RC23610 (W0505-101) May 17, 2005
Mathematics

IBM Research Report

Synchronizability of Networks of Chaotic Systems Coupled via a Graph with a Prescribed Degree Sequence

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Abstract

Generally, synchronization in a network of chaotic systems depends on the underlying coupling topology. Recently, there have been several studies conducted to determine what features of this topology contribute to the ability to synchronize. A short diameter has been proposed by several authors as a means to facilitate synchronization whereas others point to features such as the homogeneity of the degree sequence. Recently, it has been shown that the degree sequence by itself is not sufficient to determine synchronizability. The purpose of this letter is to continue this study. For a given degree sequence, we construct two connected graphs with this degree sequence whose synchronizability can be quite different. In particular, we construct a graph with low synchronizability which improves upon previous bounds under certain conditions and we construct a graph which has synchronizability that is asymptotically maximal. On the other hand, we show analytically that for a random network model, homogeneity of the degree sequence is beneficial to synchronization.

Key words: Eigenvalue analysis, nonlinear dynamics, random graphs, scale free networks, synchronization

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1 Introduction

The object of interest in this paper is a coupled network of n identical systems with linear coupling:

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f(x_1, t) \\ \vdots \\ f(x_n, t) \end{pmatrix} - (G \otimes P)x \tag{1}$$

where x_i is the state vector of the i-th system, $x = (x_1, \ldots, x_n)^T$ and $G \otimes P$ is the Kronecker product of the matrices G and P. The coupling matrix G describes the coupling topology of the network, i.e. $G_{ij} \neq 0$ if there is a coupling from system i to system j. If $G_{ij} < 0$ or $G_{ij} > 0$, we call such a coupling element *cooperative* or *competitive* respectively. The network in Eq. (1) is said to synchronize globally if $\|x_i - x_j\| \to 0$ as $t \to \infty$ for all i, j. It was shown in [1] that for symmetric matrices G and a suitable matrix P, the ability for the network to synchronize depends on the size of the smallest nonzero eigenvalue of G. We will denote this eigenvalue as $\lambda_2(G)$. An example of such suitable matrices P include the class of positive definite matrices. For other classes of matrices P [2] and the discrete time case [3,4], the ratio $r(G) = \frac{\lambda_2(G)}{\lambda_{\max}(G)}$ is important in determining synchronizability, where $\lambda_{\max}(G)$ is the largest eigenvalue of G.

The graph of a matrix G is defined as the weighted graph with an edge of weight G_{ij} from vertex i to vertex j if and only if $G_{ij} \neq 0$. We assume that G is symmetric, so the graph of G is undirected [1]. The Laplacian matrix of a weighted graph is defined as $L = D - A$ where D is the diagonal matrix of vertex degrees and A is the adjacency matrix. In this paper we assume that G is a symmetric zero row sum matrix with nonpositive off-diagonal elements. In this case, G is equal to the Laplacian matrix of the graph of G and $\lambda_2(G)$ is the algebraic connectivity of the graph [10]. There are many results relating various characteristics of the graph of G and the eigenvalues of its Laplacian matrix [11–13]. In this paper we will assume that the graphs are unweighted, i.e. the adjacency matrix is a 0-1 matrix.

Recently, various models of random graphs have been proposed that mimic more closely man-made and natural networks [14]. These random graphs differ from the classical random graph model in that the degree distribution does not have to be Poisson or binomial [15], but can have other types of distribution such as a power law distribution. Several of these random graph models

[1] See [5–9] for synchronization results when G is not symmetric and the graph of G is directed.

have two important defining characteristics. First, the graph is generated via a random procedure. Second, the graph has a particular degree distribution. Focusing on the degree distribution alone, while ignoring the random aspect or vice versa can lead to incomplete conclusions. In [16] it was shown that the heterogeneity of the degree distribution can affect λ_2 and r and hence the synchronizability. On the other hand, several papers show that the degree sequence by itself is not enough to characterize λ_2 and r. In [17–19] it was shown that randomness makes λ_2 and r large whereas local coupling makes λ_2 and r small. In [20] a graph is constructed for a prescribed degree sequence whose normalized Laplacian matrix has a small nonzero eigenvalue. In this paper we continue this study by constructing two graphs with the same degree sequence whose values for λ_2 and r differ significantly. In particular, in some cases the graph with small λ_2 improves upon the bound for λ_2 in [20] whereas the graph with large λ_2 has a value for λ_2 which is in a sense maximal. Furthermore, we show analytically that for a model of random graphs, homogeneity of the degree sequence is beneficial for synchronization.

2 Graphs with a prescribed degree sequence

A simple graph is an unweighted graph with no loops nor multiple edges [21]. The degree sequence of a graph is a list of the degrees of the vertices. A list of natural numbers is called *graphical* if there is a simple graph with this list as its degree sequence. Sufficient and necessary conditions for a list to be graphical are given in [22,23]. The following result gives upper and lower bounds for λ_2 and r of the Laplacian matrix of a graph with a given degree sequence.

Lemma 1 *For a connected graph with degree sequence $d_1 \leq d_2 \cdots \leq d_n$, λ_2 and r satisfy:*

$$2\left(1-\cos\left(\frac{\pi}{n}\right)\right)\delta \leq \lambda_2 \leq \frac{n}{n-1}\delta$$

$$\frac{\left(1-\cos\left(\frac{\pi}{n}\right)\right)\delta}{\Delta} \leq r \leq \frac{\delta}{\Delta}$$

where $\delta = d_1$ is the minimum vertex degree and $\Delta = d_n$ is the maximum vertex degree.

Proof: The first set of inequalities follows from [10]. The second set of inequalities follows from the fact that $\frac{n}{n-1}\Delta \leq \lambda_{\max} \leq 2\Delta$, also from [10]. □

In fact, if the graph is not complete, then $\lambda_2 \leq \delta$ [10]. The lower bounds in Lemma 1 are attained or approached asymptotically as $n \to \infty$ for the nearest neighbor path graph with Laplacian matrix

$$L = \begin{pmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 1 \end{pmatrix}$$

and $\lambda_2 = 2\left(1 - \cos\left(\frac{\pi}{n}\right)\right)$, $r = \frac{1-\cos\left(\frac{\pi}{n}\right)}{1+\cos\left(\frac{\pi}{n}\right)}$. The upper bounds are attained for the complete graph with Laplacian matrix

$$L = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \\ & \ddots & \ddots & \ddots \\ -1 & \cdots & -1 & n-1 \end{pmatrix}$$

and $\lambda_2 = n$, $r = 1$.

2.1 Regular graphs

A k-regular graph is a graph where every vertex has degree k. In this section, we consider two types of connected regular graphs whose values of λ_2 and r can be quite different.

2.1.1 Construction 1: graph with low λ_2 and r

For a $2k$-regular graph, the degree sequence is $(2k, 2k, \ldots, 2k)$. We consider two cases. In the first case, the vertex degree grows as $\Omega\left(n^{\frac{1}{3}-\epsilon}\right)$ for $\epsilon > 0$. If we arrange the vertices in a circle, and connect each vertex to its $2k$ nearest neighbors, then the resulting $2k$-regular graph is denoted C_{2k}. The Laplacian matrix L is a circulant matrix:

$$\begin{pmatrix} 2k & -1 & \cdots & -1 & 0 & \cdots & 0 & -1 \cdots & -1 \\ -1 & 2k & -1 & \cdots & -1 & & & & \\ & \ddots & \ddots & & \ddots & & & & \\ -1 & \cdots & -1 & 0 & \cdots & 0 & -1 & \cdots & -1 \; 2k \end{pmatrix}$$

The eigenvalues of L are given by $(\mu_0, \ldots, \mu_{n-1})$ with:

$$\mu_0 = 0$$

$$\mu_m = 2k - 2\sum_{l=1}^{k} \cos\left(\frac{2\pi m l}{n}\right) = 2k + 1 - \frac{\sin\left(\left(k+\frac{1}{2}\right)2\pi\frac{m}{n}\right)}{\sin\left(\frac{\pi m}{n}\right)}, \quad m = 1, \ldots, n-1$$

A series expansion shows that as $n \to \infty$, $\lambda_2 \le \mu_1 \approx \frac{4\pi^2(k+\frac{1}{2})^3}{3n^2}$. Since $\lambda_{\max} \ge 2k\frac{n}{n-1}$ by Lemma 1, $r \le \frac{\mu_1}{\lambda_{\max}} \approx \frac{2\pi^2(k+\frac{1}{2})^3}{3kn^2}$. This means the values λ_2 and r decrease as $\Omega\left(\frac{1}{n^{1+3\epsilon}}\right)$ and $\Omega\left(\frac{1}{n^{\frac{4}{3}+2\epsilon}}\right)$ respectively. In particular, if k is bounded, i.e. $\epsilon = \frac{1}{3}$, then λ_2, r decrease as $\Omega\left(\frac{1}{n^2}\right)$ which is asymptotically the fastest possible by Lemma 1.

In the second case, we consider graphs for which $k < \lfloor \frac{n}{2} \rfloor$ and use the same approach as in [20]. The main difference is that in [20] the smallest nonzero eigenvalue of the *normalized* Laplacian matrix $\tilde{L} = I - D^{-1}A$ is studied [2].

For a graph with adjacency matrix A, let $e(B,C) = \sum_{u \in B, v \in C} A_{u,v}$ be the number of edges between B and C. The following Lemma is shown in [11]:

Lemma 2

$$\lambda_2(L) \le \frac{e(S,\overline{S})n}{|S|(n-|S|)} \le \lambda_{\max}(L)$$

As in [20], we create a connected k-regular graph with $|S| = \lfloor \frac{n}{2} \rfloor$ and $e(S,\overline{S}) = 2$. By Lemma 2 this means that

$$\lambda_2(L) \le \frac{2n}{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil} = \begin{cases} \frac{8}{n} & , n \quad \text{even} \\ \frac{8}{n-\frac{1}{n}} & , n \quad \text{odd} \end{cases} \le \frac{8}{n - \frac{1}{n}}$$

$$r(L) \le \begin{cases} \frac{8(n-1)}{kn^2} & , n \quad \text{even} \\ \frac{8}{k(n+1)} & , n \quad \text{odd} \end{cases} \le \frac{8}{k(n+1)}$$

The reason for studying the two cases is that C_{2k} gives better bounds when k grows slower than $n^{\frac{1}{3}}$ and the construction in [20] gives better bounds when k grows faster than $n^{\frac{1}{3}}$. In either case, $\lambda_2, r \to 0$ as $n \to \infty$.

[2] For k-regular graphs, this difference between L and $\tilde{L}$ is not important since $L = k\tilde{L}$ and the eigenvalues of L and $\tilde{L}$ differ by a constant factor k. However, for non-regular graphs, the eigenvalues of $\tilde{L}$ and L will have different properties.

2.1.2 *Construction 2: graph with high λ_2 and r*

With high probability, a random $2k$-regular graph has eigenvalues $\lambda_2 = 2k - O(\sqrt{k})$, $\lambda_{\max} = 2k + O(\sqrt{k})$ as $n \to \infty$ [24]. This means that on a relative scale, $\lambda_2 \approx 2k$, and $r \approx 1$ as $k, n \to \infty$ which can be considered optimal according to Lemma 1. In conclusion, whereas λ_2 and $r \to 0$ as $n \to \infty$ for Construction 1, they remain bounded from below for Construction 2.

Lemma 3 *If $\mathcal{H}$ is a subgraph of $\mathcal{G}$ with the same set of vertices*[3]*, then*

(1) $\lambda_2(\mathcal{H}) + \lambda_2(\mathcal{G}\backslash\mathcal{H}) \leq \lambda_2(\mathcal{G})$,
(2) $\lambda_{\max}(\mathcal{H}) + \lambda_{\max}(\mathcal{G}\backslash\mathcal{H}) \geq \lambda_{\max}(\mathcal{G})$
(3) $\min(r(\mathcal{H}), r(\mathcal{G}\backslash\mathcal{H})) \leq \frac{\lambda_2(\mathcal{H})+\lambda_2(\mathcal{G}\backslash\mathcal{H})}{\lambda_{\max}(\mathcal{H})+\lambda_{\max}(\mathcal{G}\backslash\mathcal{H})} \leq r(\mathcal{G})$,
(4) $\lambda_{\max}(\mathcal{H}) \leq \lambda_{\max}(\mathcal{G})$.

Here $\mathcal{G}\backslash\mathcal{H}$ is a graph with the same set of vertices as $\mathcal{G}$ and with edges which are in $\mathcal{G}$ but not in $\mathcal{H}$. By a slight abuse of notation, we use $\lambda_i(\mathcal{H})$ to denote λ_i of the Laplacian matrix of the graph $\mathcal{H}$.

Proof: Follows from the facts that $\lambda_2(L) = \min_{\sum_i x_i=0, \|x\|=1} x^T L x$, $\lambda_{\max}(L) = \max_{\|x\|=1} x^T L x$ and $L(\mathcal{H}) + L(\mathcal{G}\backslash\mathcal{H}) = L(\mathcal{G})$. □

2.2 *Graphs with prescribed degree sequence*

For each n, consider a graphical list of degrees $0 < d_1 \leq d_2 \leq \cdots \leq d_n$. The average vertex degree is defined as $\overline{k} = \frac{\sum_i d_i}{n}$. Next we construct two connected graphs with the same list of degrees but different λ_2 and r.

2.2.1 *Construction 1: graph with low λ_2 and r*

As in Section 2.1.1 we consider two cases. In the first case, d_n grows as $\Omega\left(n^{\frac{1}{3}-\epsilon}\right)$ for $\epsilon > 0$. Given a graphical list of degrees $0 < d_1 \leq d_2 \leq \cdots \leq d_n$, the Havel-Hakimi algorithm [25,26] constructs a connected graph with this degree sequence where the vertices are arranged on a line and each vertex is connected to its nearest neighbors. Let us denote this graph as $\mathcal{G}_{hh}$ with Laplacian matrix L_{hh}. The graph $\mathcal{G}_{hh}$ is a subgraph of the $2d_n$-regular graph C_{2d_n} defined in Sec. 2.1.1. Therefore, by Lemma 3, $\lambda_2(L_{hh}) \leq \mu_1 \approx \frac{4\pi^2(d_n+\frac{1}{2})^3}{3n^2} = \Omega\left(\frac{1}{n^{1+3\epsilon}}\right)$ as $n \to \infty$. Since $\lambda_{\max}(L_{hh}) \geq \frac{nd_n}{n-1}$, this means that $r(L_{hh}) \leq \frac{\mu_1}{\lambda_{\max}} \approx \frac{4\pi^2(d_n+\frac{1}{2})^3}{3d_n n^2} =$

[3] By this we mean that $A_{ij} \leq B_{ij}$ for all i,j, where A and B are the adjacency matrices of $\mathcal{H}$ and $\mathcal{G}$ respectively.

$\Omega\left(\frac{1}{n^{\frac{4}{3}+2\epsilon}}\right)$. As in Section 2.1.1, for bounded d_n the values of λ_2 and r for this construction decrease as $\Omega\left(\frac{1}{n^2}\right)$ when $n \to \infty$. By Lemma 1, this is the lowest possible rate.

In the second case, assume that the degree sequence is of the form $(d_{\max}, \dots, d_{\max}, d_{\max}-1, \dots, d_{\max}-1, \dots, 1, \dots, 1)$ with $d_{\max} \leq \frac{n}{4}$ and the average degree satisfies $\overline{k} = 2 + \frac{d_{\max}}{n}$. We further assume that for each $1 \leq i \leq d_{\max}$, there are at least 2 vertices with vertex degree i. As in [20], we construct a graph with this degree sequence such that $|S| \geq \lceil \frac{n}{2} \rceil - d_{\max}$ and $E(S, \overline{S}) = 2$. By Lemma 2 this means that

$$\lambda_2(L) \leq \frac{2n}{\left(\lfloor \frac{n}{2} \rfloor + d_{\max}\right)\left(\lceil \frac{n}{2} \rceil - d_{\max}\right)} = \begin{cases} \frac{8}{n - \frac{4d_{\max}^2}{n}} & \text{,n} \quad \textit{even} \\ \frac{8}{n - \frac{4(d_{\max}-\frac{1}{2})^2}{n}} & \text{,n} \quad \textit{odd} \end{cases}$$

$$\leq \frac{8}{n - \frac{4d_{\max}^2}{n}} \leq \frac{10\frac{2}{3}}{n}$$

$$r(L) \leq \frac{(n-1)\lambda_2(L)}{d_{\max} n} \leq \begin{cases} \frac{8(n-1)}{n^2 d_{\max} - 4d_{\max}^3} & \text{,n} \quad \textit{even} \\ \frac{8(n-1)}{n d_{\max}\left(n - \frac{4(d_{\max}-\frac{1}{2})^2}{n}\right)} & \text{,n} \quad \textit{odd} \end{cases} \leq \frac{10\frac{2}{3}}{n d_{\max}}$$

where we have used the fact that $d_{\max} \leq \frac{n}{4}$. Again, in either case r, $\lambda_2 \to 0$ as $n \to \infty$.

2.2.2 Construction 2: graph with high λ_2 and r

For convenience, we allow loops in the graph, i.e. the graphs are not necessarily simple. Furthermore, rather than constructing graphs with a specific degree sequence, we consider a random graph model which has a prescribed degree sequence in expectation. Given a list of degrees $d_1 \leq d_2 \leq \cdots \leq d_n$ satisfying the condition

$$n(d_n - d_1)^2 \leq (n - d_1) \sum_k (d_k - d_1)$$

we construct a random graph as in [18] where an edge between vertex i and vertex j is randomly selected with probability

$$P_{ij} = \frac{d_1}{n} + \frac{(d_i - d_1)(d_j - d_1)}{\sum_k (d_k - d_1)}$$

Since $\sum_i P_{ij} = d_j$, the graph has the desired degree sequence in expectation. Let us call this graph model $\mathcal{G}_{pr}$ with Laplacian matrix L_{pr}. For the special case $d_1 = d_2 = \cdots = d_n$, we choose $P_{ij} = \frac{d_1}{n}$ and denote this graph model as $\mathcal{G}_r(d_1)$.

Since $P_{ij} \geq \frac{d_1}{n}$, this means that $\mathcal{G}_r(d_1)$ is a “subgraph” of $\mathcal{G}_{pr}$. What we mean by this is that the probability space of $\mathcal{G}_{pr}$ can be partitioned such that for each graph $\mathcal{H}$ in $\mathcal{G}_r(d_1)$ with probability p, there exists exactly one event A in $\mathcal{G}_{pr}$ with probability p, such that $\mathcal{H}$ is a subgraph of all the graphs in event A. In particular, consider the edges in $\mathcal{G}_{pr}$ to be of two types. An edge (i, j) of type 1 occurs with probability $\frac{d_1}{n}$ and and edge (i, j) of type 2 occurs with probability $P_{ij} - \frac{d_1}{n}$. To each graph $\mathcal{H}$ in $\mathcal{G}_r(d_1)$ corresponds a set of graphs $A(\mathcal{H})$ constructed as follows. If (i, j) is an edge in $\mathcal{H}$, then (i, j) is an edge of type 1 in graphs in $A(\mathcal{H})$. If (i, j) is not an edge in $\mathcal{H}$, then either (i, j) is not an edge or (i, j) is an edge of type 2 in graphs in $A(\mathcal{H})$. It is clear that these sets $A(\mathcal{H})$ over all graphs $\mathcal{H}$ in $\mathcal{G}_r(d_1)$ exactly partition the probability space of $\mathcal{G}_{pr}$. Furthermore, the probability of $A(\mathcal{H})$ in $\mathcal{G}_{pr}$ is equal to the probability of $\mathcal{H}$ in $\mathcal{G}_r(d_1)$ and $\mathcal{H}$ is a subgraph of every graph in $A(\mathcal{H})$.

By the Courant-Fischer minmax theorem, the eigenvalues of the Laplacian matrix cannot decrease as more edges are added to the graph. This means that if $\lambda_2(\mathcal{G}_r(d_1)) \geq \xi$ with high probability, then $\lambda_2(\mathcal{G}_{pr}) \geq \xi$ with high probability.

Let us assume now that $d_1 = p_1 n$ for some $0 < p_1 < 1$. In [27] it was shown that for the graph $\mathcal{G}_r(d_1)$, $|\lambda_2 - d_1|$ and $|\lambda_{\max} - d_1|$ are both on the order of $o(\sqrt{n})$ with high probability as $n \to \infty$. This means that $\lambda_2(\mathcal{G}_{pr}) \geq d_1 - o(\sqrt{n})$ with high probability. Since $\lambda_2 \leq \frac{n}{n-1} d_1$, this implies that $\lambda_2 \approx d_1$ and $r \gtrapprox \frac{d_1}{2d_n}$ for $\mathcal{G}_{pr}$, i.e. λ_2 and r are bounded away from 0 as $n \to \infty$.

The conclusion that $\lambda_2 \approx d_1$ shows that for the random graph model $\mathcal{G}_{pr}$, homogeneity of the degree sequence enhances synchronizability in terms of λ_2. In particular, for a given average degree $\overline{k}$, the highest λ_2 can be as $n \to \infty$ is approximately $\overline{k}$ since $\lambda_2 \leq \frac{n}{n-1} d_1 \leq \frac{n}{n-1} \overline{k}$. This is achieved for the random graph where $d_1 = \overline{k}$, i.e. the d_1-regular graph which has the most homogeneous degree sequence. This supports the experimental results in [16].

3 Conclusions

We present graph constructions that for the same degree sequence generate graphs with low and high synchronizability in networks of coupled systems. In particular, we construct a graph with low synchronizability whose values of λ_2 and r decreases to 0 and a graph with high snchronizability whose values of λ_2 and r are bounded away from 0 as $n \to \infty$. This further indicates that the degree sequence alone is not sufficient to determine the synchronizability of a network of coupled dynamical systems. Furthermore, the constructions further support the assertion in [18,19] that local coupling tends to have low synchronizability, whereas random coupling tends to have high synchronizability. On the other hand, for the random graph model $\mathcal{G}_{pr}$ in Section 2.2.2, homogeneity in the vertex degree is beneficial for synchronization.

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