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Synchronization and Convergence of Linear Dynamics in Random Directed Networks

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Abstract

Recently, methods in stochastic control are used to study the synchronization properties of a nonautonomous discrete-time linear system $x(k+1) = G(k)x(k)$ where the matrices $G(k)$ are derived from a random graph process. The purpose of this paper is to extend this analysis to directed graphs and more general random graph processes. Rather than using Lyapunov type methods, we use results from the theory of inhomogeneous Markov chains in our analysis. Sufficient conditions are derived that depend on the types of graphs that have nonvanishing probabilities. For instance, if a scrambling graph occurs with nonzero probability, then the system synchronizes.

I. Introduction

Synchronizing dynamics among coupled nonlinear systems where the coupling topology is expressed as a graph is an active area of research [1]–[7]. In recent years, in the context of agreement and consensus problems, there is increased interest to study the case where the dynamics are linear [8]–[10]. In [11] a discrete-time nonautonomous linear system $x(k+1) = G(k)x(k)$ is studied where the matrices $G(k)$ are derived from a random graph process. It was found that $x(k)$ converges to the subspace spanned by $(1, \ldots, 1)^T$ in probability if each edge is chosen with the same probability. The purpose of this paper is to extend this to directed graphs and more general random graph models. In contrast to [11] where Lyapunov methods from stochastic control are used, we use results from the theory of inhomogeneous Markov chains. This approach allows us to easily obtain results for general random graph models.

II. Problem formulation

We consider the following nonautonomous discrete-time linear dynamical system:

$$x(k+1) = (G(k) \otimes D(k))x(k) + \mathbf{1} \otimes v(k) \tag{1}$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $x_i \in \mathbb{R}^m$, $v(k) \in \mathbb{R}^m$, and $\mathbf{1} = (1, \ldots, 1)^T$. The matrix $G \otimes D$ is the Kronecker product or tensor product of the matrices G and D. We assume that for each k, $\|D(k)\| \leq 1$ and $G(k)$ is an n by n stochastic matrix (i.e $G(k)$ is a nonnegative matrix whose rows sum to 1). We say Eq. (1) *synchronizes* if $\|x_i(k) - x_j(k)\| \rightarrow 0$ as $k \rightarrow \infty$ for all i, j. First we show that the maximum distance between the x_i's is nonincreasing:

Theorem 2.1: Let $\kappa(k) = \max_{i,j} \|x_i(k) - x_j(k)\|$. Then $\kappa(k+1) \leq \kappa(k)$.

Proof: Let $S(k) = \{x_1(k), \ldots, x_n(k)\}$ and $T(k) = \{D(k)x_1(k) + v(k), \ldots, D(k)x_n(k) + v(k)\}$. Note that $\kappa(k)$ is the diameter of the convex hull of $S(k)$. Since $x_i(k+1)$ is a convex combination of elements of $T(k)$, it is in the convex hull of $T(k)$. Thus means that the convex hull of $S(k+1)$ is a subset of the convex hull of $T(k)$. Since $\|D(k)\| \leq 1$, the diameter of the convex hull of $T(k)$ is less than or equal to the diameter of the convex hull of $S(k)$ and thus $\kappa(k+1) \leq \kappa(k)$. $\square$

III. Scrambling stochastic matrices and Hajnal's inequality

We next summarize some results from the theory of ergodic inhomogeneous Markov chains which are useful in deriving sufficient conditions for synchronization.

Definition 3.1: A matrix A is *scrambling* if for each pair of indices (i, j) there exists k such that A_{ik} and A_{jk} are both nonzero.

Definition 3.2: For an n by n matrix A, the ergodicity coefficient $\mu(A)$ is defined as

$$\mu(A) = \min_{j,k} \sum_i \min(A_{ji}, A_{ki})$$

For stochastic matrices, it is clear that $0 \leq \mu(A) \leq 1$ with $\mu(A) > 0$ if and only if A is scrambling.

Definition 3.3: For a n by n matrix A, define $\delta(A)$ as

$$\delta(A) = \frac{1}{2} \max_{i,j} \sum_k |A_{ik} - A_{jk}|$$

If A is stochastic, it is easy to see that $\delta(A) = \max_{i,j} \sum_k \max(0, A_{ik} - A_{jk}) \geq \max_{i,j,k}(A_{ik} - A_{jk})$.

Theorem 3.1 (Hajnal's inequality [12], [13]): If A, B are stochastic matrices, then $\delta(AB) \leq (1 - \mu(A))\delta(B)$.

IV. A SYNCHRONIZATION CRITERION

We consider the set of n by n stochastic matrices along with a probability measure on this set. We study synchronization of Eq. (1) when each $G(k)$ is taken independently from this set using the corresponding probability measure.

Definition 4.1: Eq. (1) synchronizes in probability if for any $x(0)$ and any $\epsilon > 0$,

$$Pr(\kappa(x(k)) \geq \epsilon) \to 0$$

as $k \to \infty$.

Theorem 4.1: If there exists a compact set of stochastic scrambling matrices H such that $Pr(H) > 0$, then Eq. (1) synchronizes in probability.

Proof: Pick $\epsilon, \eta > 0$. Since $Pr(H) > 0$, for any N there exists K such that for all $k \geq K$, at least N matrices in the set $\{G(2), \ldots, G(k)\}$ belongs to H with probability at least $1 - \eta$. Let $B = G(k)G(k-1)\ldots G(1)$. Let $\mu = \inf_{X \in H} \mu(X)$. By Theorem 3.1, $\delta(B) \leq (1-\mu)^N \delta(G(1))$. Since H is compact, $\mu > 0$ and this means that $\delta(B)$ can be made arbitrarily small for large enough N. Since $x(k+1) = (B \otimes \Pi_{i=1}^{k} D(i))x(1) + \mathbf{1} \otimes v'(k)$ for some vector $v'(k)$, $\|x_i(k+1) - x_j(k+1)\| = \|\sum_l (B_{il} - B_{jl})\Pi_{i=1}^{k} D(i) x_l(1)\| \leq \delta(B) \sum_l \|x_l(1)\|$. This implies that with probability $1 - \eta$, $\kappa(x(k+1)) \leq \epsilon$ for large enough N. $\square$

V. GRAPHS AND DIGRAPHS

A directed graph (digraph) $\mathcal{G} = (V, E)$ consists of a vertex set V and an edge set $E \subset V^2$. We also consider (undirected) graphs where each edge is a set of two vertices. In addition, we consider weighted graphs where each edge has a nonzero weight. An undirected graph will be viewed as a digraph by replacing each undirected edge with two directed edges of half the weight and opposite orientation. For two graphs $\mathcal{G}_1 = (V, E_1)$ and $\mathcal{G}_2 = (V, E_2)$ with the same vertex set, the union is defined as $\mathcal{G}_1 \cup \mathcal{G}_2 = (V, E_1 \cup E_2)$.

Definition 5.1: A digraph is called *scrambling* if for any vertices i and j there exists a vertex k such that there is an edge from i to k and an edge from j to k.

The graph of a matrix A is defined as the weighted graph with an edge $e = (i, j)$ from vertex i to vertex j of weight A_{ij} if and only if $A_{ij} \neq 0$. It is clear that a matrix is scrambling if and only if its graph is scrambling. An unweighted graph is simple if it has no self-loops and multiple edges between vertices. If an edge (i, j) exists in a digraph, then i is called the parent of j and j is called the child of i. A spanning directed tree is a digraph with n vertices and $n-1$ edges with a vertex, called the root vertex, that has directed paths to every other vertex. A *reversal* of a digraph is obtained by reversing the orientation of all the edges. The following result shows when products of matrices are scrambling.

Theorem 5.1: If A_i are nonnegative matrices with positive diagonal elements such that for each i the reversal of the graph of A_i contains a spanning directed tree, then $A_1 A_2 \ldots A_{n-1}$ is scrambling.

Proof: Let $B(m) = A_1 A_2 \ldots A_m$. For $i \neq j$, we need to show that $B(n-1)_{ik} \neq 0$ and $B(n-1)_{jk} \neq 0$ for some k. Let $C_i(m)$ be the children of vertex i in the graph of $B(m)$. Since A_i has positive diagonal elements, $i \in C_i(m)$ and the graph of $B(m)$ (ignoring the weights) is a subgraph of the graph of $B(m+1)$. Suppose that $C_i(m)$ does not intersect $C_j(m)$. Then the root of the directed tree r in the reversal of the graph of A_{m+1} is either not in $C_i(m)$ or not in $C_j(m)$. Suppose r is not in $C_j(m)$. Since all vertices in $C_j(m)$ has a directed path to r in the graph of A_{m+1}, at least one vertex in $C_j(m)$ must have a child in the graph of A_{m+1} outside of $C_j(m)$. Since $C_j(m+1)$ is $C_j(m)$ plus the children of $C_j(m)$ in the graph of A_{m+1}, this means that $C_j(m+1)$ is strictly larger than $C_j(m)$. Similarly, if r is not in $C_i(m)$, then $C_i(m+1)$ is strictly larger than $C_i(m)$. Since the reversal of the graph of A_1 contains a spanning directed tree, one of the two sets $C_i(1)$, $C_j(1)$ has at least two elements (recall that $i \in C_i(m)$), and this means that $C_i(n-1)$ must intersect $C_j(n-1)$, say $k \in C_i(n-1) \cap C_j(n-1)$ which is the k we are looking for. $\square$

In Theorem 5.1, a product of matrices A_i of length $n-1$ results in a scrambling matrix. Using the example

$$A_i = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & \ddots & \\ & & & \ddots & \ddots \\ & & & & 1 \end{bmatrix}$$

for all i, we see that Theorem 5.1 is not true if we replace the length $n-1$ with something smaller.

A. Random graphs

For a given n, a random graph model assigns a probability measure on the set of graphs (or digraphs) with n vertices. In particular, two of the most well known *simple* random graph models are [14]:

$G(n,p)$: Each undirected edge is chosen with probability p. Thus a graph with m edges has probability $p^m(1-p)^{\frac{n(n-1)}{2}-m}$ where $\frac{n(n-1)}{2}$ is the total number of possible edges.

$G(n,M)$: Each graph of n vertices and M edges is given equal probability whereas the rest of the graphs of n vertices has probability 0.

We can generalize these models to simple directed graphs:

$G_d(n,p)$: Each directed edge is chosen with probability p. A graph with m edges has probability $p^m(1-p)^{n(n-1)-m}$.

$G_d(n,M)$: Each digraph of n vertices and M edges is given equal probability whereas the rest of the graphs of n vertices has probability 0.

VI. Synchronization in random networks

Suppose now that to each unweighted graph $\mathcal{G}$, there corresponds a stochastic matrix G such that the graph of G ignoring the weights is $\mathcal{G}$. We then consider Eq. (1) where at each k, a graph $\mathcal{G}$ is chosen independently from a random graph process and $G(k)$ is set to be the stochastic matrix corresponding to $\mathcal{G}$.

Corollary 6.1: If the random graph model is $G(n,p)$ with $p>0$, then Eq. (1) synchronizes in probability.

Proof: Follows from Theorem 4.1 and the fact that the complete graph, which is scrambling, has nonzero probability in $G(n,p)$. □

Corollary 6.1 was shown in [11] for the case $D(k)=1$ using stochastic Lyapunov theory. We show here how it follows easily from Theorem 4.1.

Corollary 6.2: If the random graph model is $G(n,M)$ with $M \geq 2n-3$, then Eq. (1) synchronizes in probability.

Proof: Consider the simple graph $\mathcal{G}$ with $2n-3$ edges as shown in Figure 1. It is clear that every pair of vertices has a common child, and therefore this graph is scrambling. Since any graph with $\mathcal{G}$ as a subgraph is scrambling, the result follows from Theorem 4.1. □

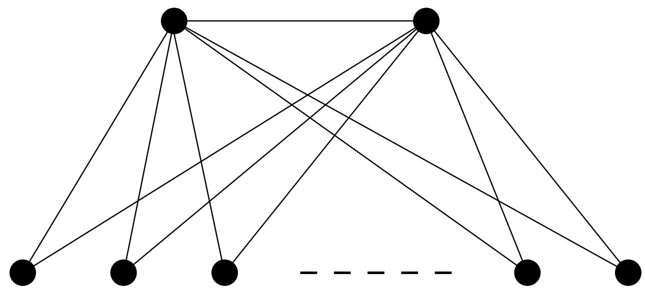

Fig. 1. A simple undirected scrambling graph.

Similar arguments show that these results hold for the digraph case as well.

Corollary 6.3: If the random graph model is $G_d(n,p)$ with $p>0$, then Eq. (1) synchronizes in probability.

Corollary 6.4: If the random graph model is $G_d(n,M)$ with $M \geq 2n-1$, then Eq. (1) synchronizes in probability.

Proof: The proof is the same as Corollary 6.2 except that we consider the simple scrambling directed graph with $2n-1$ edges in Figure 2. □

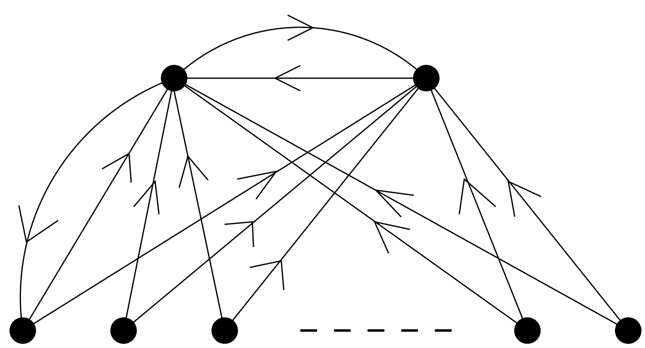

Fig. 2. A simple directed scrambling graph.

VII. Synchronization without the scrambling condition

In this section, we study a synchronization condition that does not require the existence of a scrambling matrix with nonvanishing probability. Let P be the set of stochastic matrices with positive diagonal elements.

Theorem 7.1: Consider n_p compact sets of matrices $S_i \subset P$ with $Pr(S_i) > 0$ for each $i = 1, \ldots, n_p$. Suppose that if $G_i \in S_i$, then the reversal of the union of the digraphs G_i, $i = 1, \ldots, n_p$, contains a spanning directed tree. Then Eq. (1) synchronizes in probability.

Proof: The proof is similar to Theorem 4.1. In this case we choose K large enough such that a sequence of matrices $G_{1,1}, \ldots, G_{1,n_p}, G_{2,1}, \ldots, G_{2,n_p}, \ldots, G_{n-1,1}, \ldots G_{n-1,n_p}$ where $G_{i,j} \in S_j$ can be found as a subsequence of $G(2), \ldots, G(K)$ with probability at least $1 - \eta$. Since the reversal of $G_{i,1} G_{i,2} \ldots G_{i,n_p}$ contains a spanning directed tree, by Theorem 5.1 $\Pi_{i,j} G_{i,j}$ is scrambling. Since the sets S_i are compact, $\mu(\Pi_{i,j} G_{i,j})$ is bounded away from 0. By choosing K even larger, we can make this happen at least N times. The rest of the proof is the same as that of Theorem 4.1. □

If the condition that $S_i \subset P$ is omitted, then Theorem 7.1 is not true as the following example illustrates. Let $G(k) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $D(k) = I$, and $v(k) = 0$ for all k. In this case the states oscillate with period 2.

Suppose now that to each simple graph $\mathcal{G}$, there corresponds a matrix $G \in P$ such that the graph of G minus the diagonal elements and ignoring the weights is equal to $\mathcal{G}$.

Corollary 7.1: If the random graph model is $G(n, M)$ or $G_d(n, M)$ with $M > 0$, then Eq. (1) synchronizes in probability.

Proof: Follows from Theorem 7.1 by choosing each set S_i to consist of the matrix corresponding to a single graph of M edges. □

Theorems 4.1 and 7.1 are applicable to other random graph models besides the classical models in Section V-A. In particular, it is applicable to random graph models such as small world networks [15], scale free networks [16] or random geometric graphs [17].

VIII. CONCLUSION

We have shown how results in inhomogeneous Markov chains can be useful in establishing synchronization in nonautonomous linear systems where the coupling topology at each time is drawn from a random graph model. We show that synchronization is possible if the probability of certain types of graphs is nonzero. In particular, the system synchronizes in probability if the probability of a scrambling graph is nonzero. If we impose additional conditions, the system synchronizes if a set of graphs whose union contains a spanning directed graph in its reversal occurs with nonzero probability.

REFERENCES

[1] C. W. Wu and L. O. Chua, “Application of graph theory to the synchronization in an array of coupled nonlinear oscillators,” *IEEE Transactions on Circuits and Systems–I: Fundamental Theory and Applications*, vol. 42, no. 8, pp. 494–497, Aug. 1995.

[2] C. W. Wu, “Synchronization in arrays of chaotic circuits coupled via hypergraphs: static and dynamic coupling,” in *Proceedings of the 1998 IEEE Int. Symp. Circ. Syst.*, vol. 3. IEEE, 1998, pp. III–287–290.

[3] M. Barahona and L. M. Pecora, “Synchronization in small-world systems,” *Physical Review Letters*, vol. 89, no. 5, p. 054101, 2002.

[4] X. F. Wang and G. Chen, “Synchronization in small-world dynamical networks,” *International Journal of Bifurcation and Chaos*, vol. 12, no. 1, pp. 187–192, 2002.

[5] V. N. Belykh, I. V. Belykh, and M. Hasler, “Connection graph stability method for synchronized coupled chaotic systems,” *Physica D*, vol. 195, pp. 159–187, 2004.

[6] C. W. Wu, “Synchronization in systems coupled via complex networks,” in *Proceedings of IEEE ISCAS 2004*, 2004, pp. IV–724–727.

[7] ——, “Synchronization in networks of nonlinear dynamical systems coupled via a directed graph,” *Nonlinearity*, vol. 18, pp. 1057–1064, 2005.

[8] A. Jadbabaie, J. Lin, and A. S. Morse, “Coordination of groups of mobile autonomous agents using nearest neighbor rules,” *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[9] R. Olfati-Saber and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[10] C. W. Wu, “Agreement and consensus problems in groups of autonomous agents with linear dynamics,” in *Proceedings of 2005 IEEE International Symposium on Circuits and Systems*, 2005, pp. 292–295.

[11] Y. Hatano and M. Mesbahi, “Agreement over random networks,” *43rd IEEE Conference on Decision and Control*, pp. 2010–2015, 2004.

[12] J. Hajnal, “Weak ergodicitiy in non-homogeneous Markov chains,” *Proc. Cambridge Philos. Soc.*, vol. 54, pp. 233–246, 1958.

[13] A. Paz and M. Reichaw, “Ergodic theorems for sequences of infinite stochastic matrices,” *Proc. Cambridge Philos. Soc.*, vol. 63, pp. 777–784, 1967.

[14] B. Bollobás, *Random Graphs*, 2nd ed. Cambridge University Press, 2001.

[15] D. J. Watts and S. H. Strogatz, “Collective dynamics of ‘small-world’ networks,” *Nature*, vol. 393, pp. 440–442, 1998.

[16] A.-L. Barabási, R. Albert, and H. Jeong, “Scale-free characteristics of random networks: the topology of the world wide web,” *Physica A*, vol. 281, pp. 69–77, 2000.

[17] M. Penrose, *Random Geometric Graphs*. Oxford University Press, 2003.