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Abstract

In this paper we study the network design arc set with variable upper bounds. This set appears as a common substructure of many network design problems and is a relaxation of several fundamental mixed-integer sets studied earlier independently. In particular, the splittable flow arc set, the unsplittable flow arc set, the single node fixed-charge flow set, and the binary knapsack set are facial restrictions of the network design arc set with variable upper bounds. Here we describe families of strong valid inequalities that cut off all fractional extreme points of the continuous relaxation of the network design arc set with variable upper bounds. Interestingly, some of these inequalities are also new even for the aforementioned restrictions studied earlier.

1 Introduction

We study the *network design arc set with variable upper bounds* defined as

$$\mathcal{P} = \Big\{ x \in \mathbb{R}^N_+,\ y \in \mathbb{Z}_+,\ z \in \{0,1\}^N \ :\ \sum_{i \in N} a_i x_i \ \le\ a_0 + y, \quad x \ \le\ z \Big\},$$

where $a_i > 0$ for $i \in N$ and $a_0 \ge 0$. The network design arc set with variable upper bounds This set appears as a common substructure of many network design problems.

For a multicommodity network design problem with either fixed charges or combinatorial restrictions on the paths, x_i denotes the fraction of commodity i with demand a_i flowing along an arc with capacity $a_0 + y$. The binary variables z_i's are used for modeling combinatorial restrictions on the paths, such as cardinality, disjointness, etc. as well as fixed charges. Alternatively, this model arises also when $a_0 + y$ is used to model a hub capacity with flow and fixed-charge variables (x_i, z_i) for each incoming arc $i \in N$ into the hub. We will refer to the following inequality as the *capacity constraint*:

$$\sum_{i \in N} a_i x_i \ \le\ a_0 + y. \tag{1}$$

An interesting feature of the set $\mathcal{P}$ is that it is a common relaxation of four fundamental sets that received significant attention in the literature. As such, $\mathcal{P}$ links these four sets that have been studied independently from each other. The first set is the *splittable flow arc set* [16]

$$\mathcal{Q} = \Big\{ x \in \mathbb{R}^N_+,\ y \in \mathbb{Z}_+ \ :\ \sum_{i \in N} a_i x_i \ \le\ a_0 + y, \ \ x \ \le\ \mathbf{1} \Big\},$$

which is obtained from $\mathcal{P}$ by restricting $z = \mathbf{1}$. The second relevant set is the *unsplittable flow arc set* [9]

$$\mathcal{R} = \Big\{ y \in \mathbb{Z}_+,\ z \in \{0,1\}^N \ :\ \sum_{i \in N} a_i z_i \ \le\ a_0 + y \Big\},$$

which is obtained from $\mathcal{P}$ by restricting $x = z$. The third set of interest is the *single node fixed-charge flow set* [20]

$$\mathcal{T} = \Big\{ x \in \mathbb{R}^N_+,\ z \in \{0,1\}^N \ :\ \sum_{i \in N} a_i x_i \ \le\ a_0, \ \ x \ \le\ z \Big\},$$

which is obtained from $\mathcal{P}$ by restricting $y = 0$. Finally, the fourth set is the *binary knapsack set* [6, 13, 23]

$$\mathcal{K} = \Big\{ z \in \{0,1\}^N \ :\ \sum_{i \in N} a_i z_i \ \le\ a_0 \Big\},$$

which is obtained from $\mathcal{P}$ by restricting $x = z$ and $y = 0$.

The set $\mathcal{Q}$ is the simplest one among these four sets and an explicit linear description of its convex hull description is known. Optimization over the other sets is $\mathcal{NP}$-hard and only partial descriptions of the corresponding convex hulls are known.

Note that the convex hulls of $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{T}$, $\mathcal{K}$ are faces of the convex hull of $\mathcal{P}$. Thus $\mathcal{P}$ has the characteristics of all these four sets and one can obtain strong inequalities for them from $\mathcal{P}$. We shall observe in the later sections that the seemingly unrelated inequalities given independently for $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{T}$, and $\mathcal{K}$ are just special cases of the valid inequalities for $\mathcal{P}$ when they are restricted to the appropriate faces of the convex hull of $\mathcal{P}$.

In the remainder of this section, we review some of the basic results known for the related sets $\mathcal{Q}, \mathcal{R}, \mathcal{T}$, and $\mathcal{K}$ so that we can show the connections between the inequalities for $\mathcal{P}$ and those known for the others. We also review the basic mixed-integer rounding procedure as it is used in the paper. In Section 2, we describe some of the basic polyhedral properties of $\mathcal{P}$. In Section 3, we give generalizations of the flow cover inequalities for $\mathcal{P}$ and discuss their strength as well as the fractional solutions cut off by them. In Section 4, we describe strong valid inequalities obtained through two consecutive applications of the mixed-integer rounding procedure. It turns out that these inequalities are sufficient to cut off all fractional extreme points of the continuous relaxation of $\mathcal{P}$. Interestingly, some of the strong inequalities obtained for $\mathcal{P}$ are also new even for the aforementioned restrictions studied earlier.

Throughout, the convex hull and the continuous relaxation of a set are denoted by $\operatorname{conv}(\cdot)$ and $\operatorname{relax}(\cdot)$, respectively. For $v \in \mathbb{R}^N$, we define $v(S) = \sum_{i \in S} v_i$ for $S \subseteq N$. For $a \in \mathbb{R}$, we use $(a)^+$ to denote $\max\{a, 0\}$. We let $\hat{a} = (\lceil a(N) - a_0 \rceil)^+$ and $n = |N|$. We use e_i to denote the ith unit vector, $\mathbf{0}$ and $\mathbf{1}$ to denote a vector of zeros and ones, respectively.

1.1 Splittable flow arc set

The splittable flow arc set $\mathcal{Q}$, is the relaxation of a multicommodity flow design problem for a single arc of the network. The *residual capacity inequalities* [5, 16]

$$\sum_{i \in S} a_i(1 - x_i) \quad \geq \quad \rho(\eta - y), \quad S \subseteq N, \tag{2}$$

where $\eta = \lceil a(S) - a_0 \rceil$ and $\rho = a(S) - a_0 - \lfloor a(S) - a_0 \rfloor$, are valid for $\mathcal{Q}$. For the slightly special case, where $a_0 = 0$, Magnanti et al. [16] show that adding all residual capacity inequalities to relax($\mathcal{Q}$) gives a complete description of conv($\mathcal{Q}$). Atamtürk and Rajan [5] give a polynomial separation algorithm for (2). In particular, they show that for a point $(x, y) \in \text{relax}(\mathcal{Q}) \setminus \mathcal{Q}$, a violated residual capacity inequality (2) is given by letting $S = \{i \in N : x_i > y - \lfloor y \rfloor\}$. Although stated in [5], a proof for convex hull description is not given for $\mathcal{Q}$ when $a_0 \neq 0$. For completeness, we show below that the convex hull result for $\mathcal{Q}$ follows from [16].

Lemma 1 *Adding the residual capacity inequalities* (2) *to* relax($\mathcal{Q}$) *gives* conv($\mathcal{Q}$).

Proof Given $\mathcal{Q}$, define the set

$$\mathcal{Q}_0 = \Big\{ x \in \mathbb{R}^N_+, x_0 \in \mathbb{R}_+, y_0 \in \mathbb{Z}_+ : (\lceil a_0 \rceil - a_0)x_0 + \sum_{i \in N} a_i x_i \ \leq \ y_0, \ x \leq \mathbf{1}, \ x_0 \leq 1 \Big\}.$$

From [16] adding the residual capacity inequalities to relax($\mathcal{Q}_0$) gives conv(Q_0). $X = \{(x, x_0, y_0) \in \text{conv}(\mathcal{Q}_0) \ : \ x_0 = 1\}$ is a face of conv($\mathcal{Q}_0$), and therefore it is integral. This holds true after adding a lower bound $y_0 \geq \lceil a_0 \rceil$ on the only integer variable. Then projecting out variable x_0 and defining $y = y_0 - \lceil a_0 \rceil$ gives conv($\mathcal{Q}$). Observe that residual capacity inequality $\sum_{i \in S} a_i(1 - x_i) + (\lceil a_i \rceil - a_0)(1 - x_0) \geq \rho_0(\eta_0 - y_0)$ for $\mathcal{Q}_0$ with $\eta_0 = \lceil a(S) + \lceil a_0 \rceil - a_0 \rceil$ and $\rho_0 = a(S) + \lceil a_0 \rceil - a_0 - \lfloor a(S) + \lceil a_0 \rceil - a_0 \rfloor$ equals (2) for $x_0 = 1$ and $y = y_0 - \lceil a_0 \rceil$ since $\eta_0 = \eta + \lceil a_0 \rceil$ and $\rho_0 = \rho$. ■

1.2 Single node fixed-charge flow set

The first polyhedral study of the single node fixed-charge flow set $\mathcal{T}$ is due to Padberg et al. [20]. Let $S \subseteq N$ be called a *cover* if $\lambda = a(S) - a_0 > 0$. For a cover S, the authors define the *flow cover inequality*

$$\sum_{i \in S} a_i x_i + \sum_{i \in S} (a_i - \lambda)^+ (1 - z_i) \leq a_0, \tag{3}$$

which is facet-defining for conv($\mathcal{T}$) if $\lambda < \bar{a} = \max_{i \in S} a_i$. In the same paper they also show that the *augmented flow cover inequalities*

$$\sum_{i \in S \cup T} a_i x_i + \sum_{i \in S} (a_i - \lambda)^+ (1 - z_i) \leq a_0 + \sum_{i \in T} (\bar{a} - \lambda) z_i, \tag{4}$$

where $T \subseteq \{i \in N \setminus S : a_i \leq \bar{a}\}$ define facets of *conv*($\mathcal{T}$) under the same condition as well. Gu et al. [12] obtain generalizations of (4) through sequence independent lifting of (3). A complementary class of pack inequalities for $\mathcal{T}$ and their lifting are studied in [1, 21]. Flow sets with integer variable upper bounds are studied in [2, 8, 14].

1.3 Unsplittable flow arc set

The unsplittable flow arc set $\mathcal{R}$ is studied first by Brockmüller et al. [9]. For $S \subseteq N$ they define the *c-strong inequalities*

$$\sum_{i \in S} \lceil a_i \rceil z_i + \sum_{i \in N \setminus S} \lfloor a_i \rfloor z_i \leq c_S + y, \tag{5}$$

where $c_S = \sum_{i \in S} \lceil a_i \rceil - \lceil a(S) - a_0 \rceil$. A set $S \subseteq N$ is called *maximal c-strong* if $c_{S \setminus \{i\}} = c_S$ for all $i \in S$ and $c_{S \cup \{i\}} = c_S + 1$ for all $i \in N \setminus S$. Brockmüller et al. show that a c-strong inequality (5) is facet-defining for $\text{conv}(\mathcal{R})$ if and only if S is maximal c-strong. Atamtürk and Rajan [5] generalize (5) to *k-split c-strong inequalities*

$$\sum_{i \in S} \lceil k a_i \rceil z_i + \sum_{i \in N \setminus S} \lfloor k a_i \rfloor z_i \leq c_S^k + k y, \tag{6}$$

where $c_S^k = \sum_{i \in S} \lceil k a_i \rceil - \lceil k a(S) - k a_0 \rceil$ for a positive integer k. Other strong inequalities obtained by lifting binary knapsack cover inequalities for $\mathcal{R}$ are described in [5, 22].

1.4 Binary knapsack set

The binary knapsack set $\mathcal{K}$ is the most studied restriction of $\mathcal{P}$. The basic inequalities for $\mathcal{K}$ are the so-called cover inequalities: A set $S \subseteq N$ is called a *cover* if $\lambda = a(S) - b > 0$. For a cover S, the *cover inequality* [6, 13, 23]

$$\sum_{i \in S} x_i \leq |S| - 1 \tag{7}$$

is valid for $\mathcal{K}$. Cover inequalities (7) from minimal covers induce facets of the restriction $\text{conv}\{x \in \mathcal{K} : x_i = 0,\ i \in N \setminus S\}$ and they cut off all fractional extreme points of $(\mathcal{K})$. These inequalities typically need to be lifted in order to obtain facet-defining inequalities for $\text{conv}(\mathcal{K})$ [7, 10, 12, 19, 25, 26].

1.5 MIR inequalities

Mixed-integer rounding (MIR) [18] is a general procedure for deriving valid inequalities for mixed-integer sets. Typically strong inequalities for mixed-integer sets can be derived with a single MIR application from an appropriate relaxation [17]. For completeness and ease of presentation, we next review the basic idea behind these inequalities.

Observation 2 *[24] If $x + y \geq b$ is a valid inequality for a mixed-integer set $X \subseteq \{(x, y) \in \mathbb{R}_+ \times \mathbb{Z}\}$, then the* MIR *inequality $x \geq r(\lceil b \rceil - y)$, where $r = b - \lfloor b \rfloor$ is also valid for X.*

Next we give a simple application of the MIR procedure that will be used later in the paper for obtaining strong inequalities for $\mathcal{P}$.

Lemma 3 *Consider a mixed-integer set*

$$Y = \left\{ (x, y) \in \mathbb{R}_+ \times \mathbb{Z}_+^{S \cup T} \;:\; x + \sum_{i \in S} a_i y_i - \sum_{i \in T} a_i y_i \geq b \right\},$$

with $b \geq 0$ and $a_i > 0$ for all $i \in S$. For $\alpha \geq \max\{b, \bar{a}\}$ and $\bar{a} = \max_{i \in S} a_i$, inequality

$$x + \sum_{i \in S} \min\{a_i, b\}\, y_i - \sum_{i \in T} \phi_\alpha(a_i, b) y_i \geq b, \tag{8}$$

where $\phi_\alpha(a, b) = (b\lfloor a/\alpha \rfloor + (a - \alpha\lceil a/\alpha \rceil + b)^+)$, is valid for Y.

Proof Dividing the inequality defining Y by α, one obtains

$$\frac{x}{\alpha} + \sum_{i \in S} \frac{a_i}{\alpha}\, y_i - \sum_{i \in T} \frac{a_i}{\alpha}\, y_i \geq \frac{b}{\alpha}. \tag{9}$$

Applying MIR to inequality (9) gives inequality (8). To see this, let $\rho_i = \alpha\lceil a_i/\alpha \rceil - a_i$. For $i \in T$, if $\rho_i < b$ we rewrite the coefficient of y_i in inequality (9) as $-\lceil a_i/\alpha \rceil + \rho_i/\alpha$, otherwise we relax inequality (9) by changing this coefficient to $-\lfloor a_i/\alpha \rfloor$. We then apply Observation 2 by *(i)* treating $(a_i/\alpha)y_i,\ i \in S$ as a continuous variable if $a_i < b$, and *(ii)* treating $(\rho_i/\alpha)y_i,\ i \in T$ as a continuous variable if $\rho_i < b$. ∎

2 Basic properties of conv($\mathcal{P}$)

First note that optimizing a linear function over conv($\mathcal{P}$) is $\mathcal{NP}$-hard since the binary knapsack polytope conv($\mathcal{K}$) is a face of it. We next state basic polyhedral properties of conv($\mathcal{P}$). Observe that if $a(N) \leq a_0$, then relax($\mathcal{P}$) is integral and therefore conv($\mathcal{P}$) = relax($\mathcal{P}$). This is due to the fact that when the capacity constraint in redundant, the remaining constraints defining $\mathcal{P}$ only consist of (variable) bound constraints.

Proposition 4 *The polyhedron* conv($\mathcal{P}$) *is full-dimensional.*

Proof The following $2n + 2$ points (y, z, x) of $\mathcal{P}$ are affinely independent: $(\hat{a}, \mathbf{0}, \mathbf{0})$, $(\hat{a} + 1, \mathbf{0}, \mathbf{0})$, $(\hat{a}, e_i, \mathbf{0})$ and $(\hat{a}, e_i, e_i)$ for $i \in N$. ∎

We next make some observations that help characterize the extreme points of relax($\mathcal{P}$) and conv($\mathcal{P}$).

Proposition 5 *Let $p = (y, z, x)$ be an extreme point of* relax($\mathcal{P}$)*.*

1. *if $y > 0$, then $\sum_{i \in N} a_i x_i = a_0 + y$, and $x_i, z_i \in \{0, 1\}$ for all $i \in N$;*
2. *if $1 > x_k > 0$ for some $k \in N$, then $y = 0$, $x_i, z_i \in \{0, 1\}$ for all $i \in N \setminus \{k\}$ and $z_k \in \{x_k, 1\}$.*

Proof First note that if the capacity inequality is not tight, then $y = 0$ as the non-negativity constraint is the only other constraint that variable y appears in. Similarly, at least one of $1 \geq z_i$ or $z_i \geq x_i$ has to hold as equality for all $i \in N$.

Assume that $y > 0$ and $1 > x_i > 0$ for some $i \in N$. Let $p^+ = p + (\epsilon a_i, \epsilon' e_i, \epsilon e_i)$ and $p^- = p - (\epsilon a_i, \epsilon' e_i, \epsilon e_i)$ where $\epsilon' = \epsilon$ if $z_i \neq 1$ and $\epsilon' = 0$ otherwise. Notice that for some small $\epsilon > 0$, we have $p^+, p^- \in \text{relax}(\mathcal{P})$ and therefore p can not be an extreme point.

If $1 > x_i, x_k > 0$ for distinct $i, k \in N$, then it is possible to construct two points by simultaneously perturbing x_i and x_k that give the point p as a convex combination. ∎

Based on this observation, we have the following characterization of the extreme points of $\text{relax}(\mathcal{P})$.

Corollary 6 *The point (y, z, x) is an extreme point of $\text{relax}(\mathcal{P})$ if and only if one of the following two cases holds:*

1. *There exist $S \subseteq T \subseteq N$ and $k \in S$ such that $a_k \geq \lambda = a(S) - a_0 > 0$ and*

$$x_i = \begin{cases} 1 & \text{if } i \in S \setminus \{k\} \\ 0 & \text{otherwise} \end{cases}, \quad z_i = \begin{cases} 1 & \text{if } i \in T \setminus \{k\} \\ 0 & \text{otherwise} \end{cases}, \quad y = 0, \text{ and}$$

   either $x_k = z_k = 1 - \lambda/a_k$, or $x_k = 1 - \lambda/a_k$ and $z_k = 1$.

2. *There exist $S \subseteq T \subseteq N$ such that*

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}, \quad z_i = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}, \quad y = \max\{\lambda, 0\}.$$

In Sections 3 and 4 we present valid inequalities that cut off fractional extreme points of $\text{relax}(\mathcal{P})$. We next identify basic properties of the extreme points of $\text{conv}(\mathcal{P})$.

Proposition 7 *Let $p = (y, z, x)$ be an extreme point of $\text{conv}(\mathcal{P})$. If $1 > x_k > 0$ for some $k \in N$, then*

1. *$\sum_{i \in N} a_i x_i = a_0 + y$, and $x_i \in \{0, 1\}$ for all $i \in N \setminus \{k\}$;*
2. *if $y > 0$, then either $a_k x_k < 1$ or $a_k x_k > a_k - 1$.*

Proof If $1 > x_k > 0$ and the capacity inequality is not tight, it is easy to construct two points in $\text{conv}(\mathcal{P})$ by increasing and decreasing x_k. Using a similar argument, if $1 > x_i > 0$ for $i \neq k$, then point p can not be extreme.

Assume $y > 0$, $a_k x_k \geq 1$ and $a_k(1 - x_k) \geq 1$. Let $p^+ = p + (1, \mathbf{0}, \epsilon e_k)$ and $p^- = p - (1, \mathbf{0}, \epsilon e_k)$ where $\epsilon = 1/a_k$. Clearly $p = p^+/2 + p^-/2$ and therefore p cannot be extreme. ∎

Based on this observation, we have the following characterization of the extreme points of $\text{conv}(\mathcal{P})$.

Corollary 8 *The point (y,z,x) be an extreme point of* $\mathrm{conv}(\mathcal{P})$ *if and only if one of the following three cases holds:*

1. *There exist $S \subseteq T \subseteq N$ and $k \in S$ such that $\lambda = a(S) - a_0 > 0$ and $a_k \geq \rho$, where $\rho = \lambda - \lfloor \lambda \rfloor$, and*

$$x_i = \begin{cases} 1 & \text{if } i \in S \setminus \{k\} \\ 1-\rho/a_k & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}, \quad z_i = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}, \quad y = \lfloor \lambda \rfloor.$$

2. *There exist $S \subseteq T \subseteq N$ and $k \in N \setminus S$ such that $\lambda > 0$, $a_k \geq 1-\rho$ and*

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ (1-\rho)/a_k & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}, \quad z_i = \begin{cases} 1 & \text{if } i \in T \cup \{k\} \\ 0 & \text{otherwise} \end{cases}, \quad y = \lceil \lambda \rceil.$$

3. *There exist $S \subseteq T \subseteq N$ such that*

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}, \quad z_i = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}, \quad y = \max\{\lceil \lambda \rceil, 0\}.$$

We next present some basic results on the facets of $\mathrm{conv}(\mathcal{P})$.

Proposition 9 *Trivial facets of* $\mathrm{conv}(\mathcal{P})$*.*

1. *Inequalities $0 \leq x_k$, $x_k \leq z_k$, $z_k \leq 1$ for all $k \in N$ are facet-defining for* $\mathrm{conv}(\mathcal{P})$*.*
2. *Inequality $0 \leq y$ is facet-defining for* $\mathrm{conv}(\mathcal{P})$ *if and only if $a_0 > 0$.*
3. *The capacity inequality* (1) *is facet-defining for* $\mathrm{conv}(\mathcal{P})$ *if and only if*

   (i) $a(N) - a_0 \geq \max\{1, \max_{i\in N} a_i\}$ *if* $a_0 > 0$,

   (ii) $a(N) > 1$ *if* $a_0 = 0$.

Proof For each inequality we give $2n+1$ affinely independent points (y,z,x) on the respective face of $\mathrm{conv}(\mathcal{P})$.

1. $0 \leq x_k$: $(\hat{a}, \mathbf{0}, \mathbf{0}); (\hat{a}+1, \mathbf{0}, \mathbf{0}); (\hat{a}, e_i, \mathbf{0})$ for $i \in N$; $(\hat{a}, e_i, e_i)$ for $i \in N \setminus \{k\}$.

   $x_k \leq z_k$: $(\hat{a}, \mathbf{0}, \mathbf{0}); (\hat{a}+1, \mathbf{0}, \mathbf{0}); (\hat{a}, e_k, e_k)$; $(\hat{a}, e_i, \mathbf{0}), (\hat{a}, e_i, e_i)$ for $i \in N \setminus \{k\}$.

   $z_k \leq 1$: $(\hat{a}, e_k, \mathbf{0}); (\hat{a}+1, e_k, \mathbf{0}); (\hat{a}, e_k, e_k)$; $(\hat{a}, e_k+e_i, \mathbf{0}), (\hat{a}, e_k+e_i, e_i)$ for $i \in N \setminus \{k\}$.
2. $(0, \mathbf{0}, \mathbf{0}); (0, e_i, \mathbf{0}), (0, e_i, \epsilon e_i)$ for $i \in N$, where $\epsilon > 0$, small.
3. (i) If $a(N) \leq a_0$, then (1) is implied by the bounds on the variables. If $0 < a(N) - a_0 < \max\{1, \max_{i\in N} a_i\}$, then inequality (11) with $S = N$, $z \leq \mathbf{1}$ and $y \geq 0$ imply inequality (1). For the other direction, consider following points:

   $(0, \mathbf{1}, \sum_{i\in N} \frac{a_0}{a(N)} e_i)$; $(1, \mathbf{1}, \sum_{i\in N} \frac{a_0+1}{a(N)} e_i)$;

$(0, \mathbf{1}, \sum_{i \in N \setminus \{k\}} \frac{a_0}{a(N \setminus \{k\})} e_i)$ and $(0, \mathbf{1} - e_k, \sum_{i \in N \setminus \{k\}} \frac{a_0}{a(N \setminus \{k\})} e_i)$ for $k \in N$.

(ii) If $a(N) \leq 1$ and $a_0 = 0$, then inequalities (13) imply (1). For the other direction, consider following points:

$(0, \mathbf{0}, \mathbf{0})$; $(0, e_k, 0)$, $(1, \mathbf{1}, \sum_{i \in N \setminus \{k\}} \frac{1-\delta_k}{a(N)} e_i + \frac{1+\epsilon}{a(N)} e_k)$ for $k \in N$, where $\delta_k = \frac{\epsilon a_k}{a(N \setminus \{k\})}$, $\epsilon > 0$, small if $|N| > 1$ and $\delta_k = \epsilon = 0$ if $|N| = 1$. ∎

Proposition 10 *For all non-trivial facet-defining inequalities* $\alpha x - \beta z - \gamma y \leq \delta$ *of* $\operatorname{conv}(\mathcal{P})$ *the following statements are true:*

1. $\delta \geq 0$, $\beta \geq 0$, *and* $\gamma > 0$*;*
2. $\beta_i + \lceil a_i \rceil \gamma \geq \alpha_i \geq \beta_i$ *for all* $i \in N$*;*
3. $\exists i \in N$ *such that* $\alpha_i > \beta_i$*;*
4. $a(T) > a_0$ *for* $T = \{i \in N : \alpha_i > 0\}$.

Proof As $(0, \mathbf{0}, \mathbf{0}) \in \mathcal{P}$, we have $\delta \geq 0$. Let F be the set of points in $\mathcal{P}$ that satisfy the inequality as equality. As the inequality is assumed to be different from $z_i \leq 1$, there is a point $p = (y, z, x) \in F$ with $z_i = 0$. Since setting $z_i = 1$ does not violate feasibility, $\beta_i \geq 0$ for all $i \in N$. Furthermore, the value of y can be increased without violating feasibility, and therefore $\gamma \geq 0$. Similarly, increasing z_i and x_i to one and y by $\lceil a_i \rceil$ gives another point in $\mathcal{P}$, therefore $\alpha_i \leq \beta_i + \lceil a_i \rceil \gamma$ for all $i \in N$.

On the other hand, as the inequality is assumed to be different from $z_i \geq 0$, there is point $p' = (y, z, x) \in F$ with $z_i = 1$. Decreasing z_i and x_i to zero gives another feasible point. Therefore $\beta_i \leq \alpha_i x_i \leq \alpha_i$ for all $i \in N$. If $\alpha = \beta$, then the inequality is implied by $\alpha(x - z \leq 0) + \gamma(-y \leq 0)$ and $0 \leq \delta$; hence $\alpha \neq \beta$. But now, if $\gamma = 0$, we have $\alpha = \beta$ by $\beta_i + \lceil a_i \rceil \gamma \geq \alpha_i \geq \beta_i$.

Finally, suppose $a(T) \leq a_0$ for $T = \{i \in N : \alpha_i > 0\}$ and consider a tight point $p'' = (y, z, x)$ with $y > 0$. As $a(T) \leq a_0$, the point $(0, z, x)$ is also feasible implying $\gamma \leq 0$, a contradiction. ∎

3 Capacity flow cover inequalities

For $S \subseteq N$ such that $\lambda = a(S) - a_0 > 0$, let us relax the capacity inequality as

$$a_0 + y \quad \geq \quad \sum_{i \in S} a_i x_i \quad = \quad \sum_{i \in S} a_i [1 - (1 - z_i) - (z_i - x_i)]$$

or equivalently

$$y + \sum_{i \in S} a_i (1 - z_i) + \sum_{i \in S} a_i (z_i - x_i) \quad \geq \quad \lambda. \tag{10}$$

Then by Lemma 3, the *capacity flow cover inequality*

$$\min\{1, \lambda\} y + \sum_{i \in S} \min\{a_i, \lambda\} (1 - z_i) + \sum_{i \in S} a_i (z_i - x_i) \quad \geq \quad \lambda \tag{11}$$

is valid for $\mathcal{P}$. Note that inequality (11) can also be written as

$$\sum_{i\in S} a_i x_i + \sum_{i\in S}(a_i-\lambda)^+(1-z_i) \quad \leq \quad a_0 + \min\{1,\lambda\}y. \tag{12}$$

We first note that (12) is a generalization of the well-known flow cover inequality (3).

Remark 11 *Observe that for the single node fixed-charge flow set $\mathcal{T}$, the capacity flow cover inequality* (12) *reduces to the flow cover inequality* (3) *by letting $y = 0$.*

We next identify the conditions under which the capacity flow cover inequality (12) is facet-defining for $\text{conv}(\mathcal{P})$. We study the cases when $a_0 = 0$ and $a_0 > 0$ separately as the polyhedral structure of $\text{conv}(\mathcal{P})$ depends on a_0.

Proposition 12 *Assume $a_0 > 0$. The capacity flow cover inequality* (11) *is facet-defining for* $\text{conv}(\mathcal{P})$ *if and only if one of the following three conditions hold:* (*i*) $\lambda < \max_{i\in S}\{a_i\}$, *or* (*ii*) $\lambda < 1$, *or* (*iii*) $S = N$.

Proof *Necessity.* If $\lambda \geq \max_{i\in S}\{a_i\}$, $\lambda \geq 1$, and $S \neq N$, then inequality (11) becomes $\sum_{i\in S} a_i x_i \leq a_0 + y$, which is implied by the capacity inequality (1) and $x_i \geq 0$, $i \in N \setminus S$.

Sufficiency. For a given $S \subseteq N$, we first write inequality (11) in canonical form as follows:

$$\min\{1,\lambda\}y + \sum_{i\in S\setminus S'}(a_i-\lambda)z_i - \sum_{i\in S} a_i x_i \quad \geq \quad \sum_{i\in S\setminus S'} a_i - a_0 - \lambda|S\setminus S'|,$$

where $S' = \{i \in S \;:\; a_i \leq \lambda\}$. Let F be the face induced by inequality (11) and let $\alpha y + \beta z + \gamma x = \delta$ be satisfied by all points in F. We will show that any such equality is a multiple of the inequality that induces the face by generating pairs of points $p' = (y', z', x')$, and $p'' = (y'', z'', x'')$ and using the fact that $\alpha(y'-y'') + \beta(z'-z'') + \gamma(x'-x'') = 0$ if both points are in F. We first construct a point $p^1 = (y^1, z^1, x^1) \in F$, where

$$y^1 = 0, \quad z_i^1 = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}, \quad x_i^1 = \begin{cases} 1 - \frac{\lambda}{a(S)} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}.$$

Since $a_0, \lambda > 0$ by assumption, $a(S) > \lambda > 0$, and therefore $1 > x_k^1 > 0$ for all $k \in S$.

Let $s_k = (0, \mathbf{0}, (1/a_k)e_k)$. Since $1 > x_k^1 > 0$, there exists a small enough $\epsilon > 0$, such that $p^1 + \epsilon s_k - \epsilon s_j \in F$ for all $j, k \in S$. Therefore, $\gamma_k = -a_k\bar{\sigma}$ for all $k \in S$ for some fixed constant $\bar{\sigma} > 0$.

Next, for all $k \in S \setminus S'$, we construct a point $q^k = (y^k, z^k, x^k)$, where

$$y^k = 0, \quad z_i^k = \begin{cases} 1 & \text{if } i \in S \setminus k \\ 0 & \text{otherwise} \end{cases}, \quad x_i^k = \begin{cases} 1 & \text{if } i \in S \setminus k \\ 0 & \text{otherwise} \end{cases}.$$

Using $p^1, q^k \in F$, we see that $\beta_k + (1-\lambda/a(S))\gamma_k - \sum_{i\in S\setminus\{k\}}(\lambda/a(S))\gamma_i = 0$. Substituting $\gamma_k = -a_k\bar{\sigma}$ for all $k \in S$ and simplifying the equation gives $\beta_k = (a_k - \lambda)\bar{\sigma}$ for all $k \in S \setminus S'$ as desired.

Next, for all $k \in S'$ we construct a point $q^k = (y^k, z^k, x^k)$, where

$$y^k = 0, \quad z_i^k = \begin{cases} 1 & \text{if } i \in S \setminus k \\ 0 & \text{otherwise} \end{cases}, \quad x_i^k = \begin{cases} 1 - \frac{\lambda - a_k}{a(S) - a_k} & \text{if } i \in S \setminus k \\ 0 & \text{otherwise} \end{cases}.$$

Note that $a(S \setminus \{k\}) \geq \lambda - a_k \geq 0$ for $k \in S'$. Since $\gamma x = \lambda$ for both $q^k, p^1 \in F$, we have $\beta_k = 0$ for all $k \in S'$.

Finally, we construct a point $p^2 = (y^2, z^2, x^2) \in F$, where

$$y^2 = 1, \quad z_i^2 = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}, \quad x_i^2 = \begin{cases} 1 - \frac{\lambda}{a(S)} + \frac{\min\{1,\lambda\}}{a(S)} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}.$$

Using $p^1, p^2 \in F$, we conclude that $\alpha = \min\{1, \lambda\}\bar{\sigma}$ as desired.

If $S \neq S'$, then for all $k \in S \setminus S'$ the slack of the capacity inequality (1) for point q^k is $s = a_o + y - \sum_{i \in N} a_i x_i = a_k - \lambda > 0$. If, on the other hand, $S = S'$, then by assumption, we have $1 > \lambda$, and for p^2 (1) has a slack of $s = 1 - \lambda$. In either case, we have a point $p \in P$ with slack and we can perturb it to obtain points $p + t_i^1, p + t_i^2 \in F$, where $t_i^1 = (0, 1, 0)$ and $t_i^2 = (0, 1, (s/a_i)e_i)$, to show $\beta_i = \gamma_i = 0$ for all $i \notin S$.

Using p^1, for instance, we also have $\delta = \sum_{i \in S \setminus S'} a_i - a_0 - \lambda |S \setminus S'|$. We have therefore shown that inequality $\alpha y + \beta z + \gamma x = \delta$ is a multiple of the original inequality and the points defined above are affinely independent. As $(0, \mathbf{1}, \mathbf{0}) \in \mathcal{P} \setminus F$, F is a maximal proper face of $\text{conv}(\mathcal{P})$. ∎

Therefore, when $a_0 > 0$ the capacity flow cover inequality (11) is facet-defining under mild conditions. When $a_0 = 0$, however, inequality (11) defines a facet only when it reduces to the capacity inequality (1), or to the surrogate variable upper bound inequality (13).

Proposition 13 *Assume $a_0 = 0$. The capacity flow cover inequality* (11) *is facet-defining for* $\text{conv}(\mathcal{P})$ *if and only if one of the following two conditions hold: (i) $|S| = 1$ and $a(S) < 1$, or (ii) $S = N$ and $a(S) > 1$.*

Proof *Necessity.* In this case $\lambda = a(S)$. If $a(S) < 1$ and $|S| > 1$, inequality (11) becomes $\sum_{i \in S} a_i x_i \leq a(S) y$, which is implied by individual inequalities $a_i(x_i - y) \leq 0$, $i \in S$. If $a(S) > 1$ and $S \neq N$, inequality (11) is implied by the capacity inequality (1) and $x_i \geq 0$, $i \in N \setminus S$.

Sufficiency. In the first case, inequality (11) reduces to $x_k \leq y$, $k \in N$. The following affinely independent points are clearly on the face: $(0, \mathbf{0}, \mathbf{0})$; $(0, e_i, \mathbf{0})$ for $i \in N$; $(1, \mathbf{1}, e_k)$; $(1, \mathbf{1}, \epsilon e_i + e_k)$ for $i \in N \setminus \{k\}$, where $0 < \epsilon \leq 1 - a_k$. In the second case, inequality (11) is the capacity inequality (1) and the result follows from Proposition 9. ∎

Corollary 14 *If $a_0 = 0$, then the surrogate variable upper bound inequality*

$$x_i \ \leq \ y \tag{13}$$

is facet-defining for $\text{conv}(\mathcal{P})$ *if and only if $a_i < 1$.*

We next identify the fractional extreme points of relax($\mathcal{P}$) that can be cut off using a capacity flow cover inequality.

Proposition 15 *Every fractional extreme point* (x, y, z) *of* relax($\mathcal{P}$) *with* $y < 1$ *is cut off by a capacity flow cover inequality (11).*

Proof Let $p = (x, y, z)$ be a fractional extreme point of relax($\mathcal{P}$). By Corollary 6, if $y = 0$, there exist $S \subseteq N$ and $k \in S$ such that $a_k > \lambda > 0$. Then inequality (11) with such S is violated by p as

$$\sum_{i \in S} a_i x_i + \sum_{i \in S} (a_i - \lambda)^+ (1 - z_i) = a_0 + (a_k - \lambda)\lambda / a_k > a_0 = a_0 + \min\{1, \lambda\} y.$$

On the other hand if $y > 0$, then there exist $S \subseteq N$ such that $y = \lambda \notin \mathbb{Z}$. If $\lambda < 1$, then inequality (11) with such S is violated by p as

$$\sum_{i \in S} a_i x_i + \sum_{i \in S} (a_i - \lambda)^+ (1 - z_i) = a_0 + \lambda > a_0 + \lambda^2 = a_0 + \min\{1, \lambda\} y.$$

∎

3.1 Lifting with integer capacity variable

We next describe valid inequalities obtained by first fixing the value of the y variable, and then lifting the associated basic flow cover inequality. If the y variable is fixed to $v \in \mathbb{Z}_+$, then the resulting lifted inequality has the form

$$\sum_{i \in S} a_i x_i + \sum_{i \in S} (a_i - r)^+ (1 - z_i) \leq a_0 + v + \alpha (y - v), \tag{14}$$

where $S \subseteq N$ and $r = a(S) - v - a_0 > 0$.

Proposition 16 *Let* $S \subseteq N$ *be such that* $r = a(S) - v - a_0 > 0$ *for* $v \in \mathbb{Z}_+$. *Then the lifted capacity flow cover inequality (14) is valid for* $\mathcal{P}$ *if and only if*

1. $-\Phi(-1) \leq \alpha$ *if* $v = 0$;
2. $-\Phi(-1) \leq \alpha \leq \Phi(1)$ *if* $v > 0$, *where* Φ *is defined as in* (15).

Moreover, (14) *defines a facet of* conv($\mathcal{P}$) *if* α *equals one of its bounds and* $r < \max_{i \in S} a_i$.

Proof Inequality (14) is the flow cover inequality (3) for the restriction $\mathcal{P}(v) = \{(x, y, z) \in \mathcal{P} : y = v\}$ of $\mathcal{P}$ and it is valid for $\mathcal{P}(v)$ for any α. Then (14) is valid for $\mathcal{P}$ if and only if $\underline{\alpha} \leq \alpha \leq \overline{\alpha}$, where

$$\underline{\alpha} = \max \left\{ \frac{\sum_{i \in S} a_i x_i + \sum_{i \in S} (a_i - r)^+ (1 - z_i) - a_0 - v}{y - v} : (x, y, z) \in \mathcal{P}, y > v \right\}$$

and

$$\overline{\alpha} = \min\left\{\frac{a_0 + v - \sum_{i\in S} a_i x_i - \sum_{i\in S}(a_i - r)^+(1 - z_i)}{v - y} : (x, y, z) \in \mathcal{P}, y < v\right\},$$

with $\overline{\alpha} = \infty$ if $v = 0$.

Without loss of generality suppose $S = \{1, 2, \ldots, |S|\}$ with $a_1 \geq a_2 \geq \cdots \geq a_{|S|}$. Let $p = \max\{i \in S : a_i > \lambda\}$, $A_i = \sum_{k=1}^{i} a_k$ for $i \in \{1, 2, \ldots, p\}$, and $A_0 = 0$. It is shown in [12] that the lifting function

$$\Phi(a) = \min\left\{a_0 + v - \sum_{i\in S} a_i x_i + \sum_{i\in S}(a_i - r)^+(1 - z_i) : (x, y, z) \in \mathcal{P}(v - a)\right\}$$

can be stated as

$$\Phi(a) = \begin{cases} \max\{-r, a\} & \text{if } a \leq 0, \\ ir & \text{if } \; A_i \leq a \leq A_{i+1} - r, \\ ir + (a - A_i) & \text{if } \; A_i - r \leq a \leq A_i, \\ pr + (a - A_p) & \text{if } \; A_p - r \leq a_0 + v, \\ +\infty & \text{if } \; a > a_0 + v, \end{cases} \tag{15}$$

where $i \in \{0, 1, \ldots, p-1\}$ and that Φ is superadditive on $[0, a_0+v]$ and $(-\infty, 0]$, separately.

Then for $v > 0$ we have $\overline{\alpha} = \min_{a\in\mathbb{Z}, a>0} \frac{\Phi(a)}{a} = \Phi(1)$, where the last equation follows from superadditivity of Φ over $[0, a_0 + v]$. Similarly, $\underline{\alpha} = -\min_{a\in\mathbb{Z}, a<0} \frac{\Phi(a)}{a} = -\Phi(-1)$. Finally, if $r < \max_{i\in S} a_i$, inequality (14) is facet-defining for $\text{conv}(\mathcal{P}(v))$ and in addition if $\alpha \in \{\underline{\alpha}, \overline{\alpha}\} < \infty$, the lifting is exact; hence, (14) defines a facet for $\text{conv}(\mathcal{P})$ ■

Note that $-\Phi(-1) = \min\{1, r\}$. Therefore, if we chose $v = 0$, then

$$\sum_{i\in S} a_i x_i + \sum_{i\in S}(a_i - r)^+(1 - z_i) \leq a_0 + \min\{1, r\}y \tag{16}$$

is valid for $\text{conv}(\mathcal{P})$. Notice that this inequality is identical to the capacity flow cover inequality (11). Also notice that the facet sufficient condition of Proposition 16 is more restrictive than the condition of Proposition 12. Therefore, when $v = 0$, the lifted inequalities do not lead to new inequalities.

If $v > 0$, however, the resulting inequalities are new. First observe that $\min\{1, r\} \leq \Phi(1)$ if and only if $\max_{i\in S} a_i \leq 1$. So if $v > 0$, then

$$\sum_{i\in S} a_i x_i + \sum_{i\in S}(a_i - r)^+(1 - z_i) \leq a_0 + v + r(y - v) \tag{17}$$

as well as

$$\sum_{i\in S} a_i x_i + \sum_{i\in S}(a_i - r)^+(1 - z_i) \leq a_0 + v + \Phi(1)(y - v) \tag{18}$$

are valid for $\text{conv}(\mathcal{P})$ provided that $\max_{i\in S} a_i \leq 1$. Inequalities (17) and (18) are facet-defining provided that $r < \max_{i\in S} a_i$. They are distinct if and only if $A_2 - r < 1$.

Recall that every fractional extreme point (x, y, z) of $\text{relax}(\mathcal{P})$ with $y < 1$ is cut off by a capacity flow cover inequality (11). We next show that some of the remaining ones are cut off by the lifted capacity flow cover inequality (17).

Proposition 17 *Every fractional extreme point* (x, y, z) *of* relax($\mathcal{P}$) *with* $y \geq 1$ *is cut off by a lifted capacity flow cover inequality (17) with* $v = \lfloor y \rfloor$ *and* $S = \{i \in N : x_i > 0\}$ *provided that* $a_i \leq 1$ *for all* $i \in S$.

Proof By Corollary 6, if $y \geq 1$ for a fractional extreme point, then (i) $y \notin \mathbb{Z}_+$, (ii) $\sum_{i \in N} a_i x_i = a_0 + y$ and (iii) $x_i, z_i = 1 \in \{0, 1\}$ for all $i \in N$. Therefore,

$$\sum_{i \in S} a_i x_i + \sum_{i \in S} (a_i - r)^+ (1 - z_i) = a_0 + y = a_0 + v + r > a_0 + v + r^2 = a_0 + v + r(y - v).$$

∎

3.2 Augmented capacity flow cover inequalities

We can augment inequality (11) to obtain new valid inequalities that have non-zero coefficient for variables $(x_i, z_i), i \in N \setminus S$. Let $S \subseteq N$ be such that $\lambda = a(S) - a_0 > 0$ and $T \subseteq N \setminus S$. Now let us relax the capacity constraint (1) as

$$a_0 + y \;\; \geq \;\; \sum_{i \in S} a_i [1 - (1 - z_i) - (z_i - x_i)] \; + \; \sum_{i \in T} a_i [z_i - (z_i - x_i)]$$

or equivalently

$$\left[\sum_{i \in S \cup T} a_i (z_i - x_i) \right] + \left[y + \sum_{i \in S} a_i (1 - z_i) \right] - \left[\sum_{i \in T} a_i z_i \right] \;\; \geq \;\; \lambda. \tag{19}$$

Then by Lemma 3, for $\alpha = \max\{1, \bar{a}, \lambda\}$ and $\bar{a} = \max_{i \in S}\{a_i\}$, the following inequality:

$$\sum_{i \in S \cup T} a_i (z_i - x_i) + \min\{1, \lambda\} y + \sum_{i \in S} \min\{a_i, \lambda\} (1 - z_i) - \sum_{i \in T} \phi_\alpha (a_i, \lambda) z_i \;\; \geq \;\; \lambda \tag{20}$$

is valid for $\mathcal{P}$. Inequality (20) can also be written as

$$\sum_{i \in S \cup T} a_i x_i + \sum_{i \in S} (a_i - \lambda)^+ (1 - z_i) - \sum_{i \in T} (a_i - \phi_\alpha (a_i, \lambda)) z_i \leq a_0 + \min\{1, \lambda\} y. \tag{21}$$

The coefficients of z_i, $i \in T$ approximate the ones obtained through sequence independent lifting functions in [11]. Under conditions stated in the Proposition 18 they are equal.

Proposition 18 *Augmented capacity flow cover inequality* (21) *is facet-defining for* conv($\mathcal{P}$) *provided that (1)* $a_0 > 0$*, (2)* $\lambda < \alpha$*, and (3)* $\alpha - \lambda < a_i \leq \alpha - (\lambda - a(\bar{S}))^+$ *for all* $i \in T$*, where* $\bar{S} = \{i \in S : a_i \leq \lambda\}$.

Proof Inequality (21) with $T = \emptyset$ is facet-defining for conv($\mathcal{P}$) under conditions (1) and (2) by Proposition 12. The MIR function $\phi_\alpha(., \lambda)$ is superadditive and equals the lifting function described Theorem 10 of Gu et al. [11] under condition (3). ∎

On the other hand, writing inequality (19) as

$$\left[\sum_{i\in S\cup T} a_i(z_i - x_i)\right] + \left[\sum_{i\in S} a_i(1-z_i)\right] - \left[-y + \sum_{i\in T} a_i z_i\right] \quad \geq \quad \lambda \tag{22}$$

and applying Lemma 3 for $\alpha = \max\{\bar{a}, \lambda\}$, we obtain the following valid inequality:

$$\sum_{i\in S\cup T} a_i(z_i \;-\; x_i) \;+\; \sum_{i\in S} \min\{a_i, \lambda\}(1 \;-\; z_i) \;-\; \phi_\alpha(-1,\lambda)y \;-\; \sum_{i\in T} \phi_\alpha(a_i,\lambda)z_i \quad \geq \quad \lambda$$

or equivalently

$$\sum_{i\in S\cup T} a_i x_i + \sum_{i\in S}(a_i - \lambda)^+(1-z_i) - \sum_{i\in T}(a_i - \phi_\alpha(a_i,\lambda))z_i \leq a_0 - \phi_\alpha(-1,\lambda)y \tag{23}$$

for $\mathcal{P}$.

Proposition 19 *Augmented capacity flow cover inequality* (23) *is facet-defining for* $\operatorname{conv}(\mathcal{P})$ *provided that (1)* $\lambda < \bar{a}$*, (2)* $1 \leq \bar{a}$*, and (3)* $\bar{a} - \lambda < a_i \leq A_2 - \lambda$ *for all* $i \in T$*.*

Proof Inequality (23) with $T = \emptyset$ and $y = 0$ is facet-defining for $\operatorname{conv}(\mathcal{T})$ under conditions (1) and (2) [20]. The MIR function $\phi_{\bar{a}}(.,\lambda)$ is superadditive and equals the lifting function Φ described in (15) under condition (3). ∎

Observe that inequalities (21) and (23) are equal if $\lambda < \bar{a}$ and $1 \leq \bar{a}$; however, facet condition of Proposition 19 is less restrictive in this case.

4 Mixed integer rounding inequalities

In this section we describe a family of facet-defining inequalities that cut off all fractional extreme points of $\operatorname{relax}(\mathcal{P})$. In addition all extreme points of $\operatorname{conv}(\mathcal{P})$ are extreme points of the polyhedron obtained by adding these inequalities to $\operatorname{relax}(\mathcal{P})$.

Let $S \subseteq N$ such that $\lambda = a(S) - a_0 > 0$. Relaxing the capacity flow cover inequality (11) by skipping the coefficient reduction step for $i \in S' \subseteq S$ and increasing the coefficients of the $(1 - z_i)$ terms for $i \in S \setminus S'$, we obtain

$$\left[y + \sum_{i\in S\setminus S'} \min\{\eta, \lceil a_i \rceil\}(1-z_i) + \sum_{i\in S'} \lfloor a_i \rfloor (1-z_i)\right] + \left[\sum_{i\in S} a_i(z_i - x_i) + \sum_{i\in S'} r_i(1-z_i)\right] \geq \lambda,$$

where $\eta = \lceil \lambda \rceil$ and $r_i = a_i - \lfloor a_i \rfloor$. Now applying to this inequality the MIR procedure gives the following valid inequality:

$$\begin{aligned} \sum_{i\in S} a_i(z_i - x_i) \quad &+ \quad \sum_{i\in S'} r_i(1-z_i) \\ &\geq \quad \rho\left(\eta - y - \sum_{i\in S\setminus S'} \min\{\eta, \lceil a_i \rceil\}(1-z_i) - \sum_{i\in S'} \lfloor a_i \rfloor (1-z_i)\right), \end{aligned} \tag{24}$$

where $\rho = \lambda - \lfloor \lambda \rfloor$. Notice that as the capacity flow cover inequality (11) is itself obtained by the MIR procedure, inequality (24) is the result of two iterative applications of the MIR procedure.

Remark 20 *For the splittable flow arc set $\mathcal{Q}$, inequality* (24) *reduces to residual capacity inequality* (2) *by letting $z = \mathbf{1}$.*

Proposition 21 *MIR inequality* (24) *is facet-defining for* $\text{conv}(\mathcal{P})$ *if and only if*

1. *$\eta > \lambda$, i.e., $\lambda \notin \mathbb{Z}$,*
2. *$a(S) > \rho$, i.e., either $a_0 > 0$ or $\eta > 1$,*
3. *$S' = \{i \in S \,:\, a_i < \lambda$ and $r_i < \rho\}$.*

Proof *Necessity.* 1. If $\eta = \lambda$, then (24) is implied by the capacity inequality (1) and the bounds. 2. If $a_0 = 0$ and $\eta = 1 \geq a_i$, then $a_i = r_i \leq \rho$ for all $i \in S$. Thus unless $S' = S$, inequality is weak. For $S' = S$, inequality becomes $\sum_{i \in S} a_i(1 - x_i) \geq a(S)(1 - y)$, which is implied by individual capacity flow cover inequalities (11) $x_i \leq y_i$, $i \in S$. 3. Let $S^* = \{i \in S \,:\, a_i < \lambda \text{ and } r_i < \rho\}$. If $S' \neq S^*$, then replacing S' with S^* gives a stronger inequality since $r_i + \rho\lfloor a_i \rfloor < \rho \min\{\eta, \lceil a_i \rceil\}$ for $i \in S^*$.

Sufficiency. For a given $S \subseteq N$, we first write inequality (24) in canonical form as follows:

$$\rho y + \sum_{i \in S \setminus S'} \Big(a_i - \rho \min\{\eta, \lceil a_i \rceil\}\Big) z_i + \sum_{i \in S'} \Big(a_i - r_i - \rho \lfloor a_i \rfloor\Big) z_i - \sum_{i \in S} a_i x_i$$
$$\geq \rho \left(\eta - \sum_{i \in S \setminus S'} \min\{\eta, \lceil a_i \rceil\} - \sum_{i \in S'} \lfloor a_i \rfloor \right) - \sum_{i \in S'} r_i$$

Let F be the face induced by inequality (24) and let $\alpha y + \beta z + \gamma x = \delta$ be satisfied by all points in F. We start with constructing a point $p^1 = (y^1, z^1, x^1) \in F$ and show that F is not empty:

$$y^1 = \eta(S), \quad z_i^1 = x_i^1 = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases},$$

where $\eta(S) = \lceil a(S) - a_0 \rceil$. Let $t_k = (0, e_k, 0)$ and $s_k = (0, e_k, \epsilon e_k)$, where $\epsilon = (1 - \rho)/a_k$. Since for all $k \in N \setminus S$, both p^1 and $p^1 + t_k \in F$, we have $\beta_k = 0$ for all $k \in N \setminus S$. Similarly, $p^1 + t_k$ and $p^1 + s_k \in F$ implies that $\gamma_k = 0$ for all $k \in N \setminus S$.

Next, we construct $p^2 = (y^2, z^2, x^2) \in F$, where

$$y^2 = \eta(S) - 1, \quad z_i^2 = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}, \quad x_i^2 = \begin{cases} 1 - \frac{\rho}{a(S)} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}.$$

Note that $1 > x_i^2 > 0$ for all $i \in S$. Let $t_k = (0, \mathbf{0}, (1/a_k)e_k)$. For each $i, j \in S$ and for a small enough $\epsilon > 0$, both p^2 and $p^2 + \epsilon t_i - \epsilon t_j \in F$, and therefore for some $\bar{\sigma} \in R$ we have $\gamma_k = -a_k \bar{\sigma}$ for all $k \in S$. Furthermore, $p^1, p^2 \in F$ implies that $\alpha = \rho\bar{\sigma}$.

We next observe that for any $u, v \in R$, if we let $u = \lceil u \rceil - 1 + r_u$, $v = \lceil v \rceil - 1 + r_v$ and $u + v = \lceil u + v \rceil - 1 + r_{uv}$, with $1 \geq r_u, r_v, r_{uv} > 0$, we have

(i) either $r_u + r_v > 1 \Leftrightarrow \lceil u+v \rceil = \lceil u \rceil + \lceil v \rceil \Leftrightarrow r_{uv} = r_u + r_v - 1 \leq \min\{r_u, r_v\}$, or

(ii) or $r_u + r_v \leq 1 \Leftrightarrow \lceil u+v \rceil = \lceil u \rceil + \lceil v \rceil - 1 \Leftrightarrow r_{uv} = r_u + r_v > \max\{r_u, r_v\}$.

Let $l_k = (\min\{\eta(S), \lceil a_k \rceil\}, e_k, e_k)$. Since $r_k \geq \rho$ for all $k \in S \setminus S'$, we have $\eta(S \setminus k) = \eta(S) - \lceil a_k \rceil \leq \eta(S) - \min\{\eta(S), \lceil a_k \rceil\}$ and therefore $p^1 - l_k \in P$. Using $p^1, p^1 - l_k \in F$, we obtain the equation $\min\{\eta(S), \lceil a_k \rceil\}\alpha + \beta_k + \gamma_k = 0$ implying $\beta_k = \Big(a_k - \rho \min\{\eta(S), \lceil a_k \rceil\}\Big)\bar{\sigma}$ for all $k \in S \setminus S'$.

Finally, for all $k \in S'$ we construct a point $q^k = (y^k, z^k, x^k)$, where

$$y^k = \eta(S) - \lfloor a_k \rfloor - 1, \quad z_i^k = \begin{cases} 1 & \text{if } i \in S \setminus k \\ 0 & \text{otherwise} \end{cases}, \quad x_i^k = \begin{cases} 1 - \frac{\rho - r_k}{a(S \setminus k)} & \text{if } i \in S \setminus k \\ 0 & \text{otherwise} \end{cases}.$$

Note that $r_k < \rho$ for all $k \in S'$ implying $\eta(S \setminus k) = \eta(S) - \lfloor a_k \rfloor$ and $r(S \setminus k) = \rho - r_k$. Therefore,

$$\sum_{i \in S} a_i x_i^k = a(S \setminus k)\Big(1 - \frac{\rho - r_k}{a(S \setminus k)}\Big) = a(S \setminus k) - r(S \setminus k)$$

and $q^k \in P$. Since both $p^2,\ q^k \in F$, we have

$$\begin{aligned} 0 &= \lfloor a_k \rfloor \rho\bar{\sigma} + \beta_k - a(S)\bar{\sigma}\Big(1 - \frac{\rho}{a(S)}\Big) + a(S \setminus k)\bar{\sigma}\Big(1 - \frac{\rho - r_k}{a(S \setminus k)}\Big) \\ &= \lfloor a_k \rfloor \rho\bar{\sigma} + \beta_k - a(S)\bar{\sigma} + \rho\bar{\sigma} + a(S \setminus k)\bar{\sigma} - \rho\bar{\sigma} + r_k\bar{\sigma} \\ &= \lfloor a_k \rfloor \rho\bar{\sigma} + \beta_k - a_k\bar{\sigma} + r_k\bar{\sigma} \end{aligned}$$

implying $\beta_k = \big(a_k - r_k - \rho \lfloor a_k \rfloor\big)\bar{\sigma}$ for all $k \in S'$, as desired.

We have therefore shown that inequality $\alpha y + \beta z + \gamma x = \delta$ is a multiple of the original inequality, and the points defined above are affinely independent. As $(\hat{a}+1, \mathbf{1}, \mathbf{1}) \in \mathcal{P} \setminus F$, F is a maximal proper face of conv$(\mathcal{P})$. ∎

Observe that if $\lambda \leq 1$, we have $\eta = 1$ and $\rho = \lambda$. Then by Proposition 21 facet-defining inequalities (24) satisfy $a_i < \lambda < 1$ for all $i \in S'$, in which case they are equivalent to capacity flow cover inequalities (11). Therefore, inequalities (24) are of particular interest if $\lambda > 1$ as they differ from inequalities (11) in that case.

In addition, remember the lifted capacity flow cover inequality (17) with $v \in \mathbb{Z}_+$ and $r = a(S) - v - a_0 > 0$

$$\sum_{i \in S} a_i x_i + \sum_{i \in S} (a_i - r)^+(1 - z_i) \leq a_0 + v + r(y - v)$$

which is valid and facet defining provided that $r \leq a_i \leq 1$ for all $i \in S$. Notice that, under this condition, (i) $v = \lfloor a(S) - a_0 \rfloor = \eta - 1$, (ii) $r = \rho$, and (iii) $r_i = a_i$ for all $i \in S$. In this case MIR inequality (24) becomes:

$$\sum_{i \in S} a_i(z_i - x_i) + \sum_{i \in S'} a_i(1 - z_i) \geq r(v+1) - ry - r \sum_{i \in S \setminus S'} (1 - z_i)$$

or, equivalently,

$$\sum_{i\in S} a_i + \sum_{i\in S} a_i(z_i - 1) - \sum_{i\in S} a_i x_i + \sum_{i\in S} \min\{r, a_i\}(1 - z_i) \geq rv + r - ry$$

which is identical to inequality (25) as $v + a_0 = r + a(S)$. Therefore, facet defining lifted capacity flow cover inequalities form a subclass of MIR inequalities.

We next show that all fractional extreme points of $\text{relax}(\mathcal{P})$ violate an MIR inequality.

Proposition 22 *Every fractional extreme point of* $\text{relax}(\mathcal{P})$ *is cut off by an MIR inequality (24).*

Proof Let $p = (x, y, z)$ be a fractional extreme point of $\text{relax}(\mathcal{P})$. By Corollary 6, if $y = 0$, there exist $S \subseteq N$ and $k \in S$ such that $a_k > \lambda > 0$. Consider the inequality (24) with such S and $k \in S \setminus S'$ and let rhs denote its right-hand side value for this point. This inequality is violated by p as

$$\sum_{i\in S} a_i(z_i - x_i) + \sum_{i\in S'} r_i(1 - z_i) = 0 < \rho\eta(1 - \lambda/a_k) = \text{rhs}.$$

On the other hand, if $y > 0$, there exist $S \subseteq N$ such that $y = \lambda \notin \mathbb{Z}$. Then inequality (24) with such S is invalid for (x, y, z) as

$$\sum_{i\in S} a_i(z_i - x_i) + \sum_{i\in S'} r_i(1 - z_i) = 0 < \rho(\eta - \lambda) = \text{rhs}.$$

∎

The following proposition complements Proposition 22.

Proposition 23 *If the capacity inequality* (1) *is facet-defining for* $\text{conv}(\mathcal{P})$*, then all extreme points of* $\text{conv}(\mathcal{P})$ *are extreme points of the polyhedron obtained by adding all MIR inequalities* (24) *and surrogate variable upper bound inequalities* (13) *to* $\text{relax}(\mathcal{P})$*.*

Proof Consider the extreme points defined in Corollary 8. Any point in the first case is the intersection of the following $2n+1$ facets: capacity inequality (1), MIR inequality (24) with S, $x_i \geq 0$ for $i \in N \setminus S$, $x_i \leq z_i$ for $i \in S \setminus \{k\}$ ($x_i = z_i = 1$), $z_i \leq 1$ for $i \in T$, $x_i \leq z_i$ for $i \in N \setminus T$ ($x_i = z_i = 0$). We may assume that $a_k > \rho$, since otherwise case 1 reduces to case 3. Then MIR inequality (24) is facet-defining because when $a_0 = 0$, the property $a_k > \rho$ implies that $a(S) > 1$.

Any point in the second case is the intersection of the following $2n+1$ facets: capacity inequality (1), MIR inequality (24) with S, $x_i \geq 0$ for $i \in N \setminus (S \cup \{k\})$, $x_i \leq z_i$ for $i \in S$ ($x_i = z_i = 1$), $z_i \leq 1$ for $i \in T \cup \{k\}$, $x_i \leq z_i$ for $i \in N \setminus (T \cup \{k\})$ ($x_i = z_i = 0$). In this case, if MIR inequality (24) is not facet-defining (i.e., $a_0 = 0$ and $a(S) < 1$), it is replaced with the surrogate variable upper bound inequality (13) for some $i \in S$, which is facet-defining as $a_i < 1$.

Finally, any point in the third case is the intersection of the facets defined by either $y \geq 0$ or MIR inequality (24) with S, and $x_i \geq 0$ for $i \in N \setminus S$, $x_i \leq z_i$ for $i \in S$ ($x_i = z_i = 1$), $z_i \leq 1$ for $i \in T$, $x_i \leq z_i$ for $i \in N \setminus T$ ($x_i = z_i = 0$). ∎

If the capacity inequality (1) is not facet-defining, then replacing it with the stronger capacity flow cover inequality (12) with $S = N$ in the first two cases again gives necessary $2n + 1$ facets.

4.1 Augmented MIR inequalities

For $S \subseteq N$ such that $\lambda = a(S) - a_0 > 0$ and $T \subseteq N \setminus S$, let us relax the capacity constraint as follows:

$$a_0 + y \geq \sum_{i \in S} a_i x_i + \sum_{i \in T} a_i x_i \tag{25}$$

$$= \sum_{i \in S} a_i[1 - (1 - z_i) - (z_i - x_i)] - \sum_{i \in T} a_i[(z_i - x_i) + z_i]. \tag{26}$$

Let $S' \subseteq S$ and $T' \subseteq T$. We next relax inequality (26) as follows: (i) for $i \in S'$, we split the coefficient of $(1 - z_i)$ into $\lfloor a_i \rfloor$ and r_i; (ii) for $i \in S \setminus S'$, we round up the coefficient of $(1 - z_i)$; (iii) for $i \in T'$, we rewrite the coefficient of z_i as $\lceil a_i \rceil$ and $(r_i - 1)$, and (iv) for $i \in T \setminus T'$, we relax the coefficient of x_i to $\lfloor a_i \rfloor$ and add and subtract $(a_i - \lfloor a_i \rfloor) z_i$ to the inequality. Thus the resulting inequality is

$$\left[y + \sum_{i \in S \setminus S'} \lceil a_i \rceil (1 - z_i) + \sum_{i \in S'} \lfloor a_i \rfloor (1 - z_i) - \sum_{i \in T \setminus T'} \lfloor a_i \rfloor z_i - \sum_{i \in T'} \lceil a_i \rceil z_i \right]$$
$$+ \left[\sum_{i \in S \cup T'} a_i(z_i - x_i) + \sum_{i \in T \setminus T'} \lfloor a_i \rfloor (z_i - x_i) + \sum_{i \in S'} r_i(1 - z_i) + \sum_{i \in T'} (1 - r_i) z_i \right] \geq \lambda. \tag{27}$$

Applying the MIR procedure to (27) we obtain the valid inequality

$$\sum_{i \in S \cup T'} a_i(z_i - x_i) + \sum_{i \in T \setminus T'} \lfloor a_i \rfloor (z_i - x_i) + \sum_{i \in S'} r_i(1 - z_i) + \sum_{i \in T'} (1 - r_i) z_i \geq$$
$$\rho \left(\eta - y - \sum_{i \in S \setminus S'} \lceil a_i \rceil (1 - z_i) - \sum_{i \in S'} \lfloor a_i \rfloor (1 - z_i) + \sum_{i \in T'} \lceil a_i \rceil z_i + \sum_{i \in T \setminus T'} \lfloor a_i \rfloor z_i \right). \tag{28}$$

Proposition 24 *An augmented MIR inequality* (28) *is facet-defining for* conv($\mathcal{P}$) *if*

1. $\eta > \lambda$, *i.e.*, $\lambda \notin \mathbb{Z}$,
2. $a(S) > \rho$, *i.e., either* $a_0 > 0$ *or* $\eta > 1$,
3. $S \subseteq \{i \in N : a_i \leq \eta\}$,
4. $S' = \{i \in S : r_i < \rho\}$,

5. $T = T' \subseteq \{i \in T \,:\, (1-r_i) < \rho\}$.

Proof We first rewrite inequality (28) as follows:

$$\rho y + \sum_{i \in S \setminus S'} (a_i - \rho \lceil a_i \rceil) z_i + \sum_{i \in S'} (a_i - r_i - \rho \lfloor a_i \rfloor) z_i + \sum_{i \in T \setminus T'} (a_i - \rho \lfloor a_i \rfloor) z_i$$
$$+ \sum_{i \in T'} (a_i + 1 - r_i - \rho \lceil a_i \rceil) z_i - \sum_{i \in S \cup T} a_i x_i \geq \rho \left(\eta - \sum_{i \in S \setminus S'} \lceil a_i \rceil - \sum_{i \in S'} \lfloor a_i \rfloor \right) - \sum_{i \in S'} r_i.$$

For a given $S \subseteq N$, let F be the face induced by the valid inequality and assume that all $p \in F$ satisfy the equality $\alpha y + \beta z + \gamma x = \delta$. From the proof of Proposition 21 we have $\alpha, \beta_i, \gamma_i$ for all $i \in N \setminus T$ as desired.

Let $k \in T'$. Recall that, $r_k + \rho > 1$ and therefore $\eta(S + k) = \eta(S) + \lceil a_k \rceil$. Consider $\bar{p}^1 = p^1 + \big(\lceil a_k \rceil - 1, e_k, \frac{\lceil a_k \rceil - 1 + (1-\rho)}{a_k} e_k \big)$, and note that $\lceil a_k \rceil - 1 + (1 - \rho) < a_k$. Since $p^1, \bar{p}^1 \in F$, we have $(\lceil a_k \rceil - 1)\rho\bar{\sigma} + \beta_k + \frac{\lceil a_k \rceil - \rho}{a_k} \gamma_k = 0$.

Let $t_k = (0, 0, (1/a_k) e_k)$ and $i \in S$. We have $\bar{p}^1, \bar{p}^1 + \epsilon t_k - \epsilon t_i \in F$ for a small enough $\epsilon > 0$, and therefore $\gamma_k = -a_k \bar{\sigma}$. Furthermore, when combined with above, we have $\beta_k = (\lceil a_k \rceil - \rho)\bar{\sigma} - (\lceil a_k \rceil - 1)\rho\bar{\sigma} = (1 - \rho) \lceil a_k \rceil \bar{\sigma}$, as desired. ■

Remark 25 *For the unsplittable flow arc set $\mathcal{R}$ by letting $x = z$, the augmented MIR inequalities (28) with $T = N \setminus S$ reduce to*

$$\sum_{i \in S'} r_i (1 - z_i) + \sum_{i \in T'} (1 - r_i) z_i \geq$$
$$\rho \left(\eta - y - \sum_{i \in S \setminus S'} \lceil a_i \rceil (1 - z_i) - \sum_{i \in S'} \lfloor a_i \rfloor (1 - z_i) + \sum_{i \in T'} \lceil a_i \rceil z_i + \sum_{i \in N \setminus (S \cup T')} \lfloor a_i \rfloor z_i \right). \quad (29)$$

Observe if $x = z$, inequalities (28) with $T = N \setminus S$ dominate all others with $T \subsetneq N \setminus S$; hence $T = N \setminus S$ in inequality (29).

Furthermore, if $S' = T' = \emptyset$, inequality (29) reduces to the c-strong inequality (5). Recall that a c-strong inequality is facet-defining for conv($\mathcal{R}$) *if and only if S is maximal c-strong if and only if $r_i \geq \rho$ for all $i \in S$ and $r_i \leq 1 - \rho$ for all $i \in N \setminus S$. Thus if S is not maximal c-strong, inequality (29) with $S' = \{i \in S : r_i < \rho\}$ and $T' = \{i \in T : (1 - r_i) < \rho\}$ dominates the corresponding c-strong inequality.*

The following example illustrates the strength of (29) for conv($\mathcal{R}$). Let

$$\mathcal{R} = \Big\{ y \in \mathbb{Z}_+,\ z \in \{0,1\}^5 \ : 1x_1 + 0.5x_2 + 0.75x_3 + 0.75x_4 + 0.75x_5 \ \leq \ y \Big\}.$$

For $S = \{1, 2\}$, which is not maximal c-strong, the c-strong inequality (5) is

$$x_1 + x_2 \leq y, \quad (30)$$

whereas the 2-split c-strong inequality (6) is

$$2x_1 + x_2 + x_3 + x_4 + x_5 \leq 2y. \tag{31}$$

Inequality (29) with $S = \{1, 2\}$, $S' = \emptyset$, and $T' = \{3, 4, 5\}$ ($\lambda = 1.5, \eta = 2, \rho = 0.5$)

$$0.25x_3 + 0.25x_4 + 0.25x_5 \geq 0.5(2 - y - (1 - x_1) - (1 - x_2)), \tag{32}$$

which can also be stated as

$$x_1 + x_2 + 0.5x_3 + 0.5x_4 + 0.5x_5 \leq y$$

dominates both (30) and (31). It is easily checked that (32) is facet-defining for $\text{conv}(\mathcal{R})$.

4.2 Scaled augmented MIR inequalities

For $S \subseteq N$ such that $\lambda = a(S) - a_0 > 0$ and $T \subseteq N \setminus S$, let us relax the capacity constraint as (26) and multiply the inequality with $\mu > 0$ to obtain

$$\mu a_0 + \mu y = \sum_{i \in S} \mu a_i [1 - (1 - z_i) - (z_i - x_i)] - \sum_{i \in T} \mu a_i [(z_i - x_i) + z_i]. \tag{33}$$

For $S' \subseteq S$ and $T' \subseteq T$ applying the same type of relaxation as in Section 4.1, we obtain the intermediate valid inequality

$$\left\lceil \lceil \mu \rceil y + \sum_{i \in S \setminus S'} \lceil \mu a_i \rceil (1 - z_i) + \sum_{i \in S'} \lfloor \mu a_i \rfloor (1 - z_i) - \sum_{i \in T \setminus T'} \lfloor \mu a_i \rfloor z_i - \sum_{i \in T'} \lceil \mu a_i \rceil z_i \right\rceil$$
$$+ \left\lceil \sum_{i \in S \cup T} \mu a_i (z_i - x_i) + \sum_{i \in S'} \bar{r}_i (1 - z_i) + \sum_{i \in T'} (1 - \bar{r}_i) z_i \right\rceil \;\geq\; \mu\lambda, \tag{34}$$

where $\bar{r}_i = \mu a_i - \lfloor \mu a_i \rfloor$ for $i \in N$. Now applying the MIR procedure to (34) we obtain the valid inequality

$$\sum_{i \in S \cup T} \mu a_i (z_i - x_i) + \sum_{i \in S'} \bar{r}_i (1 - z_i) + \sum_{i \in T'} (1 - \bar{r}_i) z_i \geq$$
$$\bar{\rho} \left(\bar{\eta} - \lceil \mu \rceil y - \sum_{i \in S \setminus S'} \lceil \mu a_i \rceil (1 - z_i) - \sum_{i \in S'} \lfloor \mu a_i \rfloor (1 - z_i) + \sum_{i \in T'} \lceil \mu a_i \rceil z_i + \sum_{i \in T \setminus T'} \lfloor \mu a_i \rfloor z_i \right), \tag{35}$$

where $\bar{\eta} = \lceil \mu\lambda \rceil$ and $\bar{\rho} = \mu\lambda - \lfloor \mu\lambda \rfloor$.

By simple comparison, one sees that choosing $S' = \{i \in S : \bar{r}_i < \bar{\rho}\}$ and $T' = \{i \in T : (1 - \bar{r}_i) < \bar{\rho}\}$ in (35) leads to the strongest inequalities as inequalities for all other choices for S' and T' are implied by these and $\mathbf{0} \leq z \leq \mathbf{1}$.

In addition, if $\mu - \lfloor \mu \rfloor < \bar{\rho}$, one can obtain a stronger inequality by not relaxing the term μy in inequality (33) to $\lceil \mu \rceil y$, but instead writing it as $\lfloor \mu \rfloor y + (\mu - \lfloor \mu \rfloor) y$ so that in the mixed-integer rounding procedure, the first part can be treated as an integer variable and the second part as a continuous variable. We do not write the resulting inequality explicitly in order to avoid repetition.

Remark 26 *If* $1 \leq \lambda$*, the capacity flow cover inequality* (11) *can be obtained by taking* $\mu \geq \{1, \bar{a}, \lambda\}$ *in inequality (35), where* $\bar{a} = \max_{i \in S}\{a_i\}$*. If* $1 > \lambda$*, then the strengthened version (mentioned in the above paragraph) of inequality (35) gives the capacity flow cover inequality.*

Clearly inequality (35) also subsumes the augmented MIR inequality (28) by taking $\mu = 1$ and therefore it forms a superclass of all inequalities discussed in this paper except the MIR inequality (24). When $\lceil a_i \rceil > \eta$ for some $i \in S$, the resulting MIR inequality is different from (35).

We next show that (35) reduces to some well-known inequalities for the unsplittable flow set $\mathcal{R}$ and the binary knapsack set $\mathcal{K}$.

Remark 27 *For the unsplittable flow set* $\mathcal{R}$ *by letting* $x = z$*,* $\mu = k \in \mathbb{Z}$*,* $S' = T' = \emptyset$*, and* $T = N \setminus S$*, inequality* (35) *reduces to*

$$0 \geq \lceil k\lambda \rceil - ky - \sum_{i \in S} \lceil ka_i \rceil (1 - z_i) + \sum_{i \in N \setminus S} \lfloor ka_i \rfloor z_i.$$

This is the k-split c-strong inequality (6)*, which is shown to be facet-defining for* $\text{conv}(\mathcal{R})$ *in [5] under certain conditions. Then from the observation above, if* $S' = \{i \in S : \bar{r}_i < \bar{\rho}\}$ *and* $T' = \{i \in T : (1 - \bar{r}_i) < \bar{\rho}\}$*, inequality*

$$\sum_{i \in S'} \bar{r}_i (1 - z_i) + \sum_{i \in T'} (1 - \bar{r}_i) z_i \geq$$
$$\bar{\rho}\left(\lceil k\lambda \rceil - ky - \sum_{i \in S \setminus S'} \lceil ka_i \rceil (1 - z_i) - \sum_{i \in S'} \lfloor ka_i \rfloor (1 - z_i) + \sum_{i \in T'} \lceil ka_i \rceil z_i + \sum_{i \in N \setminus (S \cup T')} \lfloor ka_i \rfloor z_i \right)$$

dominates the k-split c-strong inequality.

Remark 28 *For the binary knapsack set* $\mathcal{K}$ *by letting* $x = z$ *and* $y = 0$*, inequality* (34) *reduces to*

$$\sum_{i \in S'} \bar{r}_i (1 - z_i) + \sum_{i \in T'} (1 - \bar{r}_i) z_i \geq$$
$$\bar{\rho}\left(\bar{\eta} - \sum_{i \in S \setminus S'} \lceil \mu a_i \rceil (1 - z_i) - \sum_{i \in S'} \lfloor \mu a_i \rfloor (1 - z_i) + \sum_{i \in T'} \lceil \mu a_i \rceil z_i + \sum_{i \in T \setminus T'} \lfloor \mu a_i \rfloor z_i \right). \quad (36)$$

For $S \subseteq N$ *such that* $\bar{a} = \max_{i \in S} a_i \geq \lambda$*, letting* $T = \emptyset$ *and* $\mu = 1/\bar{a}$*, we obtain* $\bar{\eta} = 1$*,* $\bar{\rho} = \lambda / \bar{a}$*, and consequently*

$$\sum_{i \in S'} a_i (1 - z_i) \geq \lambda \left(1 - \sum_{i \in S \setminus S'} (1 - z_i) \right),$$

where $S' = \{i \in S : a_i < \lambda\}$*. Therefore, for a minimal cover* S*, i.e., for* S *such that* $a_i \geq \lambda$ *for all* $i \in S$*, this inequality reduces to the the knapsack cover inequality*

$$\sum_{i \in S} z_i \leq |S| - 1.$$

Then for a minimal cover S, inequality (36) *gives the lifted knapsack cover inequality*

$$\sum_{i \in T'} (\bar{a} - a_i + \lfloor a_i/\bar{a} \rfloor \bar{a}) z_i \geq \lambda \left(1 - \sum_{i \in S} (1 - z_i) + \sum_{i \in T'} \lceil a_i/\bar{a} \rceil z_i + \sum_{i \in T \setminus T'} \lfloor a_i/\bar{a} \rfloor z_i \right),$$

or equivalently

$$\sum_{i \in S} z_i + \sum_{i \in T} \frac{\phi_{\bar{a}}(a_i, \lambda)}{\lambda} z_i \leq |S| - 1.$$

Since $\phi(\cdot, \lambda) \geq 0$, the strongest inequality is obtained by letting $T = N \setminus S$. This is the lifted knapsack cover inequality with MIR lifting function [3, 4, 15].

5 Concluding remarks

We studied the polyhedral structure of the network design arc set with variable upper bounds. This set is a common substructure of formulations of network design problems with multicommodity fixed charges and/or combinatorial restrictions.

Several fundamental sets studied independently in the literature are facial restrictions of its convex hull. Therefore, valid inequalities for the network design arc set with variable upper bounds generalize the inequalities known for these sets. In this study we have identified facets that cut off all fractional extreme points of the continuous relaxation of the network design arc set with variable upper bounds. Interestingly, some of these facets are new even for the earlier studied restrictions.

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