RC23791 (W0511-073) November 9, 2005
Mathematics

IBM Research Report

Two Step MIR Inequalities for Mixed-Integer Programs

Sanjeeb Dash, Oktay Günlük, Marcos Goycoolea
IBM Research Division
Thomas J. Watson Research Center
P.O. Box 218
Yorktown Heights, NY 10598

Research Division
Almaden - Austin - Beijing - Haifa - India - T. J. Watson - Tokyo - Zurich

LIMITED DISTRIBUTION NOTICE: This report has been submitted for publication outside of IBM and will probably be copyrighted if accepted for publication. It has been issued as a Research Report for early dissemination of its contents. In view of the transfer of copyright to the outside publisher, its distribution outside of IBM prior to publication should be limited to peer communications and specific requests. After outside publication, requests should be filled only by reprints or legally obtained copies of the article (e.g., payment of royalties). Copies may be requested from IBM T. J. Watson Research Center , P. O. Box 218, Yorktown Heights, NY 10598 USA (email: reports@us.ibm.com). Some reports are available on the internet at http://domino.watson.ibm.com/library/CyberDig.nsf/home.

Two step MIR inequalities for mixed-integer programs

Sanjeeb Dash, Oktay Günlük and Marcos Goycoolea

Mathematical Sciences Department,
IBM T. J. Watson Research Center, Yorktown Heights, NY 10598
(sanjeebd@us.ibm.com, oktay@watson.ibm.com)

November 8, 2005

Abstract

In this paper we investigate the computational effectiveness of cutting planes based on two-step MIR inequalities. We discuss the similarities and differences between the MIR inequalities and the two-step MIR inequalities. We study the separation problem for the two-step MIR inequalities and show that it can be solved in polynomial time when the resulting inequality is required to be sufficiently different from the associated MIR inequality. We also discuss computational issues and present numerical results. Finally, we also show that for a large number of instances in the MIPLIB problem library, once MIR tableau cuts are added to the formulation, there are no other violated cuts that can be derived from Gomory's master cyclic group polyhedron.

Keywords: integer programming, mixed integer rounding, computation

1 Introduction

A useful technique in generating cutting planes for mixed integer programs is to extract a single implied constraint from the formulation of the problem and derive valid inequalities based on this constraint. More precisely, let

$$P \ = \ \Big\{ v \in R^{|J|}, x \in Z^{|I|} \ : \ Cv + Ax = d, \quad x, v \geq 0 \Big\}$$

where matrices C and A are of appropriate dimension. One can generate a relaxation of it

$$W \ = \ \Big\{ v \in R^{|J|}, x \in Z^{|I|} \ : \ \sum_{j \in J} c_j v_j + \sum_{i \in I} a_i \, x_i = b, \quad x, v \geq 0 \Big\} \tag{1}$$

where the single equation defining set W, called the *base* inequality, is obtained by taking a linear combination of the equations defining P. Valid inequalities for W can be used as cutting planes for P.

A well-known valid inequality for W is the mixed integer rounding (MIR) inequality

$$\sum_{j\in J}\max\{c_j,0\}v_j+\sum_{i\in I}\big(\,\hat{b}\,\lfloor a_i\rfloor+\min\{\hat{b}\;,\hat{a}_i\}\big)x_i\geq\hat{b}\,\lceil b\rceil,\qquad(2)$$

where $\hat{b}=b-\lfloor b\rfloor$ and $\hat{a}_i=a_i-\lfloor a_i\rfloor$. The MIR inequality can be derived using the only non-trivial facet of the set

$$Q^1=\Big\{v\in R,\ z\in Z\ :\ v+z\ \geq\ \beta,\ \ v\geq 0\Big\}$$

see [16].

The Gomory mixed integer cut (GMIC) can be viewed as a two-step procedure where the first step is to obtain the base inequality using the simplex tableau, and the second step is to write the associated MIR inequality. This has been observed by Marchand and Wolsey[15].

In a recent paper, Dash and Günlük [10] studied the following simple set

$$Q^2=\Big\{v\in R,\ y,z\in Z\ :\ v+\alpha y+z\ \geq\ \beta,\ \ v,y\geq 0\Big\}.$$

and derived a parametric family of valid inequalities for W based on a facet of Q^2. More precisely, they showed that for any $\alpha\geq 0$ that satisfies $1/\alpha\geq\lceil\hat{b}/\alpha\rceil>\hat{b}/\alpha$, the *two-step MIR inequality with parameter* α

$$\sum_{j\in J}\max\{c_j,0\}v_j+\sum_{j\in I}\big(\rho^\alpha\,\tau^\alpha\,\lfloor a_i\rfloor+\min\{\rho^\alpha\,\tau^\alpha\;,\;k_i^\alpha\rho^\alpha\;+\hat{a}_i-k_i^\alpha\alpha\;,\;l_i^\alpha\rho^\alpha\;\}\big)x_i\geq\rho^\alpha\,\tau^\alpha\,\lceil b\rceil,\quad(3)$$

where $\tau^\alpha=\lceil\hat{b}/\alpha\rceil$, $\rho^\alpha=\hat{b}-\alpha\lfloor\hat{b}/\alpha\rfloor$, $k_i^\alpha=\lfloor\hat{a}_i/\alpha\rfloor$ and $l_i^\alpha=\lceil\hat{a}_i/\alpha\rceil$, is valid for W. In the rest of the paper we will write inequality (3) without the superscript α whenever possible. We note that Q^2 has other non-trivial facets, see [2] and [4], which might also lead to interesting inequalities for W.

Our purpose in this paper is to investigate the computational effectiveness of two-step MIR inequalities. Our motivation is partly due to the practical importance of MIR inequalities (including the GMIC), which is now routinely used in MIP software and is considered one of the most important classes of cutting planes, see [5] and [7].

There are strong similarities between the MIR inequality (2) and the two-step MIR inequality (3). This is not only because their derivation is based on similar simple sets with a small number of variables, but also because they have a very similar form when written in a certain way. In the next section we discuss this in detail.

The rest of the paper is organized as follows: In Section 2, we study the separation problem for the two-step MIR inequalities and show that it can be solved efficiently when parameter α is not allowed to be very small. In Section 3, we describe how we construct the base inequality for the two-step MIR inequalities and discuss other computational issues. In Section 4, we present computational results using several data sets including MIPLIB 3.0 and the MIPLIB 2003 problem library [6, 1], Atamtürk instances [3] and some real life problems from the steel industry. Our computational results are in general encouraging and suggest that two-step MIR inequality can be useful in solving mixed integer programs. On more than half of the instances from MIPLIB 3.0, however, we observe that all two-step MIR inequalities based on tableau rows become satisfied once the MIR tableau cuts are added. In Section 5, we investigate this observation and show that for a large number of these instances, if MIR inequalities are added, not only two-step MIR inequalities but all valid inequalities that can be derived from Gomory's master cyclic group polyhedron are satisfied.

1.1 Comparing MIR and two-step MIR inequalities

To see the similarities (and differences) between the two-step MIR inequality and the MIR inequality, we normalize them so that they both have a right-hand side of $\lceil b \rceil$. After this normalization, inequality (2) has the form

$$\frac{1}{\hat{b}} \sum_{j\in J} \max\{c_j, 0\} v_j + \sum_{i\in I} (\lfloor a_i \rfloor + f(\hat{a}_i)) x_i \geq \lceil b \rceil, \tag{4}$$

where $f(\hat{a}_i) = (1/\hat{b}) \min\{\hat{b}, \hat{a}_i\}$, and inequality (3) has the form

$$\frac{1}{\rho\,\tau} \sum_{j\in J} \max\{c_j, 0\} v_j + \sum_{i\in I} (\lfloor a_i \rfloor + g^\alpha(\hat{a}_i)) x_i \geq \lceil b \rceil, \tag{5}$$

where $g^\alpha(\hat{a}_i) = (1/\rho\tau) \min\{\rho\,\tau\ ,\ k_i\rho\ + \hat{a}_i - k_i\alpha\ ,\ l_i\rho\ \}$. Functions f and g^α both take values in $[0, 1]$, they are subadditive and they are extreme in a certain sense [11]. Notice that, written in this form, MIR and two-step MIR inequalities differ from each other in (i) how they (linearly) increase the coefficients of the continuous variables that have a positive coefficient in the original equation and (ii) how they change the fractional part of the coefficients of integral variables.

Note that $\hat{b} = \rho + (\tau - 1)\alpha$ and therefore $\hat{b} - \rho\tau = (\alpha - \rho)(\tau - 1) > 0$ as $\alpha > \rho$ by definition and $\tau \geq 2$ for all admissible α. In other words, the coefficient of the continuous variables in the MIR inequality (4) is smaller than that of the two-step MIR inequality (5) and the difference between the coefficients is large if (i) τ is large, and (ii) ρ is significantly smaller than α. For the coefficients of the integral variables, however, there is no dominance relationship as both inequalities are facet defining for the master cyclic group polyhedron [10].

We next give a numerical example, to demonstrate how functions f and g^α behave.

Example 1 *Let $\hat{b} = 0.8$ and $\alpha = 0.3$, implying $\rho = 0.2$ and $\tau = 3$. In this case the function g^α is piecewise linear with breakpoints at ρ, α, $\alpha + \rho$, 2α and $\hat{b} = 2\alpha + \rho$. The function f is also piecewise linear with a single break point at $\hat{b}$. In Figure 1 we plot $g^{0.3}(s)$ and $f(s)$ for $s \in [0, 1]$, and in Figure 2, we plot the difference between the two functions, $g^{0.3}(s) - f(s)$ for $s \in [0, 1]$.*

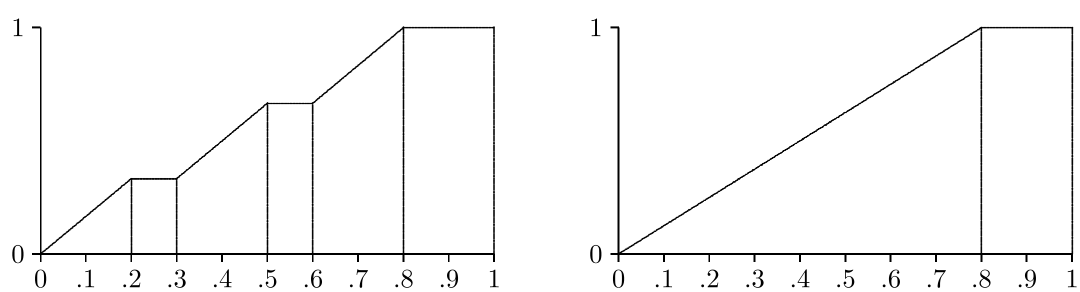

Figure 1: two-step MIR function $g^{0.3}(\cdot)$ and MIR function $f(\cdot)$

Notice that the slope of the two-step MIR function $g^{0.3}(s)$ between 0 and ρ is $(1/\rho\,\tau)$ which is precisely equal to the cut coefficient of continuous variables in inequality (5). Similarly, the slope of the MIR function $f(s)$ gives the cut coefficient of continuous variables in inequality (4).

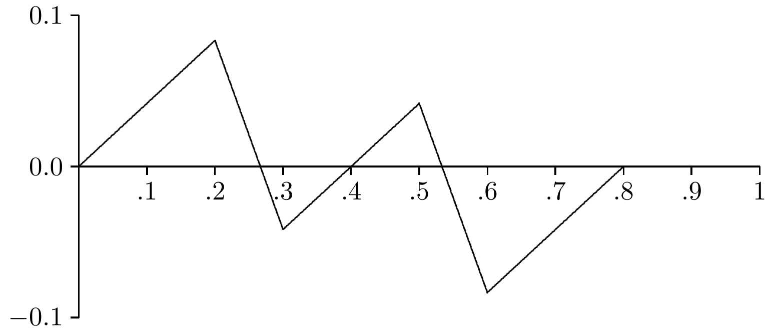

Figure 2: Difference between the two-step MIR function $g^{0.3}(\cdot)$ and MIR function $f(\cdot)$

Also notice that the difference between the functions shown in Figure 2 is symmetric around $0.4 = \hat{b}/2$*. In other words* $g^{0.3}(s) - f(s) = f(0.8 - s) - g^{0.3}(0.8 - s)$ *for* $s \in (0.0, 0.8)$*.*

We next bound the difference between the cut coefficients of the two-step MIR inequality and the MIR inequality .

Lemma 2 *Let* $c \in [0,1)$ *be given. Then,* $1/\lceil \hat{b}/\alpha \rceil \ \geq\ |g^{\alpha}(c) - f(c)|$*.*

Proof. Let $t = \lceil \hat{b}/\alpha \rceil$. For all $c \in (\hat{b}, 1)$, the claim is correct as $g^{\alpha}(c) = f(c)$. When $c \in [0, \hat{b}]$, we have

$$g^{\alpha}(c) - f(c) \quad = \quad \frac{1}{\rho^{\alpha}\, t}\ \min\{\lfloor c/\alpha \rfloor \rho^{\alpha}\ + c - \lfloor c/\alpha \rfloor \alpha\ ,\ \lceil c/\alpha \rceil \rho^{\alpha}\ \} - c/\hat{b}$$

which is a piecewise linear function on $[0, b]$ with breakpoints at $c = (k-1)\alpha + \rho^{\alpha}$ and $c = k\alpha$ for $k = 1, \cdots, t-1$. Furthermore $g^{\alpha}(0) - f(0) = g^{\alpha}(\hat{b}) - f(\hat{b}) = 0$. Therefore, it is suffices to consider the breakpoints. If c is a breakpoint, we have

$$g^{\alpha}(c) - f(c) \quad = \quad \frac{\lceil c/\alpha \rceil}{t} - \frac{c}{\hat{b}} = \frac{\lceil c/\alpha \rceil}{t} - \frac{c/\alpha}{\hat{b}/\alpha} \leq \frac{\lceil c/\alpha \rceil}{t} - \frac{c/\alpha}{t} \leq \frac{1}{t}.$$

Similarly,

$$g^{\alpha}(c) - f(c) \quad = \quad \frac{\lceil c/\alpha \rceil}{t} - \frac{c/\alpha}{\hat{b}/\alpha} \geq \frac{\lceil c/\alpha \rceil}{t} - \frac{\lceil c/\alpha \rceil}{t-1} = -\frac{\lceil c/\alpha \rceil}{t(t-1)} \geq -\frac{1}{t}.$$

∎

We note that this bound is tight, in the sense that, for each $t \geq \lceil 1/(1 - \hat{b}) \rceil$, it is possible to find an $\alpha > 0$, and $c_1, c_2 \in (0, \hat{b})$ such that $\lceil \hat{b}/\alpha \rceil = t$ and both $g^{\alpha}(c_1) - f(c_1)$ and $f(c_2) - g^{\alpha}(c_2)$ are arbitrarily close to $1/t$. This is achieved by setting $\alpha = (\hat{b} - \epsilon)/(t-1)$ and $c_1 = \epsilon$, $c_2 = \hat{b} - \epsilon$ for some small $\epsilon > 0$.

A simple observation based on Lemma 2 is the following:

Corollary 3 *Let* $c \in [0,1)$*, then* $\lim_{\alpha \to 0^+} g^{\alpha}(c) = f(c)$*.*

Therefore, if continuous variables are present, the two-step MIR inequality is dominated by the MIR inequality as α becomes very small. If there are no continuous variables, however, the inequalities become indistinguishable.

We next observe that the two-step MIR inequality becomes indistinguisable from the MIR inequality when $\rho^\alpha = \hat{b} - \alpha\lfloor \hat{b}/\alpha \rfloor$ is almost as large as α. We use $\epsilon \to \delta^+$ to denote that $\epsilon > \delta$ as ϵ tends to δ.

Lemma 4 *Let $c \in [0, 1)$ and t be a positive integer. Then, (i) $\lim_{\alpha \to (\hat{b}/t)^+} \rho^\alpha \tau^\alpha = \hat{b}$, and (ii) $\lim_{\alpha \to (\hat{b}/t)^+} g^\alpha(c) = f(c)$.*

Proof. Notice that when α satisfies $\hat{b}/(t-1) > \alpha > \hat{b}/t$, we have $\lceil \hat{b}/\alpha \rceil = t$ and $\lfloor \hat{b}/\alpha \rfloor = t - 1$. Therefore,

$$\lim_{\alpha \to (\hat{b}/t)^+} \rho^\alpha \tau^\alpha = \lim_{\alpha \to (\hat{b}/t)^+} \Big(\hat{b} - \alpha(t-1)\Big) t = t\hat{b} - \hat{b}(t-1) = \hat{b}.$$

In addition, as observed above, $\lim_{\alpha \to (\hat{b}/t)^+} \rho^\alpha = \hat{b} - (\hat{b}/t)(t-1) = \hat{b}/t$ and therefore,

$$\lim_{\alpha \to (\hat{b}/t)^+} (k_i^\alpha \rho^\alpha + c - k_i^\alpha \alpha) = c + \lim_{\alpha \to (\hat{b}/t)^+} k_i^\alpha (\rho^\alpha - \alpha) = c.$$

Finally, as $\lim_{\alpha \to (\hat{b}/t)^+} l_i^\alpha \rho^\alpha = \lim_{\alpha \to (\hat{b}/t)^+} \lceil c/\alpha \rceil \rho^\alpha \geq c$, we have $g^\alpha(c) = f(c)$. ■

2 Separating two-step MIR inequalities

In this section we discuss the separation problem for the two-step MIR inequalities. Given a point $(\bar{v}, \bar{x}) \in R^{|J|+|I|}$, we define the separation problem to be the problem of identifying an admissible parameter α that gives a most violated two-step MIR inequality (3).

For the set W, it is easy to show that parameter $\alpha \in R$ is an admissible parameter, that is, (i) $\alpha \geq 0$, (ii) $1/\alpha \geq \lceil \hat{b}/\alpha \rceil$, and (iii) $\lceil \hat{b}/\alpha \rceil > \hat{b}/\alpha$, if and only if $\alpha \in \mathcal{I} = \cup_{d=2}^\infty \mathcal{I}^\tau$ where

$$\mathcal{I}^\tau = \begin{cases} (\hat{b}/\tau, 1/\tau] & \text{if } 2 \leq \tau < \left\lceil 1/(1-\hat{b}) \right\rceil \\ (\hat{b}/\tau, \hat{b}/(\tau-1)) & \text{if } \tau \geq \left\lceil 1/(1-\hat{b}) \right\rceil. \end{cases}$$

Notice that $\mathcal{I}^\tau$ is the set of values of α such that $\lceil \hat{b}/\alpha \rceil = \tau$.

We start with showing that the coefficients of the integral variables in inequality (3) change continuously as α changes. Remember that $\hat{c}$ denotes $c - \lfloor c \rfloor$ for $c \in R$.

Lemma 5 *Given $a, b \in R$ and an integer $d \geq 2$, the function*

$$h^a(\alpha) = \rho^\alpha \, d \lfloor a \rfloor + \min\{\rho^\alpha d \, , \, \lfloor \hat{a}/\alpha \rfloor \rho^\alpha + \hat{a} - \lfloor \hat{a}/\alpha \rfloor \alpha \, , \, \lceil \hat{a}/\alpha \rceil \rho^\alpha \}$$

where $\rho^\alpha = \hat{b} - \alpha(d-1) > 0$ is continuous for $\alpha \in \mathcal{I}^d$.

Proof. Note that a function obtained by taking minimums or linear combinations of continuous functions is still continuous. As the first term of $h^a(\alpha)$ is a continiuos (linear) function of α, we only need to show that the second term is also continuous. In addition, the first term in the minimization is a linear function of α as well. Therefore, it suffices to show that $v(\alpha) = \min\{\lfloor \hat{a}/\alpha \rfloor \rho^\alpha + \hat{a} - \lfloor \hat{a}/\alpha \rfloor \alpha\ , \lceil a/\alpha \rceil \rho^\alpha\ \}$ is continuous.

As ρ^α is continuous and the terms in the minimization have discontinuities only when $\hat{a}/\alpha \in Z$, we will only consider the case when $\alpha = \hat{a}/t$, for some $t \in Z$. More precisely, we need to show that $v(\hat{a}/t) = \lim_{\epsilon \to 0^+} v(\hat{a}/t + \epsilon) = \lim_{\epsilon \to 0^+} v(\hat{a}/t - \epsilon)$, First note that $v(\hat{a}/t) = t\rho^{(\hat{a}/t)}$.

In addition,

$$\begin{aligned} \lim_{\epsilon \to 0^+} v(\hat{a}/t + \epsilon) &= \lim_{\epsilon \to 0^+} \min\{(t-1)\rho^{(\hat{a}/t+\epsilon)} + \hat{a} - (t-1)(\hat{a}/t + \epsilon)\ , t\rho^{(\hat{a}/t+\epsilon)}\ \} \\ &= \min\{(t-1)\rho^{(\hat{a}/t)} + \hat{a}/t\ , t\rho^{(\hat{a}/t)}\ \} \ = \ t\rho^{(\hat{a}/t)} \end{aligned}$$

where the last equality follows form the fact that $\rho^\alpha \leq \alpha$ for all $\alpha \in \mathcal{I}^d$.

Finally,

$$\begin{aligned} \lim_{\epsilon \to 0^+} v(\hat{a}/t - \epsilon) &= \lim_{\epsilon \to 0^+} \min\{t\rho^{(\hat{a}/t-\epsilon)} + \hat{a} - t(\hat{a}/t + \epsilon)\ , (t+1)\rho^{(\hat{a}/t-\epsilon)}\ \} \\ &= \min\{t\rho^{(\hat{a}/t)}\ , (t+1)\rho^{(\hat{a}/t)}\ \} \ = \ t\rho^{(\hat{a}/t)}. \quad \blacksquare \end{aligned}$$

Therefore, the coefficients of the integral variables change without discontinuities as $\alpha \in \mathcal{I}^d$ changes. Using this observation, we next show that for given a point the violation of the two-step MIR inequality with parameter α is a continuous function of α and it can be minimized efficiently.

Lemma 6 *Given* $(\bar{v}, \bar{x}) \in R^{|J|+|I|}$, *let* $\Delta(\alpha)$ *denote the violation of the two-step MIR inequality generated by* $\alpha \in \mathcal{I}$. *Let* $I' = \{i \in I\ :\ \bar{x}_i > 0\}$. *For some* $\bar{\alpha} \in \mathcal{I}^d$, *if* $\Delta(\bar{\alpha}) = \sup_{\alpha \in \mathcal{I}^d}\{\Delta(\alpha)\}$, *then one of the following statements hold:*

a. $\Delta(\bar{\alpha}) = \Delta(\hat{a}_i/t)$ *for some* $i \in I'$ *and* $t \in Z$, *or,*

b. $\bar{\alpha} = 1/d$, *or,*

c. $\Delta(\bar{\alpha}) = \Delta(\alpha)$ *for all* $\alpha \in \mathcal{I}^d$.

Proof. Notice that for $\alpha \in \mathcal{I}^d$, $\lceil \hat{b}/\alpha \rceil = d$. Define $J' = \{j \in J\ :\ \bar{v}_j > 0, c_j > 0\}$. The violation of the inequality generated by α is therefore

$$\Delta(\alpha) = \rho^\alpha\, d \lceil b \rceil - \sum_{j \in J'} c_j \bar{v}_j - \sum_{j \in I'} h^{a_i}(\alpha)\bar{x}_i$$

where $\rho^\alpha = \hat{b} - \alpha(d-1) > 0$ for all $\alpha \in \mathcal{I}^d$.

Let L be the set of numbers that divide some $\hat{a}_i$ for $i \in I'$. If $\bar{\alpha} \in L$ or $\bar{\alpha} = 1/d$, the claim is correct, so we assume that $\bar{\alpha} \notin L \cup \{1/d\}$. Define $\alpha^+ = \min\{\alpha \in L \cup \{+\infty\} : \alpha > \bar{\alpha}\}$ and $\alpha^- = \max\{\alpha \in L \cup \{-\infty\} : \alpha < \bar{\alpha}\}$ so that $\alpha^+ > \alpha > \alpha^-$.

Notice that $k_i = \lfloor \hat{a}_i/\alpha \rfloor$ and $l_i = \lceil \hat{a}_i/\alpha \rceil$ are constant for all $i \in I'$ when $\alpha \in (\alpha^-, \alpha^+)$. By Lemma 5 , $\Delta(\alpha)$ is continuous, and therefore, the violation of the inequality generated by α is:

$$\Delta(\alpha) = \rho^\alpha \, d \lceil b \rceil - \sum_{j \in J'} c_j \bar{v}_j - \sum_{j \in I'} \big(\rho^\alpha \, d \lfloor a_i \rfloor - \min\{\rho^\alpha \, d \ , \ k_i \rho^\alpha \ + \hat{a}_i - k_i \alpha \ , \ l_i \rho^\alpha \ \}\big) \bar{x}_i$$

for all $\alpha \in [\alpha^-, \alpha^+] \cap \mathcal{I}^d$.

Next, define a lower bound on $\Delta(\alpha)$

$$\hat{\Delta}(\alpha) = \rho^\alpha \, d \lceil b \rceil - \sum_{j \in J'} c_j \bar{v}_j - \sum_{j \in I'} \rho^\alpha \, d \lfloor a_i \rfloor \, \bar{x}_i - \sum_{j \in I_0} \rho^\alpha d \; \bar{x}_i - \sum_{j \in I_1} (k_i \rho^\alpha \ + \hat{a}_i - k_i \alpha) \; \bar{x}_i - \sum_{j \in I_2} l_i \rho^\alpha \; \bar{x}_i,$$

where $I_0 = \{j \in I' : \hat{a}_i \geq \hat{b}\}$, $I_1 = \{j \in I' \setminus I_0 : \hat{a}_i - k_i \bar{\alpha} < \bar{\rho}\}$ and $I_2 = \{j \in I' \setminus I_0 : \hat{a}_i - k_i \bar{\alpha} \geq \bar{\rho}\}$. Note that $\Delta(\alpha) \geq \hat{\Delta}(\alpha)$ for $\alpha \in [\alpha^-, \alpha^+] \cap \mathcal{I}^d$, and $\hat{\Delta}(\bar{\alpha}) = \Delta(\bar{\alpha})$.

Notice that for $\alpha \in [\alpha^-, \alpha^+] \cap \mathcal{I}^d$, the function $\hat{\Delta}(\alpha)$ is a linear function of α. Therefore for any $\alpha', \alpha'' \in [\alpha^-, \alpha^+] \cap \mathcal{I}^d$, such that $\alpha' \leq \bar{\alpha} \leq \alpha''$ we have,

$$\Delta(\bar{\alpha}) \geq \max\{\Delta(\alpha'), \Delta(\alpha'')\} \geq \max\{\hat{\Delta}(\alpha'), \hat{\Delta}(\alpha'')\} \geq \hat{\Delta}(\bar{\alpha}) = \Delta(\bar{\alpha}).$$

As $\hat{\Delta}$ is linear, $\max\{\hat{\Delta}(\alpha'), \hat{\Delta}(\alpha'')\} = \Delta(\bar{\alpha})$ implies that $\hat{\Delta}(\alpha') = \hat{\Delta}(\alpha'') = \Delta(\bar{\alpha})$. In addition, by assumption, $\bar{\alpha}$ gives the largest violation, and therefore $\Delta(\alpha') = \Delta(\alpha'') = \Delta(\bar{\alpha})$.

If $\alpha^- \in \mathcal{I}^d$, we can take $\alpha' = \alpha^-$, or, if $\alpha^+ \in \mathcal{I}^d$, we can take $\alpha'' = \alpha^+$, implying $\Delta(\bar{\alpha}) = \Delta(\hat{a}_i/t)$ for some $i \in I'$ and $t \in Z$ as claimed. On the other hand, if $\alpha^+, \alpha^- \notin \mathcal{I}^d$ we have $\alpha \in [\alpha^-, \alpha^+] \cap \mathcal{I}^d = \mathcal{I}^d$, and $\Delta(\bar{\alpha}) = \Delta(\alpha)$ for all $\alpha \in \mathcal{I}^d$. ∎

Notice that Lemma 6 only analyzes the case when $\Delta(\bar{\alpha}) = \sup_{\alpha \in \mathcal{I}^d}\{\Delta(\alpha)\}$ for some $\alpha \in \mathcal{I}^d$. If this condition does not hold, the violation function does not attain its supremum in $\mathcal{I}^d$. In this case, first observe that

$$\lim_{\alpha \to (\hat{b}/(t-1))^+} \rho^\alpha = \lim_{\alpha \to (\hat{b}/(t-1))^+} \hat{b} - \alpha \lfloor \hat{b}/\alpha \rfloor = 0,$$

and therefore the violation function is not positive. Next, note that by Lemma 4

$$\lim_{\alpha \to (\hat{b}/t)^+} \Delta(\alpha) = \hat{b} \lceil b \rceil - \sum_{j \in J'} c_j \bar{v}_j - \sum_{j \in I} \big(\hat{b} \lfloor a_i \rfloor - \min\{\hat{b} \ , \hat{a}_i\}\big) \bar{x}_i$$

which is the violation of the MIR inequality (2). Combining these observations, we conclude that if $(\bar{v}, \bar{x})$, satisfies the MIR inequality, then a violated two-step MIR inequality, if it exists, can be obtained using Lemma 6.

Also notice that for a given $\mathcal{I}^d$ and $i \in I$, there is at most one $t \in Z_+$ such that $\hat{a}_i/t \in \mathcal{I}^d$. Using the fact that $\mathcal{I}^d \subseteq (\hat{b}/d, \hat{b}/(d-1))$, t has to satisfy $\hat{b}/d > \hat{a}_i/t > \hat{b}/(d-1)$, implying (i) $t > d\hat{a}_i/\hat{b}$ and (ii) $t < (d-1)\hat{a}_i/\hat{b}$. Clearly, only one $t \in Z_+$ can satisfy this condition. Therefore, for a given $\mathcal{I}^d$, and $\hat{a}_i$, one has to consider only $t = \lceil d\hat{a}_i/\hat{b} \rceil$. It is also possible that the resulting $\alpha = \hat{a}_i / \lceil d\hat{a}_i/\hat{b} \rceil$ would not be a valid parameter. Therefore, Lemma 6 leads to a linear-time separation algorithm for two-step MIR inequalities for $\alpha \in \mathcal{I}^d$ and we have the following result.

Lemma 7 *Given $(\bar{v}, \bar{x}) \in R^{|J|+|I|}$, that satisfies the MIR inequality and a number $k \in Z^+$, a most violated two-step MIR inequality can be separated in polynomial time for $\alpha \in \cup_{d=2}^k \mathcal{I}^d$.*

Proof. Let L^d denote the set of valid parameters for $\mathcal{I}^d$ that divide some $\hat{a}_i$ for $i \in I' = \{i \in I : \bar{x}_i > 0\}$. Then we have $L^d = \cup\{\hat{a}_i/\lceil d\hat{a}_i/\hat{b}\rceil \,:\, i \in I'\} \cap \mathcal{I}^d$. To find the most violated inequality for $\alpha \in \cup_{d=2}^{k}\mathcal{I}^d$, it suffices to consider $\alpha \in \cup_{d=2}^{k}(L^d \cup \{1/d\})$, where $|L^d| \leq |I'|$. Clearly, there are at most $O(k \cdot |I|)$ candidate parameters to consider. ∎

We would like to emphasize that Lemma 7 does not give polynomial time separation algorithm for $\alpha \in \mathcal{I} \cup_{d=2}^{\infty} \mathcal{I}^d$. But based on the Lemma 2 and Corollary 3, in practice, it is not desirable to use small α's, or, in other words, one would not consider $\mathcal{I}^d$ for large d.

3 Computational Framework

The purpose of our computional experiments is to determine if two-step MIR cuts help tighten the continuous relaxations of general mixed-integer programs. As MIR cuts, which can be viewed as a subclass of two-step MIR cuts, are already known to be computionally effective, we are mainly interested in the additional gain due to two-step MIR cuts that are "sufficiently" different from MIR cuts. Therefore, in our experiments we restrict the parameter α in such a way that the resulting two-step MIR cuts are sufficiently different from MIR cuts.

In our experiments, we compare the improvement in the objective function at the root node due to MIR cuts with the improvement due to both MIR and two-step MIR cuts. In our opinion, two-step MIR cuts are most effective when used together with MIR cuts and we view our procedure as a strengthening of the MIR procedure rather than as an alternative.

In this section we describe how to create base inequalities from rows of the optimal simplex tableau and from inequalities in the formulation. We also describe how we choose the parameter α for two-step MIR cuts. Finally, we discuss the effect of different basis sequences when several rounds of cuts are added.

3.1 Transforming base inequalities

Our main subroutine for generating violated MIR and two-step MIR cuts takes as input the current fractional point (v^*, x^*) together with a valid base inequality

$$\sum_{j \in J} c_j v_j + \sum_{i \in I} a_i x_i \geq b, \tag{6}$$

and upper and lower bounds on the variables v and x:

$$l_j \leq v_j \leq u_j \quad \forall j \in J, \quad l_i \leq x_i \leq u_i \quad \forall i \in I.$$

If we start with an equality constraint, we simply relax it to obtain the inequality form in (6). Recall that in our definition of MIR (2) and two-step MIR cuts (3), we assume all variables are non-negative. Therefore, as a first step, we perform variable transformations to obtain a base inequality with non-negative variables. These transformations are

$$\bar{x}_i = x_i - l_i, \text{ or } \quad \bar{x}_i = u_i - x_i \ \forall i \in I, \tag{7}$$

$$\bar{v}_j = v_j - l_j, \text{ or } \quad \bar{v}_j = u - v_j \ \forall j \in J.$$

We do not generate cuts from a base inequality with unbounded variables. After these transformations, our base inequality becomes:

$$\sum_{j\in J} c'_j \bar{v}_j + \sum_{i\in I} a'_i \bar{x}_i \geq b', \tag{8}$$

$$0 \leq \bar{v}_j \leq u_j - l_j \ \forall j \in J, \ 0 \leq \bar{x}_i \leq u_i - l_i \ \forall i \in I,$$

where c'_j, a'_i and b' depend on the specific variable transformations performed. Dropping the upper bounds on the variables $\bar{v}_j$ and $\bar{x}_i$, we have the set W in (1).

If a variable has just one bound, we use the appropriate transformation in (7). If it has both an upper bound and a lower bound, we transform the integral variables, based on the value of (v^*, x^*):

$$\bar{x}_i = \begin{cases} x_i - l_i & \text{if } x_i^* < (l_i + u_i)/2, \\ u_i - x_i & \text{if } x_i^* \geq (l_i + u_i)/2. \end{cases}$$

We use the same rule for transforming the continuous variables v_j for all $j \in J$. Note that in option (a) of the *bound substitution* heuristic of Marchand and Wolsey [15], they transform continuous variables by the above rule, but not integer variables. The justification for this rule is relatively intuitive: when deriving inequalities (2) and (3), one uses the non-negativity of the variables to relax inequality (8) to obtain an inequality with non-negative and/or integral coefficients. If $\bar{v}_j^*$ and $\bar{x}_i^*$ are close to 0, this relaxation step does not increase the slack of inequality (8) too much and the relaxed inequality is more likely to yield violated inequalities.

After obtaining (8) and dropping the upper bounds on variables, we use (2) to get an MIR cut, as long as $\hat{b} \geq 10^{-5}$.

3.2 Choosing α for two-step MIR cuts

After we obtain inequality (8), we try a number of different values for the parameter α to generate violated cuts. From among those, we retain only two cuts (those with the highest values of violation divided by norm of cut coefficients) and add them to the formulation. We consider a candidate α acceptable if it satisfies several numerical conditions. First, Lemma 2 suggests that $\hat{b}/\alpha$ should not be too large; we restrict this number to be at most 20. (Equivalently, we require $\alpha \in \cup_{d=2}^{20} \mathcal{I}^d$.) Corollary 3 suggests that α should not be too small; we insist that $\alpha \geq .0005$. Lemma 4 suggests that α should not be too close to $\hat{b}/t$ from above; we insist that $\alpha \geq \hat{b}/t + .0005$.

From Lemma 7, it suffices to try all $\hat{a}_i/t$ as candidates for α, where $i \in I'$ – the set of non-zero integral variables, $t \in Z$ and $\hat{b}/(\hat{a}_i/t) \leq 20$, to get the most violated two-step MIR inequality. In other words, we should take the fractional parts of the coefficients in inequality (8) and divide them with small integers to obtain different α. To reduce computation time, we choose at most one α for every $i \in I'$, namely $\hat{a}_i/t$ where t is the smallest integer such that $\hat{a}_i/t$ is a valid choice for α. In addition, we only generate cuts from sufficiently different α. We consider two values α_1 and α_2 sufficiently different if $|\alpha_1 - \alpha_2| \geq .001$. Lemma 7 also suggests that one should use $1/d$ for small integers d, but the resulting two-step MIR inequalities are the same as scaled MIR inequalities and we choose not to use them as two-step MIR inequalities.

If the starting inequality is actually an equality, we also multiply it by -1 to obtain a second base inequality (8). This way, we can obtain two additional two-step MIR cuts from the same base equality. Note that the MIR cuts from (8) are identical whether one first multiplies it by -1 or not. This is not true for two-step MIR cuts. For example, assume $\hat{b} = 0.2$ and $\hat{a}_i > 0.2$ for all $i \in I$. Then we cannot derive a two-step MIR cut from (8) different from the MIR cut. However, after multiplying by -1, $\hat{b}$ becomes 0.8, and $\hat{a}_i < 0.8$ for all $i \in I$, and we can generate several two-step MIR cuts.

3.3 Formulation rows

One natural way to obtain base inequalities is to use rows of the original problem formulation. There are two main advantages to using formulation rows as base inequalities: (i) there is little loss of precision when generating cuts from these inequalities, and (ii) they are usually sparser than the rows of the simplex tableau. It is also possible to aggregate several rows to obtain base inequalities (as proposed by Marchand and Wolsey [15]) but we did not implement this.

Let the current fractional point be (v^*, x^*). To generate a base inequality (6), we pick a row of the problem formulation, and divide it by the coefficient of an integer variable from the set $I' = \{i \in I \,:\, l_i < x_i^* < u_i\}$. This way, from every row of the formultion, we obtain multiple base inequalities from which we can generate MIR and two-step MIR cuts. We do not use coefficients of all variables in I' in one round as this may lead to a very large number of base inequalities. Instead, we randomly select a small number (e.g., 20%) of them. We then transform variables as described in Section 3.1. Based on the resulting inequality, we then generate one MIR and multiple two-step MIR cuts. Once we add the generated cuts and re-optimize, the new optimal solution leads to different base inequalities due to the use of different coefficients for divisions and different variable transformations. We call cuts generated using rows of the formulation “formulation cuts”.

Though the set of all formulation cuts is finite, it can still be very large. In a pure integer problem instance with m rows, n bounded variables and k nonzero coefficients per row, one can have mk base inequalities, each of which can be transformed (via variable complementation) in 2^k possible ways giving $mk2^k$ possible base inequalties. For each base inequality there are also a number of different choices for the parameter α.

3.4 Tableau rows

Rows of an optimal simplex tableau are also natural sources of base inequalities of the form (6). We use tableau rows in which the basic variable is an integer variable with a fractional value. For such a row, let x_1 stand for the basic variable. Clearly, $a_1 = 1$, and the remaining variables are at their upper or lower bounds. Therefore, after performing the variable transformation steps in Section 3.1, all variables other than x_1 have value 0. Let the right hand side of the base inequality be $b = \lfloor b \rfloor + \hat{b}$. In this case, any admissible parameter α will give a violated two-step MIR inequality (3) with a left-hand side of $\rho\tau b$, and a right-hand side of $\rho\tau\lceil b \rceil$. Therefore the violation of any two-step MIR cut is simply $\rho\tau(1 - \hat{b})$. Observe that for all $\alpha \in \mathcal{I}^t$, $\lceil b \rceil$ and τ are constant, and $\rho^\alpha = \hat{b} - \alpha(\tau - 1)$, a decreasing function of α. Therefore, using Lemma 4, maximally violated two-step MIR cuts (based on tableau rows) are simply MIR cuts.

This does not mean that two-step MIR cuts based on table rows are useless. Suppose base inequality (6) is derived from a row of the optimal tableau and let x' stand for the corresponding optimal solution. After adding the MIR cut, we will obtain an optimal solution x''. Clearly, x'' satisfies the MIR cut based on inequality (6) but not necessarily all two-step MIR cuts based on inequality (6).

Usually x'' in fact violates some two-step MIR cuts. The above argument suggests that maximally violated two-step MIR cuts should be computed after adding violated MIR cuts and solving the strengthened LP, but without actually computing the new optimal tableau.

If the original formulation has inequalities, we implicitly append slacks before computing tableau rows. A point to note is that computing the optimal tableau from an optimal basis is an expensive operation.

3.5 Effect of basis sequences

A natural approach to measure the effectiveness of tableau based two-step MIR cuts would be to compare the effect of $k > 1$ rounds of MIR cuts with the effect of k rounds of MIR and two-step MIR cuts combined. Here a *round* of cutting means adding all cuts based on the current tableau rows and reoptimizing. One problem with this approach is that the cuts added in one round influence the subsequent tableau and resulting base inequalities and therefore the subsequent cuts. In particular, for two classes of cuts C_1 and C_2 with $C_1 \subseteq C_2$ (say C_1 = MIR, C_2 = MIR and two-step MIR combined), k rounds of cuts from C_1 may yield a better bound than k rounds of C_2. This is somewhat unexpected behaviour, but it does happen because generated cuts depend on the *basis sequences* encountered during the process. Even if the class of cuts C_2 gives a better bound after the first round of cutting, the next optimal tableau and solution do not necessarily lead to good cuts and therefore, two rounds of cutting with the smaller class C_1 can indeed give a better bound.

In Figure 3, we show the behavior of 5 rounds of MIR tableau cuts (denoted by the solid lines) and MIR+two-step MIR tableau cuts (denoted by the dotted lines) on the problems *lseu* and *rout*. These are two instances from the MIPLIB problem library [1]. For the first instance, MIR+two-step MIR cuts consistently give better objective values than MIR cuts, for each round of cuts. This is not true for the second instance where the gap closed by MIR cuts alone is consistently better from the second round onwards. In Figure 3, we show the number of rounds on the horizontal axis, and the percentage of the integrality gap closed on the other axis.

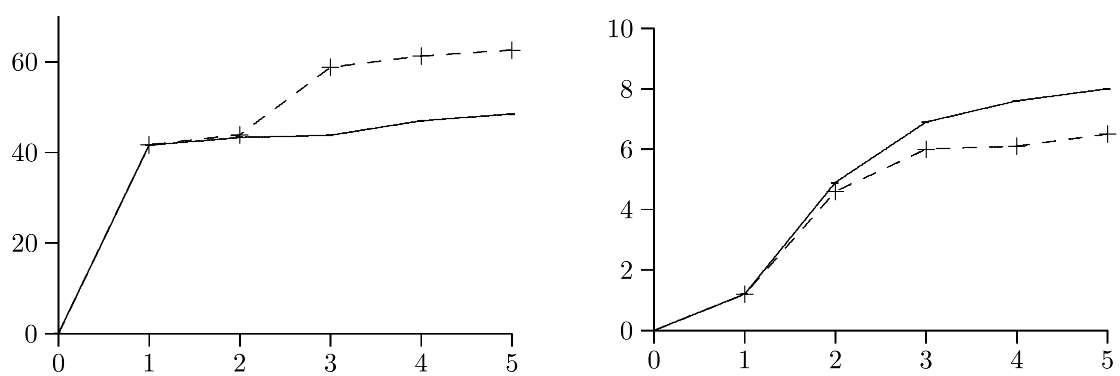

Figure 3: Gap closed by MIR and MIR + two-step MIR cuts for two (lseu and rout) instances

A starker effect of differing basis sequences is illustrated in Figure 4. We consider *fixnet6* from the MIPLIB 3.0 problem set [6] and generate 4 rounds of MIR tableau cuts. We then permute the rows of this instance (we move the first 10 rows to the end) and again generate 4 rounds of

MIR tableau cuts. In other words, we apply the same procedure to the same problem instance after changing the text representations slightly. As seen in Figure 4, the gap closed after the first round is almost the same in both cases, but after this round, the difference between the gap closed increases to approximately 5% .

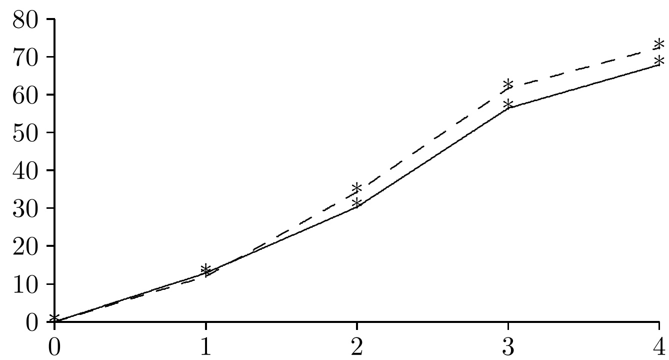

Figure 4: Effect of different basis sequences for fixnet6

This unexpected behaviour occurs because most LP solvers terminate with a *near*-optimal basis – a basis for which the primal and dual solution values are feasible within given feasibility tolerances. Therefore, even if a problem has a unique optimal solution and basis, it usually has many near-optimal bases and solutions. Different sequences of numerical operations, in this case triggered by the row permutations, can therefore lead to different near-optimal bases and therefore to different cuts.

We resolve this problem by using *list/delayed cuts* which we discuss in the next section.

3.6 List cuts

One possible solution to the undesirable behaviour described in the previous section is to generate MIR and two-step MIR cuts in sequence. More precisely, we can delay the cut generation for two-step MIR inequalities until after k rounds of MIR cuts are generated. While generating the MIR cuts, we can save the associated tableau so that the same base inequalities can be used at the end to generate the two-step MIR cuts. Using this approach, the objective value can only improve and the selected two-step MIR cuts use the same sequence of bases as the MIR cuts. In our computational experiments, we used a simplification of this approach and did not actually save all intermediate tableaus.

When generating the MIR cuts, we only considered the ones that are violated by at least some tolerance (10^{-6}). For these cuts, we save the cut and the base inequality (tableau row) it is generated from. At the end of k rounds of MIR cuts, we look at the cuts in the list , and generate two-step MIR cuts only from the base inequalities of "almost tight" MIR cuts. Typicaly, a two-step MIR cut is not violated if the associated MIR cut is satisfied with a significant slack (eg., 0.7).

A well-known property of MIR cuts from the tableau is the following: even if many cuts are generated in the different rounds, after the final round typically very few MIR cuts are active. The number of "almost tight" cuts is more than the number of active cuts, but it is still smaller then the size of the list. This fact keeps our list cut computation cost relatively low.

One important implementation detail here is that, MIR cuts are not deleted from the formulation once they are added even if some of them have large slacks in later iterations. Once a cut is added, its slack can be used in the tableau cuts generated in later rounds and therefore we might have base inequalities in the list that contain this slack variable. To write an asociated two-step MIR inequality in the original space of variables, we need to substitute out the slack variable later and therefore we might need the original cut.

4 Numerical results

In the next three sections, we present numerical results obtained from three different datasets. The first dataset is randomly generated by Atamtürk to test lifted knapsack inequalities described in [3]. These instances are publicly available at *http://ieor.berkeley.edu/∼atamturk.data*. The second dataset consists of 4 problem instances from a practical application. The last dataset is the MIPLIB 3.0 problem library which is publicly available at *http://miplib.zib.de*.

For each of the above data sets, we next present tables which show the effect of adding two-step MIR tableau cuts, and two-step MIR formulation cuts. In the Appendix, we show in detail the effect of multiple rounds of cutting based on tableau rows. In each round we solve the (strengthened) LP relaxation to optimality and obtain the base inequalities for that round of cutting from the current simplex tableau. For each problem instance we display the gap closed by MIR cuts only, and MIR and two-step MIR cuts together for 1,...,5 rounds. The two-step MIR cuts are added as list cuts as described in Section 3.6 .

4.1 Atamturk instances

These instances are randomly generated mixed integer programs, with between 250 and 500 rows, between 50 and 100 integral variables, and between 1 and 20 continuous variables. The notation $|I| : |J| : |R|$ denotes a set of 5 instances with $|I|$ integral variables, $|J|$ continuous variables and $|R|$ rows. The above instances are divided into two groups, one with upper bounds on variables and the other without. All variables are non-negative. The problems have the form:

$$\max \sum_{j\in J} h_j v_j + \sum_{i\in I} g_i\, x_i$$

$$\sum_{j\in J} c_{rj} v_j + \sum_{i\in I} a_{ri}\, x_i \le b_r, \forall r \in R,$$

$$0 \le v \le w,\ 0 \le x \le u,\ x \in Z, v \in R$$

where, the upper bounds w and u are set to infinity for unbounded instances.

Atamturk[3] has generated these instances to test the computational effectiveness of lifted knapsack inequalities and shown that these inequalities (when applied to rows of the original formulation) close the integrality gap significantly. Recently Fischetti and Saturni[8] also performed computational tests using these instances to test the computational effectiveness of *group cuts*, or cuts based on Gomory's master cyclic group polyhedra. In their paper, they present results with 1-50 scaled MIR cuts (based on the simplex tableau), and show that these simple cuts also significantly reduce the integrality gap.

Tables 1 and 2 summarize our computational results with unbounded and bounded instances respectively. In these tables, we display the percentage integrality gap closed by (*i*) MIR and two-step MIR cuts based on the simplex tableau (*ii*) MIR and two-step MIR formulation cuts, and (*iii*) scaled MIR cuts based on the simplex tableau. We also present the number of cuts added during the process. The numbers for the scaled MIR cuts are taken from [8]. Cuts based on the tableau are added in one single round using the optimal simplex tableau of the initial LP-relaxation. Formulation cuts, on the other hand, are added in several rounds until no more violated inequalities can be found. Each row represents the average over 5 instances. For example, in Table 1, the integrality gap closed by one round of MIR cuts for the 5 unbounded instances with $|I| = 250, |J| = 1$ and $|R| = 50$ is on the average 50.08%. We note that results in [3] and [8] for MIR cuts are consistent with our results.

	Tableau cuts			Formulation cuts			Scaled MIRs	
$\|I\| : \|J\| : \|R\|$	MIR	+2MIR	No. cuts	MIR	+2MIR	No. cuts	1-50 MIR	No. cuts
250:1:50	50.08	83.38	136.0	78.52	83.26	162.00	78.52	2185.4
250:1:75	54.82	79.16	203.8	76.12	79.14	228.40	76.14	3014.4
250:1:100	63.26	75.54	226.0	74.28	75.24	297.60	74.55	3187.2
500:1:50	50.68	80.40	136.0	76.66	80.26	176.40	76.67	2117.2
500:1:75	55.14	77.44	203.8	74.92	77.32	245.60	74.92	2932.4
500:1:100	63.50	75.96	269.4	75.12	75.92	316.20	75.13	3293.2
Average	56.25	78.65	195.83	75.94	78.52	237.70	75.99	2788.30

Table 1: Atamtürk's unbounded instances

	Tableau cuts			Formulation cuts			Scaled MIRs	
$\|I\| : \|J\| : \|R\|$	MIR	+2MIR	No. cut	MIR	+2MIR	No. cuts	1-50 MIR	No. cuts
250:5:100	61.32	72.34	256.2	71.16	72.30	297.60	71.17	3196.0
250:10:50	50.88	80.02	136.0	76.32	79.90	161.60	76.34	2100.6
250:10:75	56.20	74.50	203.8	72.12	74.04	227.40	72.54	2795.8
250:10:100	67.40	75.38	245.4	74.70	75.38	295.20	74.69	2948.4
250:20:50	51.58	77.86	136.0	74.76	77.74	160.00	74.77	2049.6
250:20:75	61.26	75.62	202.8	74.36	75.60	225.20	74.37	2586.6
250:20:100	70.44	75.66	228.2	75.26	75.64	297.60	75.27	2689.8
Average	59.87	75.91	201.20	74.10	75.80	237.80	74.16	2623.83

Table 2: Atamtürk's bounded instances

Looking at the bounds obtained after one round of tableau cuts, it is clear that MIR and two-step MIR cuts together close the integrality gap significantly more than MIR cuts alone. For example, in Table 1, for $|I| : |J| : |R| =$ 250:1:75, the corresponding numbers are 54.82% and 79.16%, respectively for the tableau cuts. For the formulation cuts, note that MIR cuts are significantly stronger and therefore the difference between MIR and two-step MIR cuts is less dramatic. To generate base inequalities for MIR and two-step MIR cuts in this dataset, we scale the rows of the original formulation by every variable coefficient to get base inequalities.

When compared with the scaled MIR cuts, the two-step MIR cuts also compare favorably. For example, on 250:1:50, 1 round of MIR cuts close 50.08% of the integrality gap in our experiments (the number reported in [8] is 50.07%). Scaled MIR cuts, as reported in [8], close 78.52% of the integrality gap, whereas one round of MIR and two-step MIR cuts close 83.38% of the integrality gap, with much fewer cuts added. We note that, in our experiments we add at most four violated two-step MIR cuts per tableau or formulation row, whereas, the number of cuts reported in these tables are taken from [8], where the authors seem to add all violated 1-50 scaled MIR cuts.

We also note that the integrality gap closed by two-step MIR cuts is very similar for tableau and formulation cuts. This is a quite surprising observation especially because the gap closed by MIR cuts are very different for tableau and formulation cuts. As this behavior is not seen in other datasets, it is clearly not due to the nature of two-step MIR cuts, but due to how these problem instances were constructed. In addition, two-step MIR cuts based on the tableau appear to be as strong as as the lifted knapsack cuts for unbounded instances (as reported in [3]) and weaker for the bounded instances.

We also performed experiments to see how the integrality gap changed when several rounds of tableau cuts are added. Our main observation in these experiments is that one round of MIR and two-step MIR cuts closes the same amount of integrality gap that can be closed by many rounds of MIR cuts. In Figure 5 we show how the average integrality gap changes for unbounded Atamtürk instances. The left axis denotes the integrality gap closed and the horizontal axis gives the number of rounds. The dashed line is for MIR and two-step MIR cuts together and the solid line shows the MIR cuts alone. Typically, at least five rounds of MIR cuts are necessary to exceed the improvement obtained after one round of MIR and two-step MIR cuts.

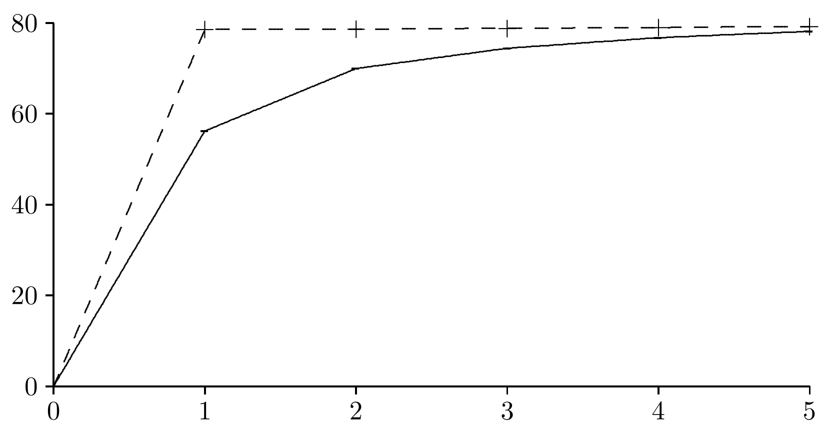

Figure 5: Multiple rounds of MIR and two-step MIR tableau cuts

4.2 Steel

These are real-life instances from the steel industry, and are essentially two-dimensional cutting stock problems with additional constraints. The problem consists of selecting the minimum "cost" set of two-dimensional patterns of rectangular plates such that the number of plates of a given "order" (with a given dimension, and desired minimum and maximum number of plates) do not exceed the maximum for that order. There are additional constraints on usage of manufacturing resources for the selected patterns. There are additional variables to count the number of *completed* orders (the selected patterns contain more than the minimum number of plates for the order). Finally, the objective function combines the goals of maximizing completed orders and reducing waste on the selected patterns. We try out our two-step MIR code on four instances having 3,000 to 7,000 constraints and 4,000 to 7,000 variables.

	Tableau cuts			Formulation cuts		
Instance	MIR	+2MIR	Δ	MIR	+2MIR	Δ
Steel 1	69.3	70.1	0.8	50.0	50.9	0.9
Steel 2	55.9	58.8	2.9	38.9	47.0	8.1
Steel 3	54.3	55.7	1.4	46.5	48.1	1.6
Steel 4	58.0	60.7	2.7	67.2	68.1	0.9
Average	59.4	61.3	1.9	50.65	53.53	2.88

Table 3: Steel instances

4.3 MIPLIB instances

We now study the performance of two-step MIR cuts on some practical problem instances given in the MIPLIB 3.0 library [6]. Table 4 lists all the problems (25/65) where two-step MIR cuts make a difference (either as tableau cuts after one round of cuts, or as formulation cuts). For the problems *mkc, p0282, qnet1-o*, two-step MIR tableau cuts make a very small but non-zero difference and we indicate this by a '0.00*'.

It is clear by looking at this table that two-step MIR cuts do not make as big a difference as in the Atamtürk test set. The average improvement in the first round of tableau cuts because of adding two-step MIR cuts is about 1.89%, which is not very large. If we exclude *arki001* and *swath*, the improvement is less than 1.0%.

In the next section, we will analyze in detail the problems for which two-step MIR cuts do not make any difference after one round of tableau cuts.

5 Group cuts for MIPLIB problems

We saw earlier that for MIPLIB problems the improvement in one round of tableau cuts because of adding two-step MIR cuts is relatively small. Does this mean that two-step MIR cuts are very weak on these problem instances ? We will argue that for a number of MIPLIB instances, the two-step MIR cuts are not necessarily weak, but rather MIR cuts are strong with respect to the tableau rows. For these instances, we will show that cuts from a much larger class of cuts – which includes MIR and two-step MIR cuts – would not have an additional effect on the integrality gap.

To make this more precise, we need to review some earlier work. Recall the set W, defined in (1). Obviously (1) can be expressed as:

$$W \;=\; \Big\{v \in R^{|J|}, x \in Z^{|I|} \;:\; (\sum_{i\in I}\lfloor a_i\rfloor x_i - \lfloor b\rfloor) + \sum_{j\in J} c_j v_j + \sum_{i \in I} \hat{a}_i\, x_i = \hat{b}, \quad x, v \geq 0.\Big\} \qquad (9)$$

Let $\hat{a}_i$ $(i \in I)$ and $\hat{b}$ be multiples of $1/n$ for some integer n, and let $b = r/n$, where $0 < r < n$. It

	Tableau cuts			Formulation cuts		
Instance	MIR	+2MIR	Δ	MIR	+2MIR	Δ
arki001	29.26	43.07	13.81	12.93	12.93	0.00
bell5	14.53	14.86	0.33	0.00	0.00	0.00
cap6000	41.65	42.27	0.62	0.00	12.43	12.43
dcmulti	47.65	48.45	0.80	0.00	0.00	0.00
fiber	63.09	66.42	3.33	88.45	90.08	1.63
gen	60.69	61.32	0.63	90.55	93.77	3.22
gesa2	28.53	28.72	0.19	68.91	70.29	1.38
gesa3	47.53	47.53	0.00	38.44	46.20	7.76
gt2	69.71	70.63	0.92	89.71	89.68	-0.03
harp2	23.11	26.74	3.63	51.38	57.62	6.24
l152lav	8.47	9.17	0.70	0.01	0.01	0.00
lseu	41.59	41.71	0.12	64.59	68.55	3.96
mas74	6.67	7.48	0.81	0.00	0.00	0.00
mas76	6.42	7.09	0.67	0.00	0.00	0.00
mitre	83.29	84.33	1.04	94.95	97.65	2.70
mkc	2.41	2.41	0.00*	4.35	11.87	7.52
mod008	20.11	20.35	0.24	47.38	63.77	16.39
mod010	93.24	93.72	0.48	18.34	18.34	0.00
p0033	56.82	57.08	0.26	56.86	57.65	0.79
p0282	3.70	3.70	0.00*	92.94	95.01	2.07
p0548	39.20	40.00	0.80	39.90	40.01	0.11
p2756	0.54	0.61	0.07	0.21	0.21	0.00
qnet1	8.81	8.88	0.07	36.85	45.96	9.11
qnet1-o	34.19	34.19	0.00*	76.58	77.24	0.66
swath	8.42	26.08	17.66	0.00	0.00	0.00
Average	33.59	35.47	1.89	38.93	41.97	3.04

Table 4: 25 MIPLIB instances where two-step MIR cuts make a diference

is clear that every point in W can be mapped to a point in the polyhedron

$$P(n,r) = conv\{v^+, v^- \in R,\ w \in Z^{n-1} : v^+ - v^- + \sum_{i=1}^{n-1} (i/n)\, w_i + z = r/n,\ \ v^+, v^-, w \geq 0, z \in Z\}, \tag{10}$$

via the mapping

$$\begin{aligned} w_k &= \sum_{k/n=\hat{a}_i} x_i, \\ v_+ &= \sum_{c_j \geq 0} c_j v_j, \quad v_- = \sum_{c_j<0} c_j v_j, \\ z &= \sum_{i \in I} \lfloor a_i \rfloor x_i - \lfloor b \rfloor. \end{aligned} \tag{11}$$

If for some $w_k, \sum_{k/n=\hat{a}_i} x_i$ has no terms, we set w_k to zero. Therefore, valid inequalities for $P(n,r)$ yield valid inequalities for W.

$P(n,r)$ is a mixed-integer extension of the *master cyclic group polyhedron* of Gomory[12]. Gomory and Johnson [13] studied $P(n,r)$ and showed that every facet of $P(n,r)$ has the form

$$n\eta_1 v^+ + n\eta_{n-1} v^- + \sum_{i=1}^{n-1} \eta_i \geq 1, \tag{12}$$

where $\eta = (\eta_1, \ldots, \eta_{n-1})$ is an extreme point of the set of inequalities

$$\eta_i + \eta_j \geq \eta_{(i+j) \bmod n} \quad \forall i, j \in \{1, \ldots, n-1\}, \tag{13}$$

$$\eta_i + \eta_j = \eta_r \quad \forall i, j \text{ such that } r = (i+j) \bmod n, \tag{14}$$

$$\eta_j \geq 0 \quad \forall j \in \{1, \ldots, n-1\}, \tag{15}$$

$$\eta_r = 1. \tag{16}$$

Further, given a facet (12), a valid inequality for W can be obtained as

$$n\eta_1 (\sum_{c_j \geq 0} c_j v_j) + n\eta_{n-1} (\sum_{c_j<0} c_j v_j) + \sum_{i \in I} f(\hat{a}_i) \geq 1,$$

where $f(\hat{a}_i) = \eta_k$ if $\hat{a}_i = k/n$. We call such inequalities *group cuts* for W. It is not difficult to show, after some transformations, that the MIR inequality (2) and the two-step MIR inequality (3) are both group cuts [10].

Gomory and Johnson [13] showed that $P(n,r)$ has exponentially many facets (in n). For say $n = 50$ and $r = 10$, we show in [11] that $P(n,r)$ has at least 74,000 facets. In our experiments with tableau rows, in one round of MIR cuts we add just one out of exponentially many group cuts for each tableau row (and two or three group cuts if we add two-step MIR cuts).

5.1 Finding violated group cuts

Let $P = \{x, v \in R : Ax + Cv = b, x, v \geq 0\}$ be the LP relaxation of the integer program we are interested in (with x standing for the integer variables). Assume we have computed an optimal basis for P with respect to the integer program's objective function, and the associated simplex tableau. We will assume that all data is rational. Let the kth tableau row be

$$x_{l_k} + \sum_{i \in I, i \neq l_k} a_{ki} x_i + \sum_{j \in J} c_{kj} v_j = b_k, \tag{17}$$

where x_{l_k} is the kth basic integer variable. Let W_k stand for the set of non-negative integral solutions of the kth tableau row. As W_k has the same form as W, we can define the corresponding master cyclic group polyhedron $P'(n_k, r_k)$, where a_{ki} and b_k are multiples of $1/n_k$ and $b_k = r_k/n_k$. Now let W'_k be the same as W_k except that x_{l_k} is not restricted to be non-negative, and let $P_k = conv(W'_k)$. It is not hard to show that P_k is essentially a face of $P'(n_k, r_k)$[12, 13]. Thus the group cuts for W_k are essentially the same as the valid inequalities for P_k.

Let (v^*, x^*) be the solution obtained after adding a round of MIR cuts. The question we are interested in is: does (v^*, x^*) satisfy all group cuts? Equivalently, is the combined effect of MIR cuts derived from the different tableau rows such that

$$(v^*, x^*) \in P \cap \{\cap_k P_k\}? \tag{18}$$

If so, then no additional valid inequalities for the different P_ks would make a difference. Surprisingly this is true for a number of MIPLIB instances, and for these instances two-step MIR cuts are of no use in addition to MIR cuts. Fischetti and Saturni independently suggest this possibility; they conjecture that (18) is true for many MIPLIB problems.

For the kth tableau row, let n_k be the smallest positive integer such that the rhs b_k and the coefficients a_{ki} of the non-zero integer variables are multiples of $1/n_k$. Let $r_k = n_k b_k$. Separating the point (v^*, x^*) from P_k is equivalent to separating the point (w^*, v^*_+, v^*_-) from $P'(n_k, r_k)$, where (w^*, v^*_+, v^*_-) is obtained via the mapping (11). The last problem is just a linear program because of the results of Gomory and Johnson mentioned earlier. Therefore, if the basic optimal solution of

$$\min\{\sum_i w^*_i \eta_i + (nv^*_+)\eta_1 + (nv^*_-)\eta_{n-1} | \eta \text{ satisfies } (13) - (16)\}, \tag{19}$$

where n and r are replaced by n_k and r_k, has value less than 1, we have found a violated facet of $P'(n_k, r_k)$, and a valid inequality for P_k cutting off (v^*, x^*). We call (19) the *separation lp*. On the other hand, if the optimum value of the separation LP is at least 1, then no valid inequalities for P_k are violated by (v^*, x^*).

5.2 Lower bounds for the separation LP

For many of tableau rows for the miplib problems, the multipliers n_k are too large to allow us to solve the separation lp exactly. For example, if $n_k = 2000$, (19) has about 2,000 variables and 2,000,000 constraints. We therefore try to obtain lower bounds on the optimal solution value of (19) without explicitly solving it, if possible. From (13), it is easy to infer that for any facet η and any index k between 1 and $n-1$

$$k\eta_1 \geq \eta_k, \ k\eta_{n-1} \geq \eta_{n-k}. \tag{20}$$

Using $\eta_r = 1$, this implies that

$$\eta_1 \geq \frac{1}{r}, \ \eta_{n-1} \geq \frac{1}{n-r}. \tag{21}$$

It follows from the inequalities above that

$$w_r^* + \frac{1}{r}(nv_+^* + w_1^*) + \frac{1}{n-r}(nv_-^* + w_{n-1}^*) \tag{22}$$

is a lower bound on the objective function value of the separation lp. For the MIR facet, $\eta_i = i/r$ if $i < r$ and $\eta_i = (n-i)/(n-r)$ if $i > r$, and the inequalities in (21) hold with equality.

We next give some observations which help us deal with the tableau rows without solving the corresponding separation lp explicitly. Consider the kth tableau row, and assume we have a point (v^*, x^*) obtained after adding MIR cuts for every tableau row. Let (w^*, v_+^*, v_-^*) be the point obtained using (11). This point obviously satisfies the MIR cut for $P'(n_k, r_k)$. Suppose we know n_k; let n stand for n_k and r for r_k.

Observation 1: If (22) has value 1 or more, then no facet of $P(n,r)$ is violated by (w^*, v_+^*, v_-^*).

Observation 2: Assume (w^*, v_+^*, v_-^*) satisfies the MIR facet inequality for $P(n,r)$. There is no violated facet of $P(n,r)$ if

$$nv_+^* + w_1^* \geq \sum_{1<i<r} (r-i)w_i^* \text{ and } nv_-^* + w_{n-1}^* \geq \sum_{r<i<n-1} (i-r)w_i^*. \tag{23}$$

Proof: Let η' stand for the MIR facet, and η'' for any other facet. We know that

$$\sum_i w_i^* \eta_i' + (nv_+^*)\eta_1' + (nv_-^*)\eta_{n-1}' \geq 1. \tag{24}$$

We will show that the conditions in Observation 2 imply that if we replace η' by η'', the left hand side of the expression above can only increase, and hence no facet is violated. Let $\eta_1'' = \eta_1' + \epsilon$. Let k be an index between 2 and $r-1$. We know that

$$\begin{aligned} 1 - \eta_k'' = \eta_{r-k}'' \leq (r-k)\eta_1'' &= (r-k)(\eta_1' + \epsilon) \\ = (r-k)\eta_1' + \epsilon(r-k) &= \eta_{r-k}' + \epsilon(r-k) \\ = 1 - \eta_k' + \epsilon(r-k) \Rightarrow \eta_k'' &\geq \eta_k' - \epsilon(r-k) \end{aligned}$$

Denote $nv_+^* + w_1^*$ by w_1^{**} and $nv_-^* + w_{n-1}^*$ by w_{n-1}^{**}. The above inequalities imply that

$$\begin{aligned} w_1^{**}\eta_1'' + \sum_{1<i<r} w_i^*\eta_i'' &\geq w_1^{**}(\eta_1' + \epsilon) + \sum_{1<i<r} w_i^*(\eta_i' - \epsilon(r-i)) \geq \\ w_1^{**}\eta_1' + \sum_{1<i<r} w_i^*\eta_i' &+ \epsilon(w_1^{**} - \sum_{1<i<r}(r-i)w_i^*) \geq w_1^{**}\eta_1' + \sum_{1<i<r} w_i^*\eta_i'. \end{aligned}$$

The last inequality follows from the first condition in Observation 2. Similarly, one can show that the second condition in Observation 2 implies that

$$w_{n-1}^{**}\eta_{n-1}'' + \sum_{r<i<n-1} w_i^*\eta_i'' \geq w_{n-1}^{**}\eta_{n-1}' + \sum_{r<i<n-1} w_i^*\eta_i'.$$

As $\eta_r' = \eta_r'' = 1$, we have shown that replacing η' by η'' in (24) causes the left hand side of to increase. ■

Observation 3: Let $T = \{1, r, n-1\} \cup \{i : w_i^* \neq 0\}$. Consider the relaxation of the separation lp consisting of all constraints where one of the indices is contained in T. If the optimal value is at least 1, then there are no violated facets.

Suppose we do not know n and r. We cannot compute w_i^* in general without knowing n; yet we can still compute the value of w_r^* without knowing n and r; this is given by

$$w_r^* = \sum_{\hat{a}_i = \hat{b}} x_i^*.$$

Further, we know that

$$n\eta_1 = \frac{1}{\hat{b}}, \; n\eta_{n-1} = \frac{1}{1-\hat{b}}.$$

This implies that

$$w_r^* + \frac{v_+^*}{\hat{b}} + \frac{v_-^*}{1-\hat{b}} \tag{25}$$

is less than or equal to (22).

Observation 4: If (25) has value 1 or more, there are no violated facets.

Observation 5: There is no violated facet of $P(n, r)$ if

$$v_+^* \geq \sum_{\hat{a}_i < \hat{b}} (\hat{b} - \hat{a}_i) x_i^* \text{ and } v_-^* \geq \sum_{\hat{b} < \hat{a}_i} (\hat{a}_i - \hat{b}) x_i^*.$$

Proof: Divide the inequalities in (23) by n. If $\hat{a}_i = k/n$, then $(\hat{b} - \hat{a}_i)x_i^* = w_k^*(r-k)/n$. Also, $w_1^*, w_{n-1}^* \geq 0$ (recall we do not know their exact values). Therefore, if the inequalities above are satisfied, then so are the inequalities in (23). ∎

5.3 Computational experiments

We now describe the precise way we deal with the tableau rows and the corresponding separation lp. We first check if n_k lies between 1 and 20,000.

1. If $n_k \leq 20,000$. We then apply observations 1 and 2. Assume observations 1 and 2 are not sufficient to show that there are no violated group cuts. If $n_k \leq 400$, we explicitly solve the separation lp. If $n_k > 400$, we instead solve the relaxation given in observation 3.
2. If $n_k > 20,000$, we do not compute it. We check if observations 4 and 5 imply that there are no violated group cuts.

Using the above observations, we can verify for 19 MIPLIB problems that once MIR tableau cuts are added, no group cuts are violated. This statement is approximately true for another 4 problems. Out of 65 problems in MIPLIB 3.0, 24 problems have violated group cuts as shown in Table 4. Of the remaining 41 problems, we can decide whether or not there are violated group cuts after adding one round of MIR cuts for 26 problems. We list these in Table 5 as the first set of problems. The second set consists of problems from the ZIB problem library with the above property.

In Table 5 the first column indicates the problem and the second column the integrality gap closed by one round of MIR cuts. The third column gives the additional reduction (denoted by Δ) in integrality gap because of two-step MIR cuts. A "> 0" in the second or third column indicates a non-zero value (which we either do not know exactly as for the ZIB problems, or is less than 0.001). The fourth (fifth) column has a 'y' if the solution after adding MIR cuts (MIR+two-step MIR cuts) does not violate any group cuts, and a 'n' otherwise. It is clear that if there is a 'y' for a problem in the fourth column, there will be a 'y' in the fifth column too, and so we omit the second 'y'. A '*' after a 'y' indicates that (18) is approximately true. For example, for *gesa3o*, after a round of MIR cuts, the lower bound on the optimal value of the separation lp is 1.0 - 0.0002 (the value in the brackets). We thus assert that if there is a violated facet of $P(n, r)$ from a tableau row for *gesa3o*, the violation is at most 0.0002. Observe that for only four problems in Table 5 – *misc03, stein27, stein45, liu* – two-step MIR cuts do not make a difference, but there are other violated group cuts. For *liu*, even though two-step MIR cuts do not change the integrality gap ($\Delta = 0$), when they are added no additional group cuts are violated.

In Table 6, we give 5 MIPLIB problems (out of 24) and 1 ZIB problem where two-step MIR cuts make a difference. The columns in Table 6 convey the same information as in Table 5. For problems in Table 6, Δ (column three) is non-zero, and column four will obviously have the value 'n'. We wish to see whether, after adding two-step MIR cuts in addition to MIR cuts, any group cuts are violated or not. There are 24 such MIPLIB problems, and we only list the ones for which we can solve the separation lps for all tableau rows. For *swath*, and *nsrand-ipx* (18) is not true after adding MIR cuts, but becomes true once two-step MIR cuts are also added. Thus for MIPLIB problems, in most cases where two-step MIR cuts are not useful, the underlying reason is the strength of the MIR cuts with respect to the tableau rows. We find this quite surprising.

References

[1] T. Achterberg, A. Martin.

[2] A. Agra and M. F. Constantino, Description of 2-integer continuous knapsack polyhedra, working paper (2003).

[3] A. Atamtürk, On the Facets of the Mixed-Integer Knapsack Polyhedron, *Mathematical Programming* **98** 145–175 (2003).

[4] A. Atamtürk and D. Rajan.

[5] E. Balas, S. Ceria, G. Cornuéjols, G. Natraj, Gomory cuts revisited, *Operations Research Letters* **19** 1–9 (1996).

[6] R. E. Bixby, S. Ceria, C. M. McZeal, M. W. .P. Savelsbergh. An updated mixed integer programming library: MIPLIB 3.0.

[7] R. E. Bixby, M. Fenelon, Z. Gu, E. Rothberg, and R. Wunderling. MIP: Theory and practice – closing the gap. In M. J. D. Powell and S. Scholtes, editors, System Modelling and Optimization: Methods, Theory, and Applications, pages 19–49. Kluwer Academic Publishers, 2000.

[8] M. Fischetti, C. Saturni, Mixed integer cuts from cyclic groups, *Manuscript*, 2005.

[9] S. Dash, J. Kalagnanam, C. Reddy, S. Song, A heirarchical bin-packing and scheduling problem from the steel industry, *Manuscript*, 2005.

[10] S. Dash and O. Günlük, Valid inequalities based on simple mixed-integer sets, *Mathematical Programming*, to appear.

[11] S. Dash and O. Günlük, Comparing valid inequalities for cyclic group polyhedra, *Mathematical Programming*, to appear.

[12] R.E. Gomory, Some polyhedra related to combinatorial problems, Journal of Linear Algebra and its Applications, **2**, 451–558 (1969).

[13] R.E. Gomory and E. Johnson, Some continuous functions related to corner polyhedra I, *Mathematical Programming* **3** 23–85 (1972).

Instance	MIR %	Δ %	MIR	MIR+2MIR
bell3a	45.10	0	y	
blend2	15.98	0	y	
dano3mip	0.11	0	y*(.005)	
dsbmip	100.00	0	y	
egout	55.63	0	y	
fixnet6	12.88	0	y	
flugpl	11.74	0	y	
gesa2o	31.03	0	y*(.005)	
gesa3	47.53	0	y*(.0002)	
gesa3o	50.54	0	y*(.0002)	
khb05250	74.91	0	y	
misc03	7.24	0	n	n
misc06	28.47	0	y	
misc07	0.00	0	y	
mod011	31.80	0	y	
noswot	0.00	0	y	
nw04	62.27	0	y	
p0201	26.71	0	y	
pp08a	54.42	0	y	
qiu	2.53	0	y	
rentacar	28.07	0	y	
set1ch	39.16	0	y	
stein27	0.00	0	n	n
stein45	0.00	0	n	n
vpm1	22.91	0	y	
vpm2	10.93	0	y	
a1c1s1	> 0	0	y	
liu	> 0	0	n	y
net12	> 0	0	y	
tr12-30	> 0	0	y	

Table 5: Problems where two-step MIR cuts are not useful

Instance	MIR %	Δ %	MIR	MIR+2MIR
fiber	63.09	3.33	n	n
lseu	41.59	0.12	n	n
p0033	56.82	0.26	n	n
p0282	3.70	> 0	n	y*(.0002)
swath	8.42	17.66	n	y
nsrand-ipx	> 0	> 0	n	y

Table 6: Problems where two-step MIR cuts are useful

[14] R.E. Gomory and E. Johnson, Some continuous functions related to corner polyhedra II, *Mathematical Programming* **3** 359–389 (1972).

[15] H. Marchand and L. Wolsey, Aggregation and mixed integer rounding to solve MIPs, *Oper. Res.* **49**, 363–371 (2001).

[16] L.A. Wolsey, *Integer Programming*, Wiley, New York (1998).

6 APPENDIX

$\|I\| : \|J\| : \|R\|$		Round 1	Round 2	Round 3	Round 4	Round 5
250:1:100	MIR	63.06	72.2	72.66	74.78	75
	+list	75.24	75.44	75.72	75.84	76
	Δ	12.18	3.24	3.06	1.06	1
250:1:50	MIR	50.08	67.26	78.86	81.04	83.28
	+list	83.38	83.38	83.44	83.7	83.86
	Δ	33.3	16.12	4.58	2.66	0.58
250:1:75	MIR	54.82	69.44	73.42	77.46	79.24
	+list	79.16	79.16	79.42	79.64	79.86
	Δ	24.34	9.72	6	2.18	0.62
500:1:100	MIR	63.5	73.5	73.78	73.98	74.38
	+list	75.96	75.96	76.16	76.28	76.56
	Δ	12.46	2.46	2.38	2.3	2.18
500:1:50	MIR	50.68	67.74	77.04	77.6	81.04
	+list	80.4	80.4	80.44	80.94	81.16
	Δ	29.72	12.66	3.4	3.34	0.12
500:1:75	MIR	55.14	69.64	70.8	75.76	76.04
	+list	77.44	77.44	77.58	77.68	77.9
	Δ	22.3	7.8	6.78	1.92	1.86
Averages	MIR	56.21	69.96	74.43	76.77	78.16
	+list	78.6	78.63	78.79	79.01	79.22
	Δ	22.38	8.67	4.37	2.24	1.06

Table 7: Multiple rounds of Tableau Cuts on Atamtürk's unbounded instances

		Tableau Cuts				
Instance		Round 1	Round 2	Round 3	Round 4	Round 5
Steel 1	MIR	69.28	86.46	88.83	89.57	90.06
	+list	70.11	87.08	88.96	89.63	90.12
	Δ	0.83	0.62	0.13	0.06	0.06
Steel 2	MIR	55.86	68.66	70.41	71.22	71.57
	+list	58.76	70.76	71.67	72.46	72.71
	Δ	2.9	2.1	1.26	1.24	1.14
Steel 3	MIR	54.31	66.02	67.66	68.65	70.81
	+list	55.71	67.44	69.99	70.61	72.02
	Δ	1.4	1.42	2.33	1.96	1.21
Steel 4	MIR	57.99	70.18	74.66	76.47	78.23
	+list	60.65	72.47	76.74	78.18	79.47
	Δ	2.66	2.29	2.08	1.71	1.24
Average	MIR	59.36	72.83	75.39	76.48	77.67
	+list	61.31	74.44	76.84	77.72	78.58
	Δ	1.95	1.61	1.45	1.24	0.91

Table 8: Multiple rounds of Tableau Cuts on Steel instances

$\lvert I\rvert : \lvert J\rvert : \lvert R\rvert$		Round 1	Round 2	Round 3	Round 4	Round 5
250:10:100	MIR	67.4	73.26	74.28	74.64	75.46
	+list	75.38	75.4	75.88	76.02	76.68
	Δ	7.98	2.14	1.6	1.38	1.22
250:10:50	MIR	50.88	68.16	74.18	77	79
	+list	80.02	80.02	80.06	80.64	80.96
	Δ	29.14	11.86	5.88	3.64	1.96
250:10:75	MIR	55.88	68.26	68.54	70.38	70.62
	+list	74.06	74.06	74.18	74.54	74.66
	Δ	18.18	5.8	5.64	4.16	4.04
250:20:100	MIR	70.44	74.18	74.9	75.16	75.42
	+list	75.66	75.68	76.06	76.16	76.4
	Δ	5.22	1.5	1.16	1	0.98
250:20:50	MIR	51.58	68.5	71.06	74.24	76.28
	+list	77.86	77.86	78.16	78.42	78.68
	Δ	26.28	9.36	7.1	4.18	2.4
250:20:75	MIR	61.26	71.68	72.66	73.26	74.06
	+list	75.62	75.62	75.8	76.1	76.68
	Δ	14.36	3.94	3.14	2.84	2.62
250:5:100	MIR	61.32	69.52	69.86	72.6	73.22
	+list	72.34	72.36	72.44	73.92	74.38
	Δ	11.02	2.84	2.58	1.32	1.16
250:5:50	MIR	50.4	67.68	76.84	81.56	82.16
	+list	81.42	81.42	81.46	81.86	82.5
	Δ	31.02	13.74	4.62	0.3	0.34
250:5:75	MIR	55.62	68.94	70.96	71.46	73.08
	+list	75.5	75.5	76	76.16	76.24
	Δ	19.88	6.56	5.04	4.7	3.16
500:10:100	MIR	66.42	72.64	73.62	75.02	75.34
	+list	74.96	75	75.78	76.74	76.92
	Δ	8.54	2.36	2.16	1.72	1.58
500:10:50	MIR	51.52	68.58	70.34	75.12	77.36
	+list	77.56	77.56	77.7	77.8	78.22
	Δ	26.04	8.98	7.36	2.68	0.86
500:10:75	MIR	56	67.9	71.88	73.2	74.28
	+list	74.24	74.24	75.78	76.26	76.7
	Δ	18.24	6.34	3.9	3.06	2.42
500:20:100	MIR	71.44	74.1	75.04	75.38	75.68
	+list	75.4	75.4	75.92	76.22	76.5
	Δ	3.96	1.3	0.88	0.84	0.82
500:20:50	MIR	51.74	68.46	70.34	73.7	75.56
	+list	77	77	77.22	77.46	77.74
	Δ	25.26	8.54	6.88	3.76	2.18
500:20:75	MIR	58.98	70.62	71.08	71.26	71.44
	+list	74.42	74.42	74.6	74.7	74.78
	Δ	15.44	3.8	3.52	3.44	3.34
500:5:100	MIR	62.02	69.02	72.58	73.68	73.96
	+list	72.9	72.92	74.88	75.6	76
	Δ	10.88	3.9	2.3	1.92	2.04
500:5:50	MIR	51.48	68.52	69.28	75.02	80.18
	+list	78.14	78.14	78.28	78.38	80.54
	Δ	26.66	9.62	9	3.36	0.36
500:5:75	MIR	55.36	67.74	71.34	72.1	72.36
	+list	73.9	73.9	75.42	75.8	75.9
	Δ	18.54	6.16	4.08	3.7	3.54
Averages	MIR	58.22	69.77	71.9	73.95	75.06
	+list	75.69	75.69	76.23	76.63	77.05
	Δ	17.47	5.92	4.33	2.69	1.99

Table 9: Multiple rounds of Tableau Cuts on Atamtürk's bounded instance
s

Instance		Round 1	Round 2	Round 3	Round 4	Round 5
blend2.mps	MIR	15.98	17.27	22.64	24.43	24.85
	+list	16.07	17.27	22.64	25.13	25.53
	Δ	0.09	0	0	0.7	0.68
mod011.mps	MIR	37.26	41.74	42.93	43.82	44.88
	+list	37.28	41.74	42.96	43.84	44.89
	Δ	0.02	0	0.03	0.02	0.01
dcmulti.mps	MIR	47.65	55.45	61.68	64.31	68.35
	+list	48.5	56.08	62.22	64.58	68.82
	Δ	0.85	0.63	0.54	0.27	0.47
set1ch.mps	MIR	39.16	75.08	87.78	92.96	93.94
	+list	39.16	75.08	87.81	92.99	93.95
	Δ	0	0	0.03	0.03	0.01
p0201.mps	MIR	26.71	39.72	46.98	51.08	55.25
	+list	26.71	40.64	47.3	52.04	57.16
	Δ	0	0.92	0.32	0.96	1.91
qnet1-o.mps	MIR	42.99	45.66	48.55	49.61	50.11
	+list	43.1	45.68	48.92	49.84	50.16
	Δ	0.11	0.02	0.37	0.23	0.05
vpm2.mps	MIR	10.93	17.19	22.55	31.45	39.1
	+list	10.93	17.21	22.57	31.53	39.1
	Δ	0	0.02	0.02	0.08	0
gesa3.mps	MIR	47.53	50.3	53.78	56.19	57.9
	+list	47.53	50.3	53.78	56.19	57.92
	Δ	0	0	0	0	0.02
harp2.mps	MIR	24.07	30.16	31.11	31.9	31.9
	+list	27.08	32.5	32.79	33.01	33.01
	Δ	3.01	2.34	1.68	1.11	1.11
vpm1.mps	MIR	24.73	34.09	39.2	40.46	54.55
	+list	24.73	34.09	39.2	40.47	54.88
	Δ	0	0	0	0.01	0.33
modglob.mps	MIR	17.28	42.38	52.93	58.75	60.25
	+list	17.28	42.38	52.93	58.88	60.32
	Δ	0	0	0	0.13	0.07
gesa2.mps	MIR	28.53	59.98	69.31	74.48	77.38
	+list	28.72	60.01	69.65	74.96	77.78
	Δ	0.19	0.03	0.34	0.48	0.4
gen.mps	MIR	60.69	64.61	65.53	67.64	68.63
	+list	61.48	64.65	65.57	67.76	68.72
	Δ	0.79	0.04	0.04	0.12	0.09
fixnet6.mps	MIR	12.88	24.85	56.4	67.98	75.73
	+list	12.88	24.85	56.41	67.98	75.78
	Δ	0	0	0.01	0	0.05
rgn.mps	MIR	3.15	8.16	16	16	16
	+list	3.15	8.16	16	16	16
	Δ	0	0	0	0	0
cap6000.mps	MIR	41.65	57.87	60.16	61.51	61.82
	+list	42.27	59.02	61.65	62.05	62.27
	Δ	0.62	1.15	1.49	0.54	0.45
pp08a.mps	MIR	54.42	69.77	76.92	81.6	83.18
	+list	54.42	69.77	76.92	81.6	83.18
	Δ	0	0	0	0	0
Average	MIR	31.51	43.19	50.26	53.77	56.7
	+list	31.84	43.5	50.55	54.05	57.03
	Δ	0.33	0.3	0.29	0.28	0.33

Table 10: Multiple rounds of Tableau Cuts on MIPLIB instances