalues of $A(t)$, we can ensure that the eigenvalues of $L(t) \otimes A(t)$ have nonnegative real parts, if the eigenvalues of $A(t)$ have positive real parts. Similarly if all eigenvalues of $L(t)$ have nonnegative real parts, then picking A to have positive eigenvalues will result in the eigenvalues of $L(t) \otimes A(t)$ having nonnegative real parts. Furthermore, it can be shown that in these cases under similar conditions as Sect. III, $x \rightarrow x^*$ where $x_i^* = x_j^*$ for all i, j. Similar statements can be made for the case when the state equations are written as $x(k+1) = (M(k) \otimes A(k))x(k)$.

VI. Conclusions

In this paper, we study in depth two consensus problems in networks of interacting agents where the dynamics are linear and possibly nonautonomous. We show that properties of the interaction graph of the linear operator is important in determining whether consensus is possible or not. In particular, the intuitive idea that if there exists an agent which influences all other agents then consensus is possible is expressed by the property of the interaction graph containing a spanning directed tree.

References

[1] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[2] R. O. Saber and R. Murray, "Agreement problems in networks with directed graphs and switching topology," in *Proceedings of 2003 Conference on Decision and Control*, 2003.

[3] C. W. Wu, "On Rayleigh-Ritz ratios of a generalized Laplacian matrix of directed graphs," *Linear Algebra and its Applications*, 2005, to appear.

[4] H. Minc, *Nonnegative Matrices*. New York: John Wiley & Sons, 1988.

[5] C. W. Wu, "On bounds of extremal eigenvalues of irreducible and m-reducible matrices," *Linear Algebra and Its Applications*, 2005, to appear.

[6] ——, "Algebraic connectivity of directed graphs," *Linear and Multilinear Algebra*, 2005, to appear.

[7] ——, "On a matrix inequality and its application to the synchronization in coupled chaotic systems," IBM Research Report RC23155, 2004, submitted for publication.

[8] ——, "Synchronization in an array of chaotic systems coupled via a directed graph," in *IEEE Int. Symp. Circ. Syst.*, 2005.

[9] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Mathematical Journal*, vol. 23, no. 98, pp. 298–305, 1973.

[10] C. W. Wu, "Perturbation of coupling matrices and its effect on the synchronizability in arrays of coupled chaotic circuits," *Physics Letters A*, vol. 319, pp. 495–503, 2003.

[11] ——, *Synchronization in coupled chaotic circuits and systems*. World Scientific, 2002.

[12] ——, "Synchronization in arrays of coupled nonlinear systems: Passivity, circle criterion and observer design," *IEEE Transactions on Circuits and Systems–I: Fundamental Theory and Applications*, vol. 48, no. 10, pp. 1257–1261, 2001.

[13] C. W. Wu and L. O. Chua, "Synchronization in an array of linearly coupled dynamical systems," *IEEE Transactions on Circuits and Systems–I: Fundamental Theory and Applications*, vol. 42, no. 8, pp. 430–447, 1995.

[14] L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized chaos in arrays of oscillators," in *Proceedings of the 1998 IEEE Int. Symp. Circ. Syst.*, vol. 4. IEEE, 1998, pp. IV–562–567.

[15] X. F. Wang and G. Chen, "Synchronization in small-world dynamical networks," *International Journal of Bifurcation and Chaos*, vol. 12, no. 1, pp. 187–192, 2002.

[16] C. W. Wu, "Synchronization in networks of nonlinear dynamical systems coupled via a directed graph," *Nonlinearity*, vol. 18, pp. 1057–1064, 2005.

[17] A. Ilchmann, D. H. Owens, and D. Prätzel-Wolters, "Sufficient conditions for stability of linear time-varying systems," *Systems and Control Letters*, vol. 9, pp. 157–163, 1987.

[18] R. Olfati-Saber and R. M. Murray, "Consensus protocols for networks of dynamic agents," in *Proceedings of 2003 American Control Conference*, 2003.

[19] L. Mureau, "Leaderless coordination via bidirectional and unidirectional time-dependent communication," preprint, 2003.

[20] J. Wolfowitz, "Products of indecomposable, aperiodic, stochastic matrices," *Proc. Amer. Math. Soc.*, vol. 14, pp. 733–737, 1963.

[21] J. Hajnal, "Weak ergodicitiy in non-homogeneous Markov chains," *Proc. Cambridge Philos. Soc.*, vol. 54, pp. 233–246, 1958.

[22] J. M. Anthonisse and H. Tijms, "Exponential convergence of products of stochastic matrices," *Journal of Mathematical Analysis and Applications*, vol. 59, pp. 360–364, 1977.

[23] J. Shen, "A geometric approach to ergodic non-homogeneous Markov chains," in *Wavelet Analysis and Multiresolution Methods*, ser. Lecture Notes in Pure and Applied Math. Marcel Dekker, 2000, vol. 212, pp. 341–366.

[24] D. Coppersmith and C. W. Wu, "Conditions for weak ergodicity of inhomogeneous Markov chains," IBM Research Report 23489, 2005.

[25] R. E. Tarjan, "Depth-first search and linear graph algorithms," *SIAM Journal on Computing*, vol. 1, no. 2, pp. 146–160, 1972.

[26] R. A. Brualdi and H. J. Ryser, *Combinatorial Matrix Theory*. Cambridge University Press, 1991.