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Relationships between Limit Cycles and Algebraic Invariant Curves for Quadratic Systems

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Relationships between limit cycles and algebraic invariant curves for quadratic systems

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Abstract

For certain classes of quadratic systems having the invariant algebraic curve $\varphi = 0$, we prove that all the limit cycles must be contained in $\varphi = 0$.

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1 Introduction and statement of the main results

We shall study polynomial vector fields in $\mathbb{R}^2$ defined by systems

$$\begin{aligned} \dot{x} &= p(x,y), \\ \dot{y} &= q(x,y), \end{aligned} \tag{1}$$

where p, q are coprime polynomials of degree 2, i.e.

$$p(x,y) = \sum_{i,j=0}^{2} p_{i,j} x^i y^j, \qquad q(x,y) = \sum_{i,j=0}^{2} q_{i,j} x^i y^j$$

We shall call these systems *quadratic systems.*

The object of our study will be the limit cycles of such systems, mainly the algebraic ones, i.e. the limit cycles contained in the zero set of some polynomial

$$\varphi(x,y) = \sum_{i,j=0}^{n} = \varphi_{i,j} x^i y^j.$$

It is well–known that each limit cycle of a polynomial vector field must surround at least one critical point, and for a quadratic system inside each limit cycle there must be precisely one critical point of focus type, see [5].

The algebraic curve $\varphi(x, y) = 0$ is an invariant algebraic curve of system (1) if and only if there exists a polynomial $\kappa = \kappa(x, y)$ satisfying

$$p\frac{\partial\varphi}{\partial x} + q\frac{\partial\varphi}{\partial y} - \kappa\varphi = 0. \tag{2}$$

The polynomial κ is called a *cofactor* of the curve $\varphi = 0$. In case of quadratic systems the degree of the cofactor can be at most 1. An invariant algebraic curve $\varphi = 0$ is called *irreducible* if the polynomial φ is irreducible.

A trajectory γ of system (1) is a *limit cycle* if it is a periodic nonconstant trajectory, and if there are no other periodic trajectories in some neighborhood of γ. The orbit γ is an *algebraic limit cycle* of system (1) if it is a limit cycle and if it is contained in some irreducible algebraic invariant curve $\varphi = 0$ of system (1). The *degree* of an algebraic limit cycle γ is the degree of φ.

Until now only seven different families of algebraic limit cycles for quadratic systems have been found: one of degree 2 in 1958 [12], and four of degree 4 (one of Yablonskii [13] in 1966, one of Filiptsov [9] in 1973, and two in [1] in 2001). It is known that there are no algebraic limit cycles of degree 3, see Evdokimenco [6, 7, 8] from 1970 to 1979, or see Theorem 11 of [3] for a short proof. It has been proved in [2], that there are no other algebraic limit cycles of degree 4. Recently, two new examples of algebraic limit cycles of degrees 5 and 6 have been found, see [4]. For a classification of all quadratic systems which can have an algebraic limit cycle see [10]. New, interesting examples of invariant algebraic curves of degrees 5 and 6 have been recently found in [11].

Our first result shows the relation between the cofactor of an invariant algebraic curve and limit cycles for a quadratic system.

Theorem 1 *Assume that system (1), with an invariant algebraic curve $\varphi = 0$ with cofactor κ, has a limit cycle γ. Then, κ is nonconstant and γ must intersect the line $\kappa = 0$.*

The next four theorema study the relationships between limit cycles and invariant algebraic curves for quadratic systems.

Theorem 2 *Let $\varphi = 0$ be an invariant algebraic curve for system (1) with cofactor $\kappa(x, y)$, $\deg\varphi > 1$. If the straight line $\kappa = 0$ is an isocline of system (1), then all the limit cycles of system (1) are contained in the set $\varphi = 0$.*

Theorem 2 together with Lemma 9 are equivalent to Theorem 1 of [1]. Nevertheless, the assumption of Theorem 2 is easier to verify and our proof is easier.

Theorem 3 *Let $\varphi = 0$ be an invariant algebraic curve for system (1) with cofactor $\kappa(x, y)$, $\deg\varphi > 1$, and assume that there is at least one critical point on the line $\kappa = 0$. Then, the system does not have limit cycles, or all the limit cycles of the system are contained in the set $\varphi = 0$, or it can be transformed into the normal form*

$$\begin{aligned} \dot{u} &= u^2 + v(au + \theta v + c), \\ \dot{v} &= \tilde{Q}(u, v). \end{aligned} \tag{3}$$

If $\theta \geq 0$, then all the limit cycles of system (1) are contained in the set $\varphi = 0$.

We do not know any nice geometrical interpretation of the quantity θ.

Theorem 4 *Let $\varphi = 0$ be an invariant algebraic curve for system (1) with cofactor $\kappa(x, y)$, $\deg \varphi > 2$. Assume that there is at least one critical point of the system on the line $\kappa = 0$. Moreover, assume that the invariant algebraic curve $\varphi = 0$ intersects the line at infinity at precisely one point, and that this point of intersection is a $2:1$ resonant node having the smallest eigenvalue associate to the infinite direction. Then, all the limit cycles of the system are contained in the set $\varphi = 0$.*

Theorem 5 *Let $\varphi = 0$ be an invariant algebraic curve for system (1) with cofactor $\kappa(x, y)$, $\deg \varphi > 2$. Assume that there is at least one critical point of the system on a line $\kappa = 0$. Moreover, assume that the invariant algebraic curve $\varphi = 0$ intersects the line at infinity at precisely one point. Then, the system does not have limit cycles, or all the limit cycles of the system are contained in the set $\varphi = 0$, or it can be transformed into the normal form*

$$\begin{aligned} \dot{x} &= x + y + xy, \\ \dot{y} &= Cx + Dy + dx^2 + exy + fy^2, \end{aligned} \tag{4}$$

If $(f-2)(f-2+d-e-C) \geq 0$, then all the limit cycles of the system are contained in the set $\varphi = 0$.

Theorem 4 is a particular case of Theorem 5. We state Theorem 4 as a separate result because its assumptions are easier to verify then the assumptions of Theorem 5, because they do not require the transformation of a system to the normal form (4).

All known examples of quadratic systems with algebraic limit cycles satisfy the assumptions of one of the above theorema. Namely, Yablonskii system satisfies the assumptions of Theorem 4, Filiptsov system satisfies the assumptions of Theorem 5. The remaining systems of degrees 4 and the unique system of degree 2 with algebraic limit cycles satisfy the assumptions of Theorem 2. Finally, the systems with algebraic limit cycles degrees 5 and 6 also satisfy the assumptions of Theorem 4.

The above results together with all the known results about algebraic limit cycles of quadratic systems motivate the following conjecture:

Conjecture *If a quadratic system has an algebraic limit cycle, then this is the unique limit cycle of the system.*

The rest of the paper is devoted to proofs of Theorema 1, 2, 3, 4 and 5.

2 Preliminaries.

We shall call the point (x, y) a *critical point of the system (1)* if and only if $p(x, y) = q(x, y) = 0$. We shall call the point (x, y) a *critical point of a function φ* if and only if $\frac{\partial \varphi}{\partial x}(x, y) = \frac{\partial \varphi}{\partial y}(x, y) = 0$.

Now immediately from the definitions it follows

Proposition 6 *All the critical points of system (1) and all the critical points of φ are contained in the union of sets $\{\kappa = 0\} \cup \{\varphi = 0\}$.*

Using that we obtain

Proposition 7 *Let γ be a limit cycle of the quadratic system (1) contained in the invariant algebraic curve $\varphi = 0$. Then γ must intersect the line $\kappa = 0$.*

Proof: Let $\mathcal{D}$ denote the compact region having γ as the boundary. $\mathcal{D}$ is a compact set, so there exist points $X, Y \in \mathcal{D}$ at which φ has a minimum and maximum respectively. Now either $\varphi(X) = \varphi(Y) = 0$, and $\varphi|_{\mathcal{D}} \equiv 0$ and the assertion is trivially satisfied, or we may assume that $\varphi^2(X) + \varphi^2(Y) \neq 0$. We consider $\varphi(X) \neq 0$, if $\varphi(X) = 0$ then we can treat the case $\varphi(Y) \neq 0$ in a similar way. In that case X is contained in the set $\mathcal{D} \setminus \gamma$. By standard facts from analysis X must be a critical point of φ, and of course it is located in the interior of $\mathcal{D}$. As $\varphi(X) \neq 0$, by Proposition 6 $\kappa(X) = 0$. The curve γ is a Jordan closed curve, so the line $\kappa = 0$ must intersect it. ∎

Proposition 8 *Suppose that system (1) has an invariant algebraic curve $\varphi = 0$ and a smooth nonconstant function H defined in the complement of the set $\varphi = 0$ satisfying either*

(i) *$\dot{H}$ has a fixed sign except for a finite set of points, or*

(ii) *$\dot{H} \equiv 0$.*

Then all the limit cycles of (1) are contained in the set $\varphi = 0$.

Proof: Let γ be a limit cycle of system (1) not contained in $\varphi = 0$. The curve $\varphi = 0$ is invariant, so γ is disjoint with $\varphi = 0$, and the function H is well–defined in the neighborhood of γ. Let $(x(t), y(t))$, $t \in [0, T]$ be a parametrization of γ. We have:

$$0 = \int_0^T \dot{H}(x(t), y(t))dt,$$

but if (i) holds $\dot{H}$ is either negative, or positive almost everywhere, which leads to a contradiction. In case (ii) H is a first integral defined in the neighborhood of γ, so γ cannot be a limit cycle. ∎

3 Isoclines

Curve γ is an *isocline* of system (1) if and only if there exist constants ζ, ξ not simultaneously equal to zero for which the equality holds

$$(\xi p(x, y) + \zeta q(x, y))|_{\gamma} \equiv 0.$$

We consider our system as a system on a complex projective space $\mathbb{C}P^2$.

$$P(X, Y, Z)\frac{\partial}{\partial X} + Q(X, Y, Z)\frac{\partial}{\partial Y} \tag{5}$$

where

$$P(X, Y, Z) = Z^2 p(\frac{X}{Z}, \frac{Y}{Z}) \quad Q(X, Y, Z) = Z^2 q(\frac{X}{Z}, \frac{Y}{Z})$$

We define the real part of a line K in $\mathbb{C}P^2$ as $\kappa = \{K \cap [X : Y : 1], \ X, Y \in \mathbb{R}\}$.

Lemma 9 *Let K be a projective line. Then the straight line $\kappa = 0$ is an isocline of system (1) if and only if there exist two points A_1, A_2 on K such that $P(A_1) = Q(A_1) = P(A_2) = Q(A_2) = 0$.*

Proof: We may assume that the line κ has the equation $y = 0$. We define

$$g_1(x) = p(x, 0), \qquad g_2(x) = q(x, 0).$$

The polynomials P, Q restricted to the line K become $P(X, 0, Z) = Z^2 g_1(X/Z)$, $Q(X, 0, Z) = Z^2 g_2(X/Z)$. In case one or both of them vanish identically the assumption and the assertion of the lemma are satisfied trivially. Therefore, from now on we shall assume that none of the functions g_1 and g_2 are identically zero, so $P(X, 0, Z)$, $Q(X, 0, Z)$ are homogenous of degree 2. The line κ is an isocline if and only if there holds $\xi g_1 + \zeta g_2 \equiv 0$ which may take place if and only if $\deg g_1 = \deg g_2 = d$ and all the roots of g_1 and g_2 coincide. Therefore we have three easy cases to consider

$d = 2$: We take $A_1 = [x_1 : 0 : 1]$, $A_2 = [x_2 : 0 : 1]$, where x_1, x_2 are two (not necessarily distinct) common roots of g_1, g_2.

$d = 1$: We take $A_1 = [x_0 : 0 : 1]$, $A_2 = [1 : 0 : 0]$, where x_0 is equal to a common root of g_1, g_2 on a real line $[X : 0 : 1]$, $X \in I\!R$.

$d = 0$: We take $A_1 = A_2 = [1 : 0 : 0]$–a double common root of P, Q on the projective line at infinity $[X : Y : 0]$.

∎

The condition of Lemma 9 means the existence of two critical points of system (5) on the line K, actually some of the points can be at infinity. We do not assume that $A_1 \neq A_2$, but we count the points with their multiplicities as zeroes of P, Q.

4 Proofs of theorema

Proof (of Theorem 1): We already know, that all the limit cycles contained in the set $\varphi = 0$ must intersect the line $\kappa = 0$ (Proposition 7). It remains to prove, that it is so for the ones not contained in $\varphi = 0$. We define $H(x, y) = \ln |\varphi|$. Now $\dot{H} = \kappa$. In case κ is constant the assumptions of Proposition 8 are satisfied, so there are no limit cycles apart from the ones contained in the set $\varphi = 0$. Now suppose that κ is nonconstant and that γ is a limit cycle not intersecting $\kappa = 0$. Then $\dot{H}|_\gamma$ has a fixed sign and we have a contradiction using the arguments from the proof of Proposition 8. ∎

Proof (of Theorem 2): After a suitable linear change of variables (translation and a rotation) we may assume that $\kappa(x, y) = y$. The assumptions of the theorem mean now, that the system takes in these coordinates the form:

$$\dot{x} = \zeta\tilde{g}(x) + y(\tilde{a}x + \tilde{b}y + \tilde{c}),$$
$$\dot{y} = \xi\tilde{g}(x) + y(\tilde{d}x + \tilde{e}y + \tilde{f}),$$

where $\tilde{g}$ is a quadratic polynomial. In case $\xi = 0$ the the line $y = 0$ is invariant and by Theorem 1 the system cannot have limit cycles.

If $\xi \neq 0$ we apply a change of coordinates $u = \xi x - \zeta y$, $v = y$, and the system becomes

$$\begin{aligned}\dot{u} &= v(au + bv + c),\\ \dot{v} &= \xi g(u) + v(du + ev + f).\end{aligned}$$

Let $\psi(u, v) = \varphi(x(u, v), y(u, v))$. Of course $\dot{\psi} = v\psi$. Now, if $a \neq 0$ we define $H(u, v) = (u + \dfrac{c}{a})\psi^{-a}(u, v)$ outside $\psi(U, v)) = 0$. The condition deg$\varphi > 1$ implies that H is nonconstant. We have

$$\dot{H} = [v(au + bv + c) - av\left(u - \frac{c}{a}\right)]\psi^{-a} = bv^2\psi^{-a}.$$

So, either $\dot{H}$ has a fixed sign (for $b \neq 0$), or H is a first integral (for $b = 0$).

In case $a = 0$ we put $H(u, v) = u - c \ln \psi$ and $\dot{H} = bv^2$. The theorem follows now from Proposition 8. ∎

Proof (of Theorem 3): We once again put $\kappa(x, y) = y$ and we move the critical point to the origin, obtaining:

$$\begin{aligned}\dot{x} &= x(\zeta x - x_1) + y\alpha(x, y),\\ \dot{y} &= x(\xi x - x_2) + y\beta(x, y),\end{aligned}$$

where α, β are some linear functions in (x, y), and x_1, x_2 are constant. If $\xi x_1 = \zeta x_2$, then the line $\kappa = 0$ is an isocline and by Theorem 2 all the limit cycles of the system are contained in the set $\varphi = 0$. We therefore assume $\xi x_1 \neq \zeta x_2$ and apply a change of coordinates $u = x_2 x - x_1 y$, $v = y$. We may assume that $x_2 \neq 0$. Otherwise, either $y = 0$ is an invariant line if $\xi = 0$, or $\xi \neq 0$ and $\dot{y}|_{y=0} = \xi x^2$ is always nonnegative or nonpositive, so the limit cycle cannot intersect $y = 0$, and Theorem 1 guarantees us, that the system cannot have any limit cycles.

After a convenient re-scaling the system in the new coordinates (u, v) becomes

$$\begin{aligned}\dot{u} &= u^2 + v(au + \theta v + c),\\ \dot{v} &= \tilde{Q}(u, v).\end{aligned}$$

As before we define $\psi(u, v) = \varphi(x(u, v), y(u, v))$ and consequently $\dot{\psi} = v\psi$. Now in case $a \neq 0$ we take $H(u, v) = (u - \frac{\theta}{a})\psi^{-a}$ and we obtain

$$\dot{H} = [u^2 + v(au + \theta v + c) - av\left(u - \frac{c}{a}\right)]\psi^{-a} = [u^2 + \theta v^2]\psi^{-a}$$

does not change sign in the complement of $\psi = 0$. In case $a = 0$ we take $H(u, v) = u - c \ln \psi$ and $\dot{H}(u, v) = u^2 + \theta v^2$. The theorem follows now from Proposition 8. ∎

For the proof of remaining two theorema we shall need the following lemma.

Lemma 10 *Let $\varphi = 0$ be an invariant algebraic curve for system (1) with cofactor $\kappa(x, y)$. Assume that there is at least one critical point of the system on $\kappa = 0$. Assume moreover that the invariant algebraic curve $\varphi = 0$ intersects the line at infinity at precisely one point, and that the line $\kappa = 0$ and the curve $\varphi = 0$ do not intersect at infinity. Then the system can be transformed into the normal form*

$$\begin{aligned}\dot{x} &= Ax + By + xy,\\ \dot{y} &= Cx + Dy + dx^2 + exy + fy^2,\end{aligned}$$

with $A, B \in \{0, 1\}$, while the cofactor has the form $\kappa(x, y) = ny$. In these coordinates $\varphi(x, y) = x^n + \ldots$, where $\ldots$ denote the terms of degree $n - 1$ and smaller.

Proof: First we choose a linear change of coordinates which transforms κ into the form $\kappa(x,y) = ny$, and that puts the critical point at the origin (recall, that we assume that there is at least one finite critical point at the line $\kappa(x,y) = 0$). Let the (unique) point of intersection of the line at infinity and the curve $\varphi = 0$ be $[\zeta : \xi : 0]$. We apply the change of coordinates $x \to x - \frac{\zeta}{\xi}y$ (by assumption $\kappa = 0$ and $\varphi = 0$ do not intersect at infinity, so $\xi \neq 0$), so the point of intersection becomes $[0 : 1 : 0]$. Now $\varphi = x^n + \ldots$ and the system has the form

$$\begin{aligned} \dot{x} &= Ax + By + ax^2 + bxy, \\ \dot{y} &= Cx + Dy + dx^2 + exy + fy^2. \end{aligned}$$

We have $\dot{\varphi} - \kappa\varphi = nx^{n-1}(ax^2 + bxy) - nyx^n + \ldots$, where $\ldots$ denote terms of degree at most n, so there must hold $a = 0$, and $b = 1$, thus

$$\begin{aligned} \dot{x} &= Ax + By + xy, \\ \dot{y} &= Cx + Dy + dx^2 + exy + fy^2. \end{aligned}$$

Finally, by scaling the variables x, y and the time we can put both A, B equal to 0 or 1, depending on if they are 0, or nonzero. ∎

Proof (of Theorem 4): Obviously every point of intersection of $\varphi = 0$ and the infinity must be a critical point of system (5). If the line $\kappa = 0$ and the curve $\varphi = 0$ intersect at infinity, there must be one finite (by assumption) and one infinite critical point on the line $\kappa = 0$, and the assertion follows from Lemma 9 and Theorem 2.

It remains to prove the theorem when $\kappa = 0$ and $\varphi = 0$ do not intersect at infinity. Then the assumptions of Lemma 10 are satisfied, so we may assume that the system is in the form (4) with $\kappa(x,y) = ny$, where n is the degree of φ. We deal with the trivial cases first:

If $A = 0$ then $(x + B)\varphi^{-\frac{1}{n}}$ is a first integral of (4) for all values of f.

If $B = 0$, the assertion of theorem is true for all values of f, because taking $H = x^2\varphi^{-\frac{2}{n}}$ we have $\dot{H} = 2A\varphi^{-\frac{2}{n}}$.

From now on we shall assume that $A = B = 1$. We define

$$H(x,y) = \Big((D - C) - 2Cx - (C + e)x^2 + 2y\Big)\,\varphi^{-\frac{2}{n}}.$$

We have $\dot{H} = 2\left((f - 2)y^2 - (C + e - d)x^2\right)\varphi^{-\frac{2}{n}}$. The ratio of eigenvalues at $[0 : 1 : 0]$ is now equal to $f/(f - 1) = 2$, so $f = 2$ and therefore $\dot{H}$ either has a fixed sign in the complement of $\varphi = 0$ for $(C + e - d) \neq 0$, or H is a first integral for $(C + e - d) = 0$. H is nonconstant, because we have assumed that $\deg\varphi > 2$.

In each of the cases the assumptions of Proposition 8 are satisfied and the theorem follows. ∎

Proof (of Theorem 5): From the discussion in the proof of Theorem 4 it follows that it is sufficient to prove the theorem for the system

$$\begin{aligned} \dot{x} &= x + y + xy, \\ \dot{y} &= Cx + Dy + dx^2 + exy + fy^2, \end{aligned} \tag{6}$$

with $\kappa(x,y) = ny$, where n is the degree of φ. We define

$$H(x,y) = \Big(D - C - 2Cx + (2f - 4 - C - e)x^2 + 2y\Big)\,\varphi^{-\frac{2}{n}}.$$

We have $\dot{H} = 2\left((f-2)y^2 + 2(f-2)xy + (2f-4-2C+d-e)x^2\right)\varphi^{-\frac{2}{n}}$. The condition for the quadratic form appearing in the above expression to be always nonnegative or always nonpositive is

$$(f-2)(f-2+d-e-C) \geq 0,$$

and the theorem follows from Proposition 8. ∎

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