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Observations and Computations in Sylvester-Gallai Theory

Hervé Brönnimann
Department of Computer and Information Science
Polytechnic University

Jonathan Lenchner
IBM Research Division
Thomas J. Watson Research Center
P.O. Box 704
Yorktown Heights, NY 10598

Research Division
Almaden - Austin - Beijing - Haifa - India - T. J. Watson - Tokyo - Zurich

Observations and Computations in Sylvester-Gallai Theory

Hervé Brönnimann * Jonathan Lenchner †

Abstract

We bring together several new results related to the classical Sylvester-Gallai Theorem and its recently noted sharp dual. In 1951 Dirac and Motzkin conjectured that a configuration of n not all collinear points must admit at least $n/2$ ordinary connecting lines. There are two known counterexamples, when $n = 7$ and $n = 13$. We provide a construction that yields both counterexamples and note that the common construction cannot be extended to provide additional counterexamples. We give examples to show that the Sylvester-Gallai Theorem and its sharp dual are both false on the Torus.

1 Introduction

In 1893 J. J. Sylvester posed the following celebrated problem [13]: Given a collection of points in the plane, not all lying on a line, prove that there exists a line which passes through precisely two of the points. Sylvester's problem, which today is usually referred to as Sylvester's Theorem or the Theorem of Sylvester and Gallai, was reposed by Erdös in 1944 [5] and then solved the same year by T. Gallai [7]. Sylvester's Theorem holds equally in the Euclidean and Real Projective Planes. We write $\mathbb{RP}^2$ to denote the real projective plane. See [1] and [6] for excellent treatments of Sylvester-Gallai theory.

A line which passes through precisely two points in a configuration of points is referred to as an **ordinary line**. Analogously, given an arrangement of lines, a point lying at the intersection of precisely two lines is referred to as an **ordinary point**.

Much work has gone into obtaining lower bounds on the number of ordinary lines in a collection of points satisfying the hypothesis of Sylvester's Theorem. Dirac [4] and Motzkin [12] separately conjectured that given n points as in the statement of Sylvester's Theorem, there must be at least $n/2$ ordinary lines. However, in 1958 Kelly and Moser [8] found an example of 7 points with just 3 ordinary lines. They also showed that a set of n not all collinear points must admit at least $3n/7$ ordinary lines. Then in 1968 McKee [2] found an example of 13 points with just 6 ordinary lines. Finally in 1993 Csima and Sawyer [3] showed that except for the case of $n = 7$ there must be at least $6n/13$ ordinary lines in a configuration of n not all collinear points.

The Dirac-Motzkin conjecture thus stands as follows: For n not all collinear points, $n \neq 7, 13$, there must be at least $n/2$ ordinary lines.

By projective duality, Sylvester's Theorem is equivalent to the statement that in an arrangement of lines in $\mathbb{RP}^2$, not all of which pass through a single point, there must be an ordinary point. For given n, a lower bound on the number of ordinary lines amongst not all collinear point configurations (in $\mathbb{R}^2$ or $\mathbb{RP}^2$) of size n, corresponds to the same lower bound on the number of ordinary points amongst not all coincident line arrangements in $\mathbb{RP}^2$ of the same size n. In [10, 11] Lenchner showed that a sharper dual version of Sylvester's Theorem actually holds, namely that given an arrangement of lines in $\mathbb{R}^2$, not all of which are parallel and not all of which are coincident (i.e. pass through a common point), then there must be an ordinary point - indeed, that given n such lines, there must be at least $(5n+6)/39$ ordinary points.

In this paper we look at the sharp dual results in [10, 11] from a different angle, prove a couple of related theorems, provide a common construction for the $n = 7$ and $n = 13$ counterexamples to the Dirac-Motzkin conjecture, and look at Sylvester's Theorem and its sharp dual on the Torus.

2 Asymptotic and other Examples

To obtain our results, we use the following key lemmas and definitions. In what follows we shall consider Sylvester's problem in its dual formulation in the real projective plane.

Definition Say that a point p is **attached** to a line l not containing p if l together with two lines crossing at p form a triangular face of the arrangement.

The following two lemmas are due to Kelly and Moser [8]. We recommend the proofs from Felsner's recent book [6].

Lemma 1 *If a line l of an arrangement $\mathcal{A}$ contains no ordinary points, then there are at least* 3 *ordinary points attached to l.*

Lemma 2 *If a line l of an arrangement $\mathcal{A}$ contains a single ordinary point, then the line l has at least* 2 *ordinary points attached to it.*

*Dept. of Computer and Information Science, Polytechnic University, hbr@poly.edu.

†IBM T.J. Watson Research Institute, lenchner@us.ibm.com.

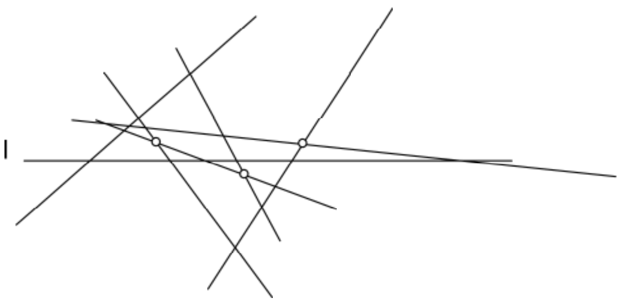

Figure 1: An example of a line l with three points attached.

Definition A line is said to be of **type** (n, m) if it contains n ordinary points and has m ordinary points attached to it.

The following lemma is due to Csima and Sawyer [3]:

Lemma 3 *Suppose $\mathcal{A}$ is a finite arrangement of lines in $\mathbb{RP}^2$ having two lines of type $(2,0)$ that intersect in an ordinary point. Then $\mathcal{A}$ must be graph theoretically isomorphic to the Kelly-Moser arrangement (Figure 2).*

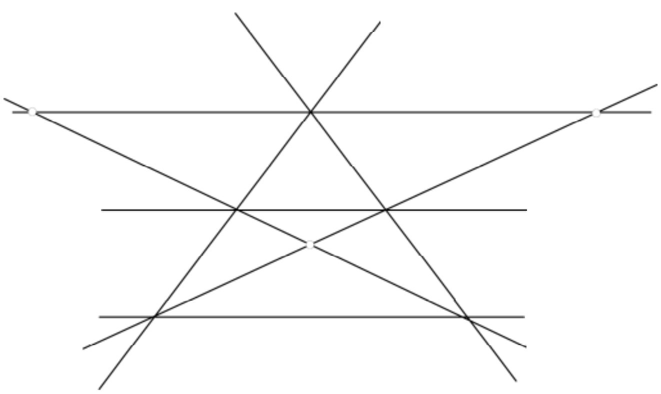

Figure 2: The Kelly-Moser arrangement of 7 lines with just 3 ordinary points.

The following simple observation is what is used to drive home the Kelly-Moser and Csima-Sawyer results.

Lemma 4 *In an arrangement of lines in $\mathbb{RP}^2$, an ordinary point can have at most 4 lines counting that point as an attachment.*

Proof. An ordinary point is contained in 2 crossing pseudolines, and hence a vertex of 4 faces; it can therefore be attached to at most 4 lines. □

Definition An arrangement of **pseudolines** in $\mathbb{RP}^2$ is a family of simple closed curves each pair of which has exactly one point in common, and at this common point the curves cross. Furthermore, a pseudoline must not separate $\mathbb{RP}^2$ into two distinct open sets.

We note that both the Csima-Sawyer and Kelly-Moser Theorems, in their dual formulations, are true if we replace lines in $\mathbb{RP}^2$ by pseudolines in $\mathbb{RP}^2$. The pseudoline version of the Kelly-Moser Theorem was proved by Kelly and Rottenberg [9] in 1972 and the pseudoline version of the Csima-Sawyer Theorem was remarked to hold without changing any of the basic lemmas in the original Csima-Sawyer 1993 article [3]. Lemmas 1 - 4 are valid both for lines and pseudolines. Lenchner's Theorem 4 from [11], which relies solely on these results, can be reinterpreted in the context of pseudolines as follows:

Theorem 5 *Given an arrangement of n pseudolines in $\mathbb{RP}^2$, one can find at least $\lceil (5n+6)/39 \rceil$ ordinary points off any pseudoline not already part of the arrangement.*

We can also say something about ordinary points off pseudolines that *are* part of the arrangement.

Theorem 6 *Given an arrangement of n pseudolines in $\mathbb{RP}^2$, no $n-1$ of which pass through a common point, one can find at least $\lceil (5n-6)/39 \rceil$ ordinary points off any pseudoline in the arrangement.*

Proof. Given an arrangement $\mathcal{A}$ as in the statement of the Theorem, Csima-Sawyer again guarantees at least $\lceil 6n/13 \rceil$ ordinary points as long as $n \neq 7$ and, as earlier, we consider the case $n = 7$ separately.

If our result were false, then more than $\frac{6n}{13} - \frac{5n-6}{39} = \frac{n}{3} + \frac{2}{13}$ of those ordinary points would have to lie on a fixed pseudoline $l \in \mathcal{A}$. Now consider the arrangement $\mathcal{A}^*$ consisting of the $n-1$ elements of $\mathcal{A}$ other than l. The removal of l "kills off" at least $\frac{n}{3} + \frac{2}{13}$ ordinary points and creates at most $\frac{n-1-(\frac{n}{3}+\frac{2}{13})}{2} = \frac{n}{3} - \frac{15}{26}$ new ones since an ordinary points can only be created where previously two pseudolines from $\mathcal{A}$ intersected in l.

Now, by assumption the pseudolines of $\mathcal{A}^*$ do not all pass through a common point, so as long as $n \neq 6$ Csima-Sawyer guarantees there are at least $6(n-1)/13$ ordinary points. Thus, there must have been at least $\frac{6(n-1)}{13} - (\frac{n}{3} - \frac{15}{26}) > \frac{5n-6}{39}$ ordinary points off of l, contradicting our choice of l.

Finally, if $n = 6$ or 7 then $\lceil (5n-6)/39 \rceil = 1$ so if the Theorem were false all ordinary points of $\mathcal{A}$ would have to lie on a single pseudoline l. Removing l kills all the the at least 3 ordinary points and so creates at most 2 new ones, leaving at least one which all along must have been off of l. □

One is led to ask whether, in "Sylvester-critical" arrangements it is possible that a small number of pseudolines can actually contain all the ordinary points. We focus on the cases where there are $\leq n/2$ ordinary points in total. If $\mathcal{A}$ is an arrangement, we use the notation $|\mathcal{A}|$ to denote the number of lines or pseudolines in $\mathcal{A}$. A sub-collection $\mathcal{B} \subset \mathcal{A}$ **spans the ordinary points**, if all ordinary points of $\mathcal{A}$ are contained in $\mathcal{B}$.

Theorem 7 *Let $\{\mathcal{A}_i\}_{i=1}^{\infty}$ be a family of arrangements in $\mathbb{RP}^2$ with $|\mathcal{A}_i| \nearrow \infty$ and such that if $|\mathcal{A}_i| = n_i$ then the number of ordinary points in $\mathcal{A}_i$ is $\leq n_i/2$. The size of sub-collections $\mathcal{B}_i \subset \mathcal{A}_i$ spanning the ordinary points is unbounded.*

Proof. Suppose we could always find a spanning sub-collection of pseudolines of size $\leq k$. Consider $N = |\mathcal{A}|$ with $N \gg k^2$. Then almost all ordinary points are contained in the intersection of one of the k lines and a line of type $(1, 2+), (2, 0+), (3, 0+)$ etc. where the notation $x+$ means that the given line has at *least* x ordinary points attached.

We show that in all such cases we would end up with too many attachments. First, suppose all $m \leq n/2$ ordinary points result from intersections of the k lines with $(1, 2)$ lines. We use the notation $\sim$ to denote "on the order of." There would have to be m such lines, leaving $\sim n-m$ lines of the form $(0, 3+)$. But this would give rise to at least, on the order of $2m + 3(n - m) = 3n - m$ total attachments. But $3n - m > 4m$ since $m \leq n/2$, which contradicts the fact that we can have at most 4 attachments per ordinary point (Lemma 4).

Next suppose that all $m \leq n/2$ ordinary points result from intersections of the k lines with $(2, 0+)$ lines. There must be $\sim m/2$ of these, leaving on the order of $n-m/2$ of type $(0, 3+)$, yielding at least $\sim 9n/4$ attachments, which again is too many.

Intersections of the k lines with $(3, 0+)$ lines require even more attachments. It is easy to see that intersection of the k lines with lines of type $(2, 0)$ requires the fewest attachments, but as noted even these are not adequate. The theorem is thus established. $\square$

In light of the argument in the preceding proof, the following lemma is not too surprising. It is asserted to be true without proof in Borwein And Moser's 1990 survey article [1].

Lemma 8 *An arrangement of n not all coincident lines in $\mathbb{RP}^2$ with fewer than $n/2$ ordinary points must have at least one line of type $(2, 0)$.*

Proof. The idea is that without lines of type $(2, 0)$, if one first writes down the lines contributing to k ordinary points through intersection, then because of the constraint of at most 4 attachments per ordinary point (Lemma 4), one is forced to have at least $2k$ lines.

Suppose first that we had just lines of type $(1, 2+)$. It is plain that $2k$ of these form exactly k ordinary points, with no room for additional lines because of the at least $4k$ attachments. How about lines of type $(2, 1+)$? k of these lines contribute k ordinary points and any more than k additional lines of type $(0, 3+)$ would yield more than $4k$ total attachments. Finally, k lines of type $(3, 0+)$ yield $3k/2$ ordinary points, and $2k$ additional lines of type $(0, 3+)$ would already yield at least $6k$ attachments, which is capacity. It is easy to verify that lines of type $(4, 0+)$ and higher do worse. We thus conclude that an arrangement of lines as in the hypothesis of the lemma must contain lines of type $(2, 0)$. $\square$

In fact, the preceding argument shows that it is only lines of type $(2, 0)$ that really contribute to an arrangement of n lines having fewer than the *critical* number $(n/2)$ of ordinary points.

3 A Unified View of the $n = 7$ and $n = 13$ Examples

There are various ways to view the Kelly-Moser example of 7 lines with just 3 ordinary points [8] and the McKee example of 13 lines with just 6 ordinary points [2]. Both of these examples live in the real projective plane and provide the only known examples of n not all coincident lines in $\mathbb{RP}^2$ having less than $n/2$ ordinary points, and hence by projective duality the only known counter-examples to the Dirac-Motzkin $n/2$ conjecture. In the various treatments of this subject, these two examples are considered to be "sporadic" counterexamples. However, they can be viewed in a common light. First, realize the Kelly-Moser example as the vertices of two equilateral triangles glued together along a common edge and add the midpoint of the common edge, along with two points at infinity. Analogously, realize the McKee example, by taking two regular pentagons, repeating basically the same procedure, and this time adding four points at infinity. See Figure 3.

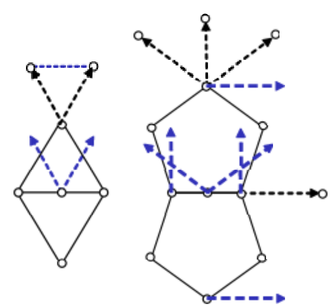

Figure 3: A common view of the Kelly-Moser and McKee examples.

Lenchner has created a program which he has dubbed "the Arrangement Workbench" which among other things, shows that analogous constructions obtained by gluing 7, 9, or 11-gons do not yield examples with fewer than $n/2$ ordinary lines.

4 Sylvester and its Sharp Dual on the Torus

The following example (Figure 4) shows that Sylvester's Theorem is false on the torus, T^2. We consider a "line" in T^2 to be any curve that is everywhere locally geodesic and maximal in extent.

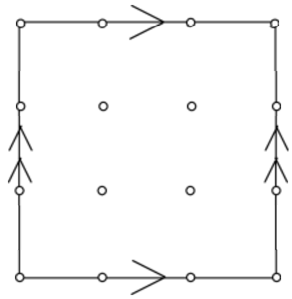

Figure 4: A collection of points on the torus with no ordinary line.

Since opposite edges are identified, note that of the 16 points drawn, there are only 9 distinct points. Any line through two of the points goes through precisely one additional point, so there are no ordinary lines.

The example in Figure 5 shows that the sharp dual of Sylvester's Theorem is also false on the torus. Here we call lines on T^2 parallel if they do not intersect.

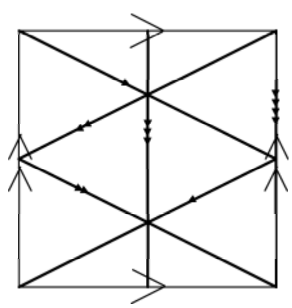

Figure 5: 4 not all parallel, not all coincident, lines on the torus with no ordinary point. (The lines are those with small arrows.)

5 Conclusion

The Dirac-Motzkin $n/2$ conjecture remains. We know that one way of trying to extend the $n = 7$ and $n = 13$ examples is not fruitful, but perhaps there is another. In addition we have the intriguing question of whether the asymptotic $5n/39$ bound is tight for the number of ordinary points avoiding any particular pseudoline. We conjecture that the $5n/9$ bound is not tight, and in fact should be the same asymptotic bound as for the number of total ordinary points.

Finally we have considered Sylvester's Theorem and its *sharp* dual on the Torus. How about the "*weak*" dual: Must a set of lines on the torus, no *two* of which are parallel contain an ordinary point? How about the situation for Sylvester and its various duals on other simple surfaces, such as the cylinder and Möbeus Strip? Is there a reasonable notion of duality for any of these surfaces?

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