RC23848 (W0601-093) January 27, 2006
Electrical Engineering

IBM Research Report

Identical Synchronization in Networks of Coupled Nonlinear Circuits and Systems

Chai Wah Wu
IBM Research Division
Thomas J. Watson Research Center
P.O. Box 218
Yorktown Heights, NY 10598

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Chai Wah Wu, *Fellow, IEEE*

***Abstract*— Synchronization is a ubiquitous phenomenon and is crucial in many coupled physical systems. We present a theory of identical synchronization in networks of coupled nonlinear dynamical systems. We first consider the case of two coupled systems, and demonstrate chaotic communication systems as an application. Next the general case of a network of coupled systems is considered. We show how the coupling topology can influence the ability of the network to synchronize. In particular, we study algebraic quantities related to the graph in order to characterize this relationship. Furthermore, we show that random coupling facilitates synchronization, whereas local coupling does not.**

***Index Terms*— Markov chains, nonlinear dynamics, graph theory, synchronization.**

I. INTRODUCTION

ACCORDING to the New Oxford American Dictionary, a definition of *to synchronize* is to occur at the same time or rate. This is how synchronization is usually understood in the sciences; e.g. the synchronization of fireflies [1], [2], timing synchronization in digital circuits [3] and phase lock loops [4], synchronization of pendulum clocks [5], and synchronization in neural networks occurring in the visual cortex [6]–[8] and during epilepsy [9]. Since there is an inherent rate in the definition, the systems that synchronize are generally periodic or quasi-periodic and this type of synchronization is referred to as *phase synchronization*. The reader is referred to [10] for a recent tutorial on some aspects of phase synchronization. One of the first people to study synchronization is Huygens in the seventeenth century where he noticed that two pendulum clocks mounted on the same frame will synchronize after some time [5]. At the beginning of the twentieth century, the synchronization of clocks plays an important role in Einstein's derivation of the theory of relativity [11].

This article deals with a different and stronger kind of synchronization. This concept first came about when studying chaotic systems. Systems operating in the chaotic regime are aperiodic and exhibit sensitive dependence on initial conditions, i.e. a small change in initial conditions or parameters leads to locally divergent or uncorrelated trajectories. There is no specific fixed period or frequency in the system and thus the traditional definition is not applicable. In this case, synchronization is meant to indicate that the behavior between two systems approaches each other. This type of synchronization is called *identical synchronization*[1]. This is almost the antithesis of the sensitive dependence on initial conditions property in chaotic systems. Pecora and Carroll were the first to report that surprisingly, identical synchronization is possible in two chaotic systems coupled in a master-slave configuration [12]. Identical synchronization is a stronger form of synchronization than phase synchronization and as a result is easier to study and analyze. As many natural, biological and man-made systems can exhibit chaotic behavior, identical synchronization has been observed in many physical systems [13], [14].

The purpose of this article is to give a concise presentation of recent results on identical synchronization and describe various applications to designing and understanding networks of complex coupled systems. The main theoretical results require a mixture of linear algebra, dynamical systems theory and graph theory.

II. BASIC FRAMEWORK AND NOTATION

We assume that the coupled systems under consideration are either ordinary differential equations for the continuous-time case or maps for the discrete-time case.

For the continuous-time case, consider a collection of n systems. The i-th system is written as $\dot{x}_i = \tilde{f}_i(x_i, t)$ where x_i is the state vector of the i-th system.

We consider coupling among the n systems such that the state equation of the coupled ensemble can be written as:

$$\dot{x}_i = f_i(x_1, \ldots, x_n, t), i = 1, \ldots, n \tag{1}$$

For the discrete-time case, each system is defined as $x_i(t+1) = \tilde{f}_i(x_i(t), t)$ and the state equations of the coupled systems are given by:

$$x_i(t+1) = f_i(x_1(t), \ldots x_n(t), t), i = 1, \ldots, n \tag{2}$$

In this case t are natural numbers.

We assume that for each set of initial conditions at t_0 there exists a unique trajectory for all time $t > t_0$. For an initial condition $x(t_0)$, we denote the corresponding trajectory as $x(t)$.

Definition 1: The coupled system (Eq. (1) or Eq. (2)) is said to *synchronize* if $\|x_i - x_j\| \to 0$ for $t \to \infty$ and all i,j.

An equivalent way to define synchronization is that as $t \to \infty$, the states approach the linear *synchronization manifold* $\mathcal{M}$, defined as the set $\{(x_1, \ldots, x_n) : x_i = x_j, \forall i, j\}$. A third way to characterize synchronization is to say that the diameter of the convex hull of $x_1, \cdots, x_n$ vanishes as $t \to \infty$. This third characterization will be useful in Section X-D.

Throughout this article we will use $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ to denote the aggregate vector of all the state vectors x_i of the n systems, where $x_i \in \mathbb{R}^m$ for each i. We say a matrix A is positive

C. W. Wu is with IBM T. J. Watson Research Center, Yorktown Heights, NY, USA.

[1]It is also referred to as complete synchronization.

semidefinite if $x^T Ax \geq 0$ for all x. We denote this as $A \succeq 0$. Similarly, we write $A \succ 0$ to say that A is positive definite ($x^T Ax > 0$ for all $x \neq 0$). The vector of all 1's is denoted $\mathbf{1}$. For a matrix A with real eigenvalues, we list them in increasing order as:

$$\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$$

In this case, we also use $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ to denote $\lambda_1(A)$ and $\lambda_n(A)$ respectively.

Synchronization is closely related to notions of stability and controllability. To this end, let us first recall some standard notations and definitions, starting with the following definition of asymptotic stability[2]:

Definition 2: The system $\dot{x} = f(x,t)$ or $x(t+1) = f(x(t),t)$ is asymptotically stable if given two initial conditions $x(t_0)$ and $\tilde{x}(t_0)$, the corresponding trajectories approach each other, i.e. $\|x(t) - \tilde{x}(t)\| \to 0$ as $t \to \infty$.

Many classical results in absolute stability theory such as passivity theory [16] and Luré-Postnikov theory [17] give conditions under which a system is asymptotically stable.

Definition 3 ([18]): For an n by n matrix A and n-vectors w and b, (A, w^T) is observable if the observability Grammian $\begin{pmatrix} w^T \\ w^T A \\ \vdots \\ w^T A^{n-1} \end{pmatrix}$ is nonsingular. (A,b) is controllable if the controllability Grammian $(b|Ab|\cdots|A^{n-1}b)$ is nonsingular.

We will concentrate on the continuous time case and defer the discrete-time case to Section IX.

III. Synchronization between two coupled systems

The simplest case occurs when there are only two identical systems, i.e. $n = 2$. Starting with two identical systems $\dot{x}_1 = \tilde{f}(x_1,t)$ and $\dot{x}_2 = \tilde{f}(x_2,t)$, if the state equations of the coupled systems can be written as:

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_1, x_2, t) \\ \dot{x}_2 &= f(x_2, x_1, x_2, t) \end{aligned} \tag{3}$$

then the following synchronization theorem follows directly from Definition 2.

Theorem 1 ([19], [20]): Eq. (3) synchronizes if $\dot{x}_1 = f(x_1, u(t), v(t), t)$ is asymptotically stable for every $u(t)$ and $v(t)$.

At the synchronized state, $x_2(t) = x_1(t)$ and thus $\dot{x}_1 = f(x_1, x_1, x_1, t)$. Therefore if the *consistency condition*

$$f(x_1, x_1, x_1, t) = \tilde{f}(x_1, t) \quad \text{for all } x_1,\ t$$

is satisfied, then when synchronization is reached, each system will follow the dynamics of the uncoupled system $\dot{x}_1 = \tilde{f}(x_1, t)$.

[2]In systems theory [15], [16], asymptotically stability would also include the additional ϵ-δ condition of stability, i.e. for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x(t_0) - \tilde{x}(t_0)\| \leq \delta$, then $\|x(t) - \tilde{x}(t)\| \leq \epsilon$ for all $t \geq t_0$. We can develop synchronization theory with or without this additional condition; we choose to not include this condition in this article.

Two configurations of Eq. (3) are of particular interest: the master-slave subsystem decomposition configuration originally studied in [21] and the additive coupling configuration.

Theorem 2 (Master-slave subsystem decomposition): Let the master system be decomposed into two subsystems as:

$$\begin{aligned} \dot{x}_1(t) &= g_1(x_1, y_1, t) \\ \dot{y}_1(t) &= g_2(x_1, y_1, t) \end{aligned}$$

Then a slave system with the same state equations as the second subsystem and driven by the state of the first subsystem x_1:

$$\dot{y}_2(t) = g_2(x_1, y_2, t)$$

synchronizes to the master subsystem, i.e. $\|y_1 - y_2\| \to 0$ as $t \to \infty$ if $\dot{y}_1(t) = g_2(v(t), y_1, t)$ is asymptotically stable for all functions $v(\cdot)$.

Theorem 3 (Synchronization through additive coupling): Two coupled systems

$$\begin{aligned} \dot{x}_1 &= f(x_1,t) + K_1(x_2,t) - K_1(x_1,t) \\ \dot{x}_2 &= f(x_2,t) + K_2(x_1,t) - K_2(x_2,t) \end{aligned}$$

synchronizes, if $\dot{x}_1 = f(x_1,t) - K_1(x_1,t) - K_2(x_1,t) + \eta(t)$ is asymptotically stable for all functions $\eta(\cdot)$.

One way the stability condition in Theorem 3 is satisfied is through quadratic stabilizability (see Section IV). The case when $K_1 = 0$ and K_2 is linear, i.e.

$$\begin{aligned} \dot{x}_1 &= f(x_1,t) \\ \dot{x}_2 &= f(x_2,t) + K_2(t)(x_1 - x_2) \end{aligned}$$

was studied in [22] via feedback control.

In a master slave configuration, the construction of a synchronizing system can be related to the observer problem in control systems [23]. In particular, given a plant $\dot{x}_1 = f(x_1,t)$, the dynamical system $\dot{x}_2 = g(x_1, x_2, t)$ is called an *full state observer* for the plant if x_2 approaches x_1 as $t \to \infty$.

In master-slave applications such as communication systems (Sect. III-A), it is desirable that the coupling signal from the master system to the slave system be a scalar signal. Under certain conditions, synchronization via a scalar coupling signal can be achieved by appealing to eigenvalue assignment results in linear feedback control. In particular, we have the following result.

Theorem 4 ([23], [24]): Two identical systems of the form $\dot{x} = Ax + f_1(w^T x)$ where f_1 is a function from $\mathbb{R}$ to $\mathbb{R}^n$ and w is an n-vector can be synchronized via a scalar coupling signal if (A, w^T) is an observable pair. Two identical systems of the form $\dot{x} = Ax + bf_2(x) + d$ where b and d are n-vectors and f_2 is a scalar-valued function can be synchronized via a scalar coupling signal if (A,b) is a controllable pair.

In particular, the synchronization is achieved through the following coupling schemes. If (A, w^T) is observable, the following two systems will synchronize:

$$\begin{aligned} \dot{x}_1 &= Ax_1 + f_1(w^T x_1) \\ \dot{x}_2 &= Ax_2 + f_1(w^T x_2) \\ &\quad -f_1(w^T x_2) + f_1(w^T x_1) - bw^T(x_1 - x_2) \end{aligned} \tag{4}$$

where $f_1(w^T x_1) - bw^T(x_1 - x_2)$ is the scalar signal from the first system that is coupled into the second system. If (A,b)

is controllable,

$$\begin{array}{rcl}\dot{x}_1 & = & Ax_1 + bf_2(x_1) + d \\ \dot{x}_2 & = & Ax_2 + bf_2(x_2) + d \\ & & +bw^T x_2 - bf_2(x_2) + b(f_2(x_1) - w^T x_1)\end{array} \tag{5}$$

synchronizes where $f_2(x_1) - w^T x_1$ is the scalar signal from the first system that is coupled into the second system.

Many chaotic systems [25], [26] and hyperchaotic systems [27]–[29] can be written in the Luré form required by Theorem 4.

A. *Chaotic communication systems*

One of the first applications of synchronized chaotic systems is their use in communication systems [30]. The idea is to add a chaotic signal to the information signal in the transmitter and send the combined signal. At the receiving end, the chaotic signal is retrieved through synchronization and subtracted from the combined signal to obtained the information signal. In [31], [32] this idea is refined by using the information signal to modulate the chaotic system and eliminating one source of error in the setup in [30].

Generally, we can create a communication system using synchronization and modulation of chaos in the following way:

$$\dot{x}_1 = f(x_1, s(t), t) \tag{6}$$

$$y = h(x_1, s(t), t) \tag{7}$$

$$\dot{x}_2 = g(x_2, y, t) \tag{8}$$

Here $s(t)$ is the information signal which is fed into the transmitter (Eq. (6)) to modulate the chaos. The transmitted signal y is based upon the modulated chaotic signal x_1 and the information signal s via the encoding function h (Eq. (7)). The transmitted signal y is then sent to the receiver in Eq. (8). The transmitter and receiver (defined by the vector fields f and g and the function h) is constructed to satisfy the following relationship: $g(x_1, h(x_1, s(t)), t) = f(x_1, s(t), t)$ for all signals $s(t)$. If $\dot{x}_1 = f(x_1, s(t), t)$ is asymptotically stable for all $s(t)$, then Theorem 1 shows that the receiver is synchronized to the transmitter, i.e. $x_2(t) \to x_1(t)$ as $t \to \infty$. If the encoding function h is "invertible" in the sense that given x_1 and $h(x_1, s, t)$ we can determine s, then[3] we can recover $s(t)$ from the transmitted signal y and the state vector x_2 in the receiver. Schematically, this is shown in Figure 1, where d is the decoding or inversion operation which recovers s from x_1 and $h(x_1, s, t)$. One of the simplest form for h is adding the chaotic signal to the information signal and transmitting the sum, i.e. $h(x_1, s, t) = x_1 + s$ [32]. Other forms of h can be found in [33], [19].

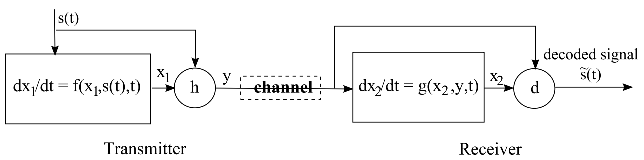

Fig. 1. Communication system via synchronization of chaos.

[3]assuming continuity of h.

Consider the synchronization via a scalar signal discussed earlier. If (A, w^T) is observable and the uncoupled system $\dot{x}_1 = Ax_1 + f_1(w^T)$ is chaotic, we can construct a communication system as follows:

$$\dot{x}_1 = Ax_1 + f_1(h(x_1, s, t)) + b(w^T x_1 - h(x_1, s, t)) \tag{9}$$

$$\dot{x}_2 = Ax_2 + f_1(h(x_1, s, t)) + b(w^T x_2 - h(x_1, s, t)) \tag{10}$$

where Eq. (9) is the transmitter and Eq. (10) is the receiver. In order for the transmitter (with state vector x_1) to have dynamics similar to an uncoupled system, we require $h(x_1, s, t) \approx w^T x_1$.

If (A, b) is controllable and $\dot{x}_1 = Ax_1 + b(f_2(x_1)) + d$ is chaotic, we construct a system as follows:

$$\begin{array}{rcl}\dot{x}_1 & = & Ax_1 + b(h(x_1, s, t) + w^T x_1) + d \\ \dot{x}_2 & = & Ax_2 + b(h(x_1, s, t) + w^T x_2) + d\end{array}$$

In order for the transmitter to have dynamics similar to an uncoupled system, we require $h(x_1, s, t) \approx f_2(x_1) - w^T x_1$.

The field of chaotic communication systems is an active research area [34] and includes other communication systems that exploit chaos without the use of synchronization[4] [35]. Recently, a communication system based on synchronized chaotic lasers was demonstrated to transmit information through a commercial fiber optic network in downtown Athens [36].

IV. Network of coupled dynamical systems

A generalization of Theorem 1 to more than two coupled systems is the following:

Theorem 5: The network of coupled systems with state equations

$$\begin{array}{rcl}\dot{x}_1 & = & f(x_1, x_1, x_2, \ldots, x_n) \\ \dot{x}_2 & = & f(x_2, x_1, x_2, \ldots, x_n) \\ & \vdots & \\ \dot{x}_n & = & f(x_n, x_1, x_2, \ldots, x_n)\end{array} \tag{11}$$

synchronizes if $\dot{x}_1 = f(x_1, \sigma_1(t), \ldots, \sigma_n(t))$ is asymptotically stable for all σ_i's.

State equations of the form Eq. (11) occur in situations where all the systems receives exactly the same coupling, for instance in globally coupled networks.

A natural way to express the coupling topology in a network of coupled systems is through the language of graph theory. A graph (V, E) consists of a set of vertices V and a set of edges $E \subset V \times V$. We consider weighted graphs where a positive weight is associated to each edge. The adjacency matrix of a graph with n vertices is the n by n matrix A where $A_{ij} = w$ if there is an edge of weight w between vertex i and vertex j and $A_{ij} = 0$ otherwise. The *reversal* of a graph is obtained by reversing the orientation of all the edges. In other words, if a graph has adjacency matrix A, then its reversal has adjacency matrix A^T. A directed tree is a directed graph with n vertices and $n-1$ edges such that there exists a vertex with directed paths to all other vertices. A spanning directed

[4]For example, chaos is used as a source of decorrelated signals and as an alternative to pseudo-random sequences in spread spectrum communication systems.

tree is a subgraph which is a directed tree with the same set of vertices. The *edge connectivity* of a graph is defined as the smallest weighted sum among all subsets of edges such that its removal results in a graph that is not weakly connected. The graph of a matrix A is the directed graph with an edge (v_i, v_j) if and only if $A_{ij} \neq 0$. The average degree of an undirected graph is defined as the total sum of edge weights divided by $2n$.

For state equations of the form Eq. (1) we can characterize the coupling topology by defining an interaction graph [24].

Definition 4: The *interaction graph* of a coupled network in Eq. (1) is defined as the directed graph with vertex set $\{v_i\}$ with an edge (v_i, v_j) if and only if $i \neq j$ and f_j depends on x_i.

The case where the additive coupling terms between systems is linear and uniform is more amenable to analysis. Consider a networks of coupled dynamical system with state equations:

$$\dot{x} = \begin{pmatrix} f(x_1, t) \\ \vdots \\ f(x_n, t) \end{pmatrix} + (G(t) \otimes D(t))x + u(t) \qquad (12)$$

The matrix $G(t)$ describes the coupling topology of the network which changes with time whereas the matrix $D(t)$ describes the coupling between two systems. $G \otimes D$ is the Kronecker product or tensor product of the matrix G and D. The term $(G(t) \otimes D(t))x$ is the additive linear coupling in the network. We call this type of coupling uniform since the matrix $D(t)$ is the same between any two systems. In general we choose $G(t)$ to be a zero row sums matrix for each t. For $u = 0$, this implies that when the network is synchronized, the coupling term vanishes and the consistency condition is satisfied. Thus the dynamics of each system follow those of an uncoupled system $\dot{x}_1 = f(x_1, t)$. If $D(t) \preceq 0$ and $G_{ij}(t) \leq 0$ for $i \neq j$ we call the coupling *cooperative*. If $D(t) \preceq 0$ and $G_{ij}(t) \geq 0$ for $i \neq j$ we call the coupling *competitive*. The case of symmetric G is called *reciprocal* coupling.

The following theorem gives sufficient conditions under which the network of coupled dynamical systems in Eq. (12) synchronizes.

Theorem 6 ([37], [38]): Suppose $u(t)$ is such that $\|u_i - u_j\| \to 0$ as $t \to \infty$. The network in Eq. (12) synchronizes if the following conditions are satisfied:

1) There exists a (time-varying) matrix $H(t)$, a symmetric positive definite matrix V and $c > 0$ such that $(x - y)^T V(f(x,t) + H(t)x - f(y,t) - H(t)y) \leq -c\|x - y\|^2$ for all x, y, t.
2) There exists an irreducible zero row sums matrix with nonpositive off-diagonal elements U such that the matrix $(U \otimes V)(G(t) \otimes D(t) - I \otimes H(t))$ is negative semidefinite for all t.

The first condition is related to linear feedback stabilizability. In particular, by using the Lyapunov function $x^T V x$, it is easy to see that the first condition implies that $f(x,t) + H(t)x + \eta(t)$ is asymptotically stable for all $\eta(\cdot)$, i.e. $H(t)x$ is a stabilizing linear feedback. This property is also known as quadratically stabilizable in the control literature. For instance, the classical Chua's chaotic circuit [39] with state equations

$$\begin{aligned} \dot{v}_1 &= \tfrac{1}{RC_1}(v_2 - v_1) - \tfrac{1}{C_1} f(v_1) \\ \dot{v}_2 &= \tfrac{1}{RC_2}(v_1 - v_2) + \tfrac{1}{C_2} i_3 \\ \dot{v}_3 &= -\tfrac{1}{L} v_2 \end{aligned}$$

can be quadratically stabilized by choosing $V = I$ and $H(t) = -\kappa \ \mathrm{diag}(1,0,0)$ for a large enough scalar $\kappa > 0$ [19]. For differentiable f, the Mean Value Theorem shows that the condition above is equivalent to $V\left(\frac{\partial f(x,t)}{\partial x} + H(t)\right) + \delta I$ being negative definite for some $\delta > 0$ [40].

Assuming the first condition is satisfied, this sufficient condition amounts to finding an appropriate matrix U that satisfies the matrix inequality $(U \otimes V)(G \otimes D - I \otimes H) \preceq 0$. If we assume that $H(t) = \alpha(t)D(t)$ for all t, then this reduces to the inequality $(U \otimes V)((G(t) - \alpha(t)I) \otimes D(t)) \preceq 0$. Let us further assume that $VD(t)$ is symmetric negative semidefinite for all t. This last assumption is true for the following scenarios which are found in practice:

1) V and D are both diagonal and D is negative semidefinite;
2) D is a nonpositive multiple of the identity matrix;
3) V is a positive multiple of the identity matrix and D is symmetric negative semidefinite.

This reduces the inequality to $U(G(t) - \alpha(t)I) \succeq 0$. To focus on this condition, we introduce the following definition

Definition 5: For a given $\alpha(t)$, $G(t)$ satisfies *synchronization condition A* if there exists an irreducible zero row sums matrix with nonpositive off-diagonal elements U such that the matrix $U(G(t) - \alpha(t)I) \succeq 0$ for all t.

The reason for extracting this part of the synchronization condition to focus on, is that this condition is related to the connectivity of the coupling graph, as we will show in Section V.

The case of nonlinear additive coupling

$$\dot{x} = \begin{pmatrix} f(x_1, t) \\ \vdots \\ f(x_n, t) \end{pmatrix} + N(x) + u(t)$$

can be recast into Eq. (12) if the nonlinear term $N(x)$ can be written as $(G(x) \otimes D(x))x$ for each x. One sufficient condition for this to occur is the following.

Theorem 7 ([41]): Suppose $m = 1$, i.e. x_i are scalars and $N(x)$ can be written as

$$N(x) = \begin{pmatrix} \sum_j \phi_{1j}(x_j - x_1) \\ \sum_j \phi_{2j}(x_j - x_2) \\ \vdots \\ \sum_j \phi_{nj}(x_j - x_n) \end{pmatrix}$$

where $\phi_{ij}(0) = 0$, then $N(x) = G(x)x$ for some matrix-valued function $G(\cdot)$ where $G(x)$ is a matrix with zero row sums for each x.

V. Algebraic connectivity and synchronization

The Laplacian matrix of a graph is given by $L = D - A$, where A is the adjacency matrix and D is the diagonal matrix with the row sums of A on the diagonal. Thus the Laplacian matrix is a zero row sums real matrix with nonpositive off-diagonal elements. Let us denote this class of matrices with zero row sums and nonpositive off-diagonal elements as $\mathcal{W}$. We denote $\mathcal{W}_s$ as the set of symmetric irreducible matrices in $\mathcal{W}$. It is important to note that if $G(t)$ is a matrix in $\mathcal{W}$, then it is the Laplacian matrix of the reversal of the interaction graph of Eq. (12). Therefore if L is the Laplacian matrix of a graph, we define the interaction graph of L to be the reversal of this graph. A matrix L can be written in Frobenius normal form [42]:

$$L = P \begin{pmatrix} B_1 & B_{12} & \cdots & B_{1k} \\ & B_2 & \cdots & B_{2k} \\ & & \ddots & \vdots \\ & & & B_k \end{pmatrix} P^T \tag{13}$$

where P is a permutation matrix and B_i are square irreducible matrices. The Frobenius normal form is generally not unique. Finding the permutation matrix P of the Frobenius normal form is related to finding the strongly connected components of the corresponding directed graph, a problem which can be solved in linear time using depth-first search [43]. For a Laplacian matrix L, each matrix B_i can be decomposed as $B_i = L_i + D_i$ where L_i is a zero row sums matrix in $\mathcal{W}$ and D_i is a positive semidefinite diagonal matrix. Let w_i be the unique positive vector[5] such that $\|w_i\|_\infty = 1$ and $w_i^T L_i = 0$ and W_i be the diagonal matrix with w_i on the diagonal. Let w be a nonnegative vector such that $\|w\|_\infty = 1$ and $w^T L = 0$ and W be the diagonal matrix with w on the diagonal. We can then define the following quantities related to the graph and its Laplacian matrix L:

- $a_1(L) = \min_{x \perp \mathbf{1}, \|x\|=1} x^T L x$;
- $a_2(L) = \min_{x \perp \mathbf{1}, \|x\|=1} x^T W L x$;
- $a_3(L) = \min_{x \neq 0} \frac{x^T W L x}{x^T \left(W - \frac{ww^T}{\|w\|_1}\right) x}$;
- $a_4(L) = \min_{1 \leq i \leq k} \eta_i$ where $\eta_i = \min_{x \neq 0} \frac{x^T W_i B_i x}{x^T W_i x}$ for $1 \leq i \leq k-1$ and $\eta_k = \min_{x \neq 0} \frac{x^T W_k B_k x}{x^T \left(W_k - \frac{w_k w_k^T}{\|w_k\|_1}\right) x}$;

It is easy to see that $a_1(L) = a_2(L) = a_3(L) = a_4(L) = \lambda_2(\frac{1}{2}(L + L^T))$ when the graph is undirected or vertex balanced[6]. Furthermore, a_4 does not depend on the specific form of the Frobenius normal form. The numbers a_1 and a_2 can be expressed as eigenvalues. In particular, $a_1(L) = \lambda_{\min}\left(\frac{1}{2}K^T(L + L^T)K\right)$ and $a_2(L) = \lambda_2\left(\frac{1}{2}(WL + L^T W)\right) = \lambda_{\min}\left(\frac{1}{2}K^T(WL + L^T W)K\right)$ where K is an n by $n-1$ matrix whose columns form an orthonormal basis of the linear subspace orthogonal to $\mathbf{1}$. The quantities a_1 to a_4 can be considered as algebraic connectivities of directed graphs, a concept first introduced in [45] for undirected graphs. The case of undirected graphs has been extensively studied in the past (see [46]–[48] for survey papers). The following result lists some known graph-theoretical properties related to the algebraic connectivity of an undirected graph that are useful to synchronization:

[5]The existence of this vector is guaranteed by Frobenius-Perron theory [44].

[6]We define a directed graph to be vertex balanced if for each vertex, the sum of the weights of incoming edges is equal to the sum of the weights of outgoing edges.

Theorem 8 ([45], [49]–[53], [38]): For a graph with Laplacian matrix L, let $\Delta_{\min}$ and $\Delta_{\max}$ be the smallest and the largest vertex degrees respectively. Let e be the edge connectivity, diam the diameter and $\overline{\rho}$ be the average distance between two vertices. The following is true of $\lambda_2(L)$:

1) $\lambda_2(L) \geq 2\Delta_{\min} - n + 2$
2) $\lambda_2(L) \geq 2e(1 - \cos(\frac{\pi}{n}))$
3) $\frac{\Delta_{\max} \ln(n-1)}{2(\text{diam}-2) - \ln(n-1)} \geq \lambda_2(L) \geq \frac{4}{n\,\text{diam}}$
4) $\lambda_2(L) \geq \frac{2}{(n-1)\overline{\rho} - \frac{n-2}{2}}$
5) There exists $c > 0$ such that almost every k-regular random graph satisfies $\lambda_2(L) > ck$ as $n \to \infty$.

The following result relates the algebraic connectivity to various properties of the directed graph.

Theorem 9 ([54]–[56]): Let L be the Laplacian matrix of a graph. Let δ_i and δ_o be the minimum indegree and outdegree respectively and Δ_i and Δ_o be the maximum indegree and outdegree respectively.

1) $a_1(L) \leq \min\left\{\delta_o + \frac{\Delta_i}{n-1}, \Delta_o + \frac{\delta_i}{n-1}\right\}$;
2) $a_1(L) \leq d_o(v) + \frac{1}{n-1} d_i(v)$ where $d_o(v)$ and $d_i(v)$ is the outdegree and the indegree of vertex v respectively;
3) $a_2(L) \geq \frac{\left(1 - \cos\left(\frac{\pi}{n}\right)\right)e}{\rho}$, where e is the edge connectivity and $\rho = \frac{\max_i w_i}{\min_i w_i}$;
4) $a_i(L) \leq 0$, $1 \leq i \leq 4$ if the graph is not weakly connected;
5) If the graph is strongly connected, then $a_3(L) \geq a_2(L) > 0$.
6) $a_4(L) > 0$ if and only if the reversal of the graph of L contains a spanning directed tree;
7) (Super-additivity) For G, $H \in \mathcal{W}$, $a_1(G + H) \geq a_1(G) + a_1(H)$;
8) If the off-diagonal elements of G are random variables chosen independently according to $P(G_{ij} = 1) = p$, $P(G_{ij} = 0) = 1 - p$, then $a_1(G) \approx pn$ in probability as $n \to \infty$.

We separate the coupling topology into two cases, the case where the coupling topology is static and the case where the coupling topology is dynamic.

A. Static coupling topology

In this case $G(t) = G$ does not change with time and the state equation is given by:

$$\dot{x} = \begin{pmatrix} f(x_1, t) \\ \vdots \\ f(x_n, t) \end{pmatrix} + (G \otimes D(t))x + u(t) \tag{14}$$

The quantities a_i provide sufficient conditions for the network to synchronize:

Theorem 10 ([54]–[56]): G satisfies synchronization condition A if one of the following conditions is satisfied for all t:

1) $a_1(G) \geq \alpha(t)$;
2) $a_4(G) \geq \alpha(t)$;
3) The graph of G is strongly connected and $a_3(G) \geq \alpha(t)$.

Since the algebraic connectivity was first defined to quantify connectivity in a graph [45], a way to paraphrase Theorem 10 is that the network is more amenable to synchronization if the underlying graph is more connected. The sufficient condition for synchronization $a_4(G) \geq \alpha$ along with the fact that $a_4(G)$ is positive if and only if the interaction graph contains a spanning directed tree implies the following:

Proposition 1: The network synchronizes if the interaction graph contains a spanning directed tree and the cooperative coupling is large enough.

This result is intuitive since the existence of a spanning directed tree in the interaction graph implies the existence of a system (located at the root of the tree) which influences directly or indirectly all other systems. The converse is also true for networks of chaotic systems. If there does not exist a spanning directed tree, then there are two groups of systems which are not influenced by any other systems [56]. If these two coupled groups remain chaotic, they will not synchronize to each other.

If the graph of G does not contain a spanning directed tree, then G is not irreducible and it is k-reducible for some $1 \leq k \leq n$ [57]. A consequence is that there are k directed trees (and no less) which together span the interaction graph. If $G \in \mathcal{W}$, then this corresponds to G having k zero eigenvalues (Theorem 19). The k strongly connected components (SCC) of the graph which contain the k roots of these trees will not have coupling from other systems and thus they will synchronize within themselves (when the coupling is sufficiently cooperative). Thus there are at least k groups or clusters of systems which synchronize among themselves.

One interpretation for the synchronization occurring in coupled systems where the interaction graph contains a spanning directed tree is that all the systems are synchronized to the system at the root of the spanning directed tree (which we will call the *root system*). This suggests that any control to the system should be applied to such a root system for maximum efficiency since its effect will be felt throughout the network. Of course, in general graphs, the spanning directed tree is not unique and neither is the root system. Furthermore, since the coupling topology can change with time, the set of spanning directed trees can also change with time. It is clear that a root system is unique if and only if there is coupling from the root system to other systems and not vice versa (Fig. 2).

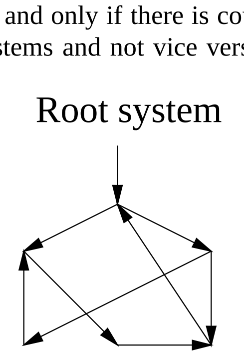

Fig. 2. Interaction graph with a unique root system.

There exists a lower bound on $a_4(G)$ which is related to the structure of the graph. In particular, in [57] a lower bound was given in terms of quantities a_2 and a_3 of the SCC's of the graph and the number of edges between them. The larger the values of a_2 of the SCC's, the larger the value of this bound. Similarly, the more coupling between the SCC, the larger the value of this bound. This is an intuitive conclusion as the a_i's are measures of connectivity of the graph.

More precisely, the SCC's of the graph correspond to the matrices B_i in the Frobenius normal form (Eq. (13)) with decomposition $B_i = L_i + D_i$. D_i describes the coupling to the i-th SCC from other systems. If we remove this coupling, we obtain L_i which corresponds to the coupling within the i-th SCC. Then η_i, $1 \leq i \leq k-1$ in the definition of a_4 can be bounded as

$$\eta_i \geq \frac{a_2(L_i)}{\left(1+\sqrt{1+\frac{a_2(L_i)}{w_i^T D_i \mathbf{1}}}\right)^2 n+1}$$

whereas $\eta_k = a_3(B_k) \geq a_2(B_k)$.

In scenarios where the dynamical systems correspond to physical systems, these systems are arranged in space with respect to some metric. In this case, it make sense to talk about local coupling where each system is only coupled to systems in a local neighborhood of radius δ. In [53], [55] it was shown that if the diameter of the graph grows faster than $\ln(n)$, then $a_2(L) \rightarrow 0$ as $n \rightarrow \infty$. In [53] the systems are located on a regular integer lattice with fixed δ and since the diameter grows as an d-th root of n, local coupling implies that $a_2(L) \rightarrow 0$ as $n \rightarrow \infty$. When the systems are located randomly, the resulting locally coupled graph is called a random geometric graph [58]. If we choose a random geometric graph such that the locations are randomly chosen on a ball of radius $r(n)$ where $r(n)$ grows faster than $\ln(n)$, then it was shown in [59] that the diameter also grows faster than $\ln(n)$ almost always and thus in this case we have $a_2(L) \rightarrow 0$ as $n \rightarrow \infty$ almost always. In general, we can conclude that for a locally coupled graph where the local neighborhood is such that the number of vertices in the neighborhood is uniformly bounded for all n, then $a_2(L) \rightarrow 0$ as $n \rightarrow \infty$.

On the other hand, the last property in Theorem 9 says that $a_1(G) \approx pn$ for random coupling. By the first property in Theorem 9, this is asymptotically the highest possible. This thus indicates that local coupling and random coupling form two extremes in the corresponding network's ability to synchronize [38], [60].

This also points out that the distribution of the vertex degrees is itself not enough to determine synchronizability. A locally connected k-regular graph and a random k-regular graph have the same degree distribution, yet their ability to synchronize are at opposite extremes. Thus how the graph is generated is also an important property of the graph in characterizing its synchronizability. Analogous to the k-regular graph case, one can construct a locally connected undirected graph with a prescribed vertex degree sequence such that $a_1(G) \rightarrow 0$ as $n \rightarrow \infty$ and an undirected random graph with the same prescribed degree sequence such that $a_1(G)$ is bounded away from zero [61], [62].

Because of the super-additivity of λ_2 and the nonnegativity of λ_2 for reciprocal coupling (see [45] and Theorem 9), adding additional reciprocal coupling does not decrease λ_2. In other words, adding cooperative reciprocal coupling can only help improve synchronizability in the network. An important difference between reciprocal and nonreciprocal coupling is that this property is not longer true for nonreciprocal coupling. In particular, consider the following directed path graph (Fig. 3) with corresponding coupling matrix

$$G = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & & 0 \end{pmatrix}.$$

In this case $a_4(G) = 1$.

Fig. 3. Directed path graph.

By adding an additional coupling edge, we obtained the directed cycle graph which is vertex balanced (Fig. 4) and has a corresponding coupling matrix

$$G = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ -1 & & & & 1 \end{pmatrix}$$

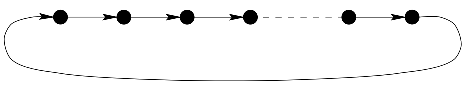

Fig. 4. Directed cycle graph.

For this graph, $a_4(G) = 1 - \cos(\frac{2\pi}{n})$ which decreases to 0 for large n. Thus by adding a *single* coupling element, we have reduced its synchronizability significantly. This can be explained by noting that the SCC's in the first graph are small; they are single vertices. In other words, each vertex influences directly the next vertex and thus synchronization is first achieved with the first 2 vertices, and then synchronization is achieved between vertex 2 and 3, etc. In other words, synchronization can be reached in stages, each time considering a subgraph of two vertices. On the other hand, the SCC is large in the second graph; it is the entire graph with a large diameter, with each vertex influencing and influenced by all other vertices, some of which are far apart. These long paths of communication between vertices make the network harder to synchronize. This phenomena is similar to Braess' paradox in traffic routing, where adding a single road can increase rather than decrease traffic congestion [63].

For a fixed coupling matrix G, what is the largest α such that G satisfies synchronization condition A, i.e. there is a matrix $U \in \mathcal{W}_s$ for which $U(G - \alpha I)$ is positive semidefinite? This value of α, which we denote as $\mu(G)$ gives another measure on how easy it is to synchronize a network with the static coupling topology expressed as G. The following theorem gives some upper and lower bounds on $\mu(G)$.

Theorem 11 ([64]): For a matrix $G \in \mathcal{W}$,

1) $\lambda_{\min}\left(\frac{1}{2}(G + G^T)\right) \leq \mu(G)$;
2) $a_1(G) \leq \mu(G)$ and $a_4(G) \leq \mu(G)$;
3) If the graph of G is strongly connected, $a_2(G) \leq a_3(G) \leq \mu(G)$;
4) $\mu(G) \leq \gamma(G)$; where $\gamma(G) = \min_\lambda Re(\lambda)$, with λ ranging over the eigenvalues of G not belonging to the eigenvector e;
5) If G has both zero rows and zero column sums, then $\gamma(\frac{1}{2}(G + G^T)) \leq \mu(G)$;
6) If G is a normal matrix, then $a_1(G) = \mu(G) = \gamma(G)$;
7) If G is a triangular matrix after simultaneous permutation of its rows and columns (i.e. the graph of G is acyclic), then $\mu(G) = \gamma(G)$.

The value of $\mu(G)$ can be computed efficiently by solving the following semidefinite programming problem for different values of μ_i:

$$\begin{array}{ll} \text{Find } U = U^T & \text{such that} \\ & U(G - \mu_i I) \succeq 0 \\ & U\mathbf{1} = 0, \\ & U_{i,j} \leq 0 \quad \forall i \neq j \\ & \text{and } K^T U K \succeq I. \end{array} \tag{15}$$

where K is as defined before and μ_i's are obtained via a bisection process converging to $\mu(G)$ [64]. Semidefinite programming problems are known to be solvable in polynomial time [65] and there are many public domain and commercial software packages available for solving them (see for instance `http://www-user.tu-chemnitz.de/~helmberg/sdp_software.html`). In [64] some numerical simulations were conducted to determine the difference between $\mu(G)$ and $\gamma(G)$. It was observed that $\mu(G)$ is more likely to be closer to $\gamma(G)$ when all the eigenvalues of G are real versus the case when G has complex eigenvalues. One direction for future research is analyzing more precisely the relationship between $\mu(G)$ and $\gamma(G)$.

B. Dynamic coupling topology

In the static coupling topology case, in order to prove that $a_3(G) \geq \alpha$ implies synchronization, we choose a matrix $U \in \mathcal{W}_s$ that depends on G. In contrast to the constant coupling case, the synchronization condition in the time-varying coupling case is not necessarily satisfied even if $a_3(G(t)) \geq \alpha(t)$ for all t since the matrix U is not allowed to change for different t. By choosing $U = I - \frac{ee^T}{n}$, it was shown in [54] that $U(G(t) - \alpha(t)I) \succeq 0$ for all t and thus:

Theorem 12: $G(t)$ satisfies synchronization condition A if $a_1(G(t)) \geq \alpha(t)$ for all t.

Since a_1 is generally smaller than a_4, comparing Theorem 6 with Theorem 12 shows that the synchronization criterion is more conservative for time-varying coupling topologies.

Note that the synchronization criterion does not depend on the rate of change of $G(t)$ and $D(t)$. For a criterion that does depend on the rate of change in $G(t)$, see [66].

VI. Algebraic connectivity of complex networks

Many man-made and natural networks have a complicated structure that can not be explained or modelled through classical random graph models such as those in [67]. Recently, graph models have been developed that try to mimic the coarse structure of such networks [68]–[71]. In order to understand synchronization in networks with such coupling topologies, we study the algebraic connectivity of such graph models. The Strogatz-Watts small world network is an undirected graph built by taking a nearest neighbour graph and *replacing* edges with randomly chosen edges. In the Newman-Watts model, randomly chosen edges are *added* to a nearest neighbor graph. The superadditivity property combined with the fact that the algebraic connectivity of local coupling vanishes shows that for large n, the algebraic connectivity of these small world networks is essentially the algebraic connectivity of the subgraph formed by the random edges [38]. Consider the small world networks in [68], [69], where starting from a locally connected graph with t edges, pt random edges (with $0 < p \leq 1$) are substituted or inserted into the graph respectively, with t growing linearly with respect to n, i.e. $t = kn$ for some fixed k. The algebraic connectivity of these small world networks is dominated by the algebraic connectivity of the random graph, which is approximately $2kp$ with high probability for large n.

Consider a set of n integers $0 \leq d_1 \leq d_2 \leq \cdots \leq d_n$. We can create a random undirected graph model $\mathcal{G}_{pr}(d_1, \ldots d_n)$ [38], [62] where an edge (v_i, v_j) is chosen with probability

$$P_{ij} = \frac{d_1}{n} + \frac{(d_i - d_1)(d_j - d_1)}{\sum_k (d_k - d_1)}$$

This is a modification of the random graph model in [72]. Since $\sum_i P_{ij} = d_j$, the expected degree of vertex v_i is d_i, i.e. this random graph has the degree sequence d_i in expectation. For the special case $d_1 = d_2 = \cdots = d_n$, we choose $P_{ij} = \frac{d_1}{n}$ and this is equivalent to the classical random graph model $G(n, p)$ with $p = \frac{d_1}{n}$ [73].

In [38], [62] it was shown that each instance of the random graph model $G(n, p)$ with $p = \frac{d_1}{n}$ is a subgraph of a set of graphs of the random graph model $\mathcal{G}_{pr}(d_1, \ldots d_n)$ with the same probability. Since $G(n, p)$ has $a_1 \approx d_1$ with high probability as $n \rightarrow \infty$ [74], by the superadditivity of a_1 and Theorem 9, the algebraic connectivity a_1 of the random graph model $\mathcal{G}_{pr}(d_1, \ldots, d_n)$ is approximately equal to d_1 with high probability as $n \rightarrow \infty$. Another conclusion we can make is that for a fixed average degree, the algebraic connectivity of the random graph is the highest when the degree sequence is uniform (i.e. $d_1 = \cdots = d_n$). In other words, homogeneity in the degree sequence is beneficial for synchronization.

In [70] a scale free random graph model was studied where new edges are added at each iteration and vertices with a high vertex degree is more likely to connect to the new edges. This process is called *preferential attachment*. A similar model of generating random graphs was considered in [75]–[77] earlier in the context of citation networks. Consider the sequence where the degree sequence satisfies a power law, i.e. $d_i \approx \rho i^{\alpha}$ for some exponent α. This random graph can be considered a model for scale free random networks. Many naturally occurring and man-made graphs appear to have a power law degree sequence with various exponents α [78]. Consider a sequence of graphs with increasing number of vertices n where the average degree d_a is constant for all n. It was numerically shown in [79] that for scale free networks and for each fixed d_a, λ_2 converges to a constant value $\lambda_2^{\infty}(d_a)$ as $n \rightarrow \infty$. Using the model $\mathcal{G}_{pr}$, it was shown analytically in [38] that $\lambda_2^{\infty}(d_a)$ is approximately $\left(1 - \frac{1}{\alpha - 1}\right) d_a$ for $\alpha \geq 2$ and 0 for $\alpha < 2$. This means that for $\alpha = 3$, which is the case considered in [70], $\lambda_2^{\infty}(d_a) = \frac{1}{2} d_a$, which was also verified numerically in [38].

VII. Lyapunov exponents approach to synchronization

In [80], local synchronization criteria are derived based on numerical estimates of Lyapunov exponents. In this case, for appropriate D (e.g., $D = I$), the network synchronizes if the nonzero eigenvalues of G has real parts which are large enough. Corollary 9 in Section X says that all nonzero eigenvalues of G has nonzero real parts if and only if the interaction graph contains a spanning directed tree. Thus the Lyapunov exponents based synchronization criterion shows that the network synchronizes locally if and only if the interaction graph contains a spanning directed tree and the cooperative coupling is large enough and this statement is qualitatively the same as Proposition 1.

The synchronization criterion for any arbitrary static network topology can also be obtained experimentally by measuring the synchronization property of a coupled network of only 3 systems [81]–[83]. The main idea here is that the nonzero eigenvalues of a parameterized graph of 3 vertices can be arbitrarily placed and thus varying the parameters of a network of 3 coupled systems suffices to determine the synchronization properties of an arbitrary network. In particular, by studying whether a network with the coupling topology in Figure 5 synchronizes for various parameters α, β we can determine whether an arbitrary network synchronizes.

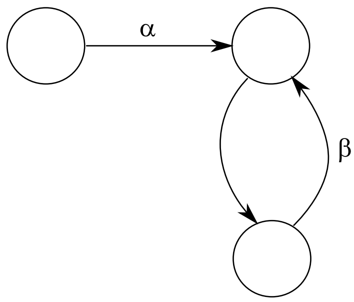

Fig. 5. Studying a network with this coupling topology is sufficient to characterize synchronization in network with arbitrary topology.

VIII. Coupling between delayed state variables

Consider the case of additive linear coupling where coupling terms are present between delayed state variables. The state equations are written as [84], [85]:

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} f(x_1, t) \\ \vdots \\ f(x_n, t) \end{pmatrix} + (G(t) \otimes D(t))x(t) \\ &\quad + (G_\tau(t) \otimes D_\tau(t))x(t - \tau) + u(t) \end{aligned} \tag{16}$$

We assume that $\|u_i - u_j\| \rightarrow 0$ for all i, j as $t \rightarrow \infty$. The following result gives sufficient conditions for the synchronization of Eq. (16).

Theorem 13 ([85]): Let $V \succ 0$ be a symmetric matrix such that $(y-z)^T V(f(y,t)+H(t)y-f(z,t)-H(t)z) \leq -c\|y-z\|^2$ for some $c>0$. Let $U \in \mathcal{W}_s$, $(B_1(t), B_2(t))$ a factorization of $UG_\tau(t) \otimes VD_\tau(t) = B_1(t)B_2(t)$, and $Y(t) \succ 0$ and symmetric for all t. The network in Eq. (16) synchronizes if

$$\begin{array}{rcl} R & \stackrel{\triangle}{=} & (U\otimes V)(G(t)\otimes D(t) - I\otimes H(t)) \\ & & +\frac{1}{2}B_1(t)Y(t)B_1^T(t) + \frac{1}{2}B_2^T(t)Y^{-1}(t)B_2(t) \preceq 0 \end{array} \tag{17}$$

for all t.

The application of Theorem 13 has several degrees of freedom: the choice of $(B_1(t), B_2(t))$, the choice of $Y(t)$ and the choice of U. Choosing them properly will simplify the condition in Eq. (17). There are several ways to choose the factorization (B_1, B_2). Depending on the factorization, the matrix $Y(t)$ can have different dimensions than $G\otimes D$ and $G_\tau \otimes D_\tau$. When the delay coupling term is absent ($G_\tau \otimes D_\tau = 0$), we can pick $B_1 = B_2 = 0$ and the synchronization theorem reverts back to the nondelay case in Theorem 6. The factorization should be chosen such that the synchronization manifold $\mathcal{M}$ is in the kernel of both B_1^T and B_2. Otherwise, as $\mathcal{M}$ is in the kernel of $(U\otimes V)$, this would mean that the matrix R in Eq. (17) is never negative semidefinite. If G_τ does not have constant row sums, then $UG_\tau \mathbf{1} \neq 0$ and $\mathcal{M}$ is not in the kernel of $UG_\tau \otimes VD_\tau$ and thus is also not in the kernel of B_2. Therefore if Eq. (17) is satisfied, then G_τ has constant row sums.

Let $J = \mathbf{1}\mathbf{1}^T$ be the matrix of all 1's and $Q = I - \frac{1}{n}J \in \mathcal{W}_s$. Note that nQ is the Laplacian matrix of the complete graph. The eigenvalues of Q are 0 and 1. If X is a matrix with zero column sums, then $JX = 0$ and thus $QX = X$. In particular, $Q^2 = Q$, and $QU = UQ = U$ for $U \in \mathcal{W}_s$. By choosing the factorizations $(B_1, B_2) = (U\otimes V, G_\tau \otimes D_\tau)$ and $(B_1, B_2) = (Q\otimes I, UG_\tau \otimes VD_\tau)$ we get the following result:

Corollary 1: Let $V \succ 0$ be some symmetric matrix such that $(y-z)^T V(f(y,t)+H(t)y-f(z,t)-H(t)z) \leq -c\|y-z\|^2$ for some $c>0$. Let $U \in \mathcal{W}_s$ and $Y(t) \succ 0$ and symmetric for all t. The network synchronizes if one of the following 2 conditions is satisfied for all t:

$$\begin{array}{l} (U\otimes V)(G(t)\otimes D(t) - I\otimes H(t)) \\ +\frac{1}{2}(G_\tau(t)\otimes D_\tau(t))^T Y^{-1}(t)(G_\tau(t)\otimes D_\tau(t)) \\ +\frac{1}{2}(U\otimes V)Y(t)(U\otimes V) \preceq 0 \end{array} \tag{18}$$

$$\begin{array}{l} (U\otimes V)(G(t)\otimes D(t) - I\otimes H(t)) \\ +\frac{1}{2}(UG_\tau(t)\otimes VD_\tau(t))^T Y^{-1}(t)(UG_\tau(t)\otimes VD_\tau(t)) \\ +\frac{1}{2}(Q\otimes I)Y(t)(Q\otimes I) \preceq 0 \end{array} \tag{19}$$

The condition in Eq. (18) was obtained in [84] for the case of constant coupling. If G_τ has zero row sums, we can choose the factorization $(B_1, B_2) = (UG_\tau \otimes VD_\tau, Q\otimes I)$ to get:

Corollary 2: Let $V \succ 0$ be some symmetric matrix such that $(y-z)^T V(f(y,t)+H(t)y-f(z,t)-H(t)z) \leq -c\|y-z\|^2$ for some $c>0$. Let $U \in \mathcal{W}_s$ and $Y(t) \succ 0$ and symmetric for all t. If G_τ has zero row sums, then the network synchronizes if the following condition is satisfied for all t:

$$\begin{array}{l} (U\otimes V)(G(t)\otimes D(t) - I\otimes H(t)) + \frac{1}{2}(Q\otimes I)Y(t)(Q\otimes I) \\ +\frac{1}{2}(UG_\tau(t)\otimes VD_\tau(t))Y^{-1}(t)(UG_\tau(t)\otimes VD_\tau(t))^T \preceq 0 \end{array}$$

A. Choosing the matrix U

By choosing $U = Q$ as was done in [38] and using the fact that $QX = X$ when X is a zero column sum matrix, the synchronization condition can be further simplified:

Corollary 3: Let $V \succ 0$ be some symmetric matrix such that $(y-z)^T V(f(y,t)+H(t)y-f(z,t)-H(t)z) \leq -c\|y-z\|^2$ for some $c>0$. Let $Y(t) \succ 0$ and symmetric for all t. Suppose G_τ and G are zero column sums matrices. The network in Eq. (16) synchronizes if one of the following conditions is satisfied for all t:

$$\begin{array}{l} G(t)\otimes VD(t) - Q\otimes VP(t) + \frac{1}{2}(Q\otimes V)Y(t)(Q\otimes V) \\ +\frac{1}{2}(G_\tau(t)\otimes D_\tau(t))^T Y^{-1}(t)(G_\tau(t)\otimes D_\tau(t)) \preceq 0 \end{array} \tag{20}$$

$$\begin{array}{l} G(t)\otimes VD(t) - Q\otimes VH(t) + \frac{1}{2}(Q\otimes I)Y(t)(Q\otimes I) \\ +\frac{1}{2}(G_\tau(t)\otimes VD_\tau(t))^T Y^{-1}(t)(G_\tau(t)\otimes VD_\tau(t)) \preceq 0 \end{array} \tag{21}$$

B. Choosing the matrix Y

By choosing Y appropriately we can show the following:

Corollary 4: Let $V \succ 0$ be some symmetric matrix such that $(y-z)^T V(f(y,t)+H(t)y-f(z,t)-H(t)z) \leq -c\|y-z\|^2$ for some $c>0$. Suppose $G_\tau(t)$ and $VD_\tau(t)$ are symmetric for all t, and G and G_τ are zero column sums matrix. Suppose further that G_τ has a simple zero eigenvalue and D_τ is nonsingular for all t. The network in Eq. (16) synchronizes if the following condition is satisfied for all t:

$$G(t)\otimes VD(t) - Q\otimes VH(t) + \sqrt{(G_\tau\otimes VD_\tau)^2} \preceq 0 \tag{22}$$

In particular, if in addition $G_\tau(t)\otimes VD_\tau(t) \succeq 0$ and symmetric for all t, then the network synchronizes if

$$G(t)\otimes VD(t) + G_\tau(t)\otimes VD_\tau(t) - Q\otimes VH(t) \preceq 0$$

C. The case $D = 0$

Consider the case where all the coupling involves delayed state variables, i.e. $D = 0$.[7] Assume that each individual system $\dot{x} = f(x,t)$ is asymptotically stable such that the addition of small positive feedback of the form $H(t)x$ where $H(t) \succeq 0$ does not change its stability. Then this stability is not destroyed if the delay coupling is small, i.e. each system x_i still converges towards the unique trajectory. More precisely, we have the following Corollary to Theorem 13:

Corollary 5: Let $V \succ 0$ be some symmetric matrix such that $(y-z)^T V(f(y,t)+H(t)y-f(z,t)-H(t)z) \leq -c\|y-z\|^2$ for some $c>0$. Let $U \in \mathcal{W}_s$, $(B_1(t), B_2(t))$ a factorization of $UG_\tau(t)\otimes VD_\tau(t) = B_1(t)B_2(t)$, and $Y(t) \succ 0$ and symmetric for all t. The network in Eq. (16) synchronizes if

$$U\otimes VH(t) \succeq \frac{1}{2}B_1(t)Y(t)B_1^T(t) + \frac{1}{2}B_2^T(t)Y^{-1}(t)B_2(t) \tag{23}$$

for all t.

[7] This case was first studied in [86].

D. Algebraic connectivity and synchronization

Similar to the nondelay case, the algebraic connectivity can be useful in characterizing synchronizability of the network.

Theorem 14: Let $V \succ 0$ be a symmetric matrix such that $(y-z)^T V(f(y,t)+D(t)y-f(z,t)-D(t)z) \leq -c\|y-z\|^2$ for some $c>0$. The network in Eq. (16) synchronizes if the following conditions is satisfied for all t and some $\alpha(t)>0$:

$$\begin{array}{rcl} F & \stackrel{\triangle}{=} & (QG(t)-Q)\otimes VD(t)+\frac{\alpha(t)}{2}(Q\otimes I) \\ & & +\frac{(QG_\tau(t)\otimes VD_\tau(t))^T(QG_\tau(t)\otimes VD_\tau(t))}{2\alpha(t)} \preceq 0 \end{array} \tag{24}$$

If in addition $VD(t) \prec 0$ for all t and G is a zero row sums matrix, then the network synchronizes if

$$a_1(G(t)) \geq 1+\|QG_\tau(t)\|\|VD_\tau(t)\|\|(VD(t))^{-1}\| \tag{25}$$

When G is a zero row and column sums matrix, $QG = G$. Furthermore, $a_1(G) = \gamma(\frac{1}{2}(G+G^T))$ and we have the following Corollary to Theorem 14:

Corollary 6: Let $V \succ 0$ be some symmetric matrix such that $(y-z)^T V(f(y,t)+D(t)y-f(z,t)-D(t)z) \leq -c\|y-z\|^2$ for some $c>0$. Suppose G_τ and G are zero column sums matrices. The network in Eq. (16) synchronizes if the following condition is satisfied for all t and some $\alpha(t)>0$:

$$\begin{array}{l} (G(t)-Q)\otimes VD(t)+\frac{\alpha(t)}{2}(Q\otimes I) \\ +\frac{1}{2\alpha(t)}(G_\tau(t)\otimes VD_\tau(t))^T(G_\tau(t)\otimes VD_\tau(t)) \preceq 0 \end{array} \tag{26}$$

If in addition $VD(t) \prec 0$ for all t and G is a zero row sums matrix, then the network synchronizes if

$$a_1(G(t)) \geq 1+\|G_\tau(t)\|\|VD_\tau(t)\|\|(VD(t))^{-1}\| \tag{27}$$

When the nondelay coupling topology $G(t)$ does not change with time, Theorem 14 can be further improved:

Theorem 15: Let $V \succ 0$ be a symmetric matrix such that $(y-z)^T V(f(y,t)+D(t)y-f(z,t)-D(t)z) \leq -c\|y-z\|^2$ for some $c>0$. Let G be an irreducible zero row sums matrix and w be a positive vector such that $w^TG=0$ and $\|w\|_\infty = 1$. Let W be a diagonal matrix with w on the diagonal. Assume that $VD(t) \prec 0$ for all t. Then the network in Eq. (16) synchronizes if

$$a_3(G) \geq 1+\|UG_\tau(t)\|\|VD_\tau(t)\|\|(VD(t))^{-1}\|$$

where $U = W - \frac{ww^T}{\|w\|_1}$.

In Section V it was concluded that a coupled network synchronizes if the underlying graph has a large algebraic connectivity. Theorem 14, Corollary 6 and Theorem 15 are extensions of this result to the case of coupling among delayed state variables.

IX. DISCRETE-TIME SYSTEMS

Analogous to continuous-time systems we have the following condition for quadratically stabilizability in discrete-time systems:

Theorem 16: If $f(x,t)$ satisfies

$$(f(x,t)-f(y,t))^T V (f(x,t)-f(y,t)) \leq c(x-y)^T V(x-y) \tag{28}$$

for some $0 \leq c < 1$ and some positive definite matrix V, then $x(t+1) = f(x(t),t)+\eta(t)$ is asymptotically stable for all $\eta(t)$.

Consider the following state equation of a network of coupled discrete-time systems:

$$\begin{array}{rcl} x(t+1) & = & \begin{pmatrix} x_1(t+1) \\ \vdots \\ x_n(t+1) \end{pmatrix} \\ & = & (I-G(t)\otimes D(t)) \begin{pmatrix} f(x_1(t),t) \\ \vdots \\ f(x_n(t),t) \end{pmatrix} \\ & = & (I-G(t)\otimes D(t))F(x(t),t) \end{array} \tag{29}$$

where the matrix $G(t)$ is a zero row sums matrix for each t. Similar to the continuous-time case, when the network synchronizes, the dynamics of each systems follow that of an uncoupled system. When the individual systems are autonomous (i.e. f does not depend on t) and can be represented as maps they are generally known as coupled map lattices [87].

Similar to Theorem 6, synchronization in a network of coupled discrete-time systems can be deduced from the eigenvalues of the coupling matrix G:

Theorem 17 ([88], [24]): Consider the network of coupled discrete-time systems with state equation Eq. (29) where $G(t)$ is a normal[8] matrix for each t. Let $V \succ 0$ be a symmetric matrix with decomposition[9] $V = C^TC$ such that Eq. (28) is satisfied for $c>0$ and all t, x, y.

If for each time t, $\|I - \lambda CD(t)C^{-1}\|_2 < \frac{1}{c}$ for every eigenvalue λ of $G(t)$ not corresponding to the eigenvector $\mathbf{1}$, then the coupled network synchronizes.

When D is also a normal matrix, Theorem 17 can be further simplified.

Theorem 18: Consider the coupled network in Eq. (29) where $G(t)$ and $D(t)$ are normal matrices for each t. Suppose f is Lipschitz continuous in x with Lipschitz constant c. If for each t, $|1-\lambda\mu| < \frac{1}{c}$ for every eigenvalue λ of $G(t)$ not corresponding to $\mathbf{1}$ and every eigenvalue μ of $D(t)$ then the coupled network synchronizes.

The criterion in Theorem 18 is shown graphically in Fig. 6. The coupled system synchronizes if all the eigenvalues of G not belonging to $\mathbf{1}$ lie in the interior of the intersection of circles of radii $\frac{1}{c|\mu_i|}$ centered at $\frac{1}{\mu_i}$ in the complex plane for the eigenvalues μ_i of D. A dual interpretation can be obtained by interchanging the roles of μ and the λ.

When $G(t)$ and $D(t)$ are symmetric, their eigenvalues are real, and we have the following result:

Corollary 7: Let c be the Lipschitz constant of f. If for each t, $D(t)$ is symmetric and has only positive eigenvalues between $\mu_1(t)$ and $\mu_2(t)$ (with $0 < \mu_1(t) \leq \mu_2(t)$) and $G(t)$ is a symmetric matrix with zero row sums and a zero eigenvalue of multiplicity 1 and the nonzero eigenvalues of $G(t)$ are in the interval $\left(\frac{1-\frac{1}{c}}{\mu_1(t)}, \frac{1+\frac{1}{c}}{\mu_2(t)}\right)$ then the coupled network in Eq. (29) synchronizes.

Note the difference between Corollary 7 and the results in Section V. When G is symmetric, in the continuous-time case the synchronization condition is a condition on the smallest

[8]i.e. $G^*G = GG^*$.

[9]An example of such a decomposition is the Cholesky decomposition.

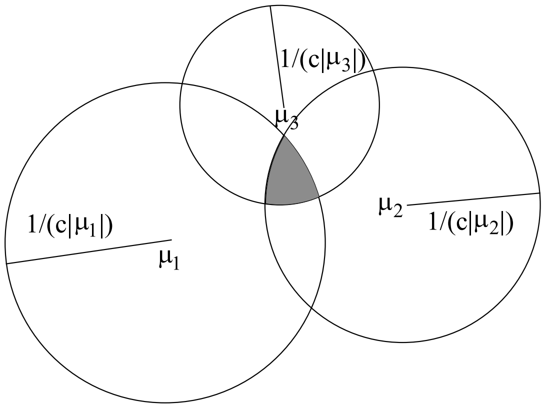

Fig. 6. Graphical interpretation of synchronization criterion in networks of coupled discrete time systems. Coupled network Eq. (29) synchronizes if all eigenvalues of G not corresponding to $\mathbf{1}$ lie inside the intersection of the circles (shown shaded), where μ_i are the eigenvalues of D.

(in magnitude) nonzero eigenvalue of G, while in the discrete-time case the synchronization condition is a condition on both the smallest and the largest nonzero eigenvalues of G. This is similar to the fact that the open left half plane stability condition in continuous time systems is mapped to the interior of the unit circle in the discrete time case via the mapping $z \to e^z$.

Next we consider the case where $G(t) = \epsilon L(t)$ with $\epsilon > 0$ and $L(t)$ is the Laplacian matrix of an undirected graph.

Corollary 8: Let c be the Lipschitz constant of f. Let the graph of $L(t)$ be connected and let $\lambda_2(t)$ and $\lambda_n(t)$ be the smallest and the largest nonzero eigenvalues of the Laplacian matrix $L(t)$ of the graph respectively. If

- $D(t)$ is symmetric and has only positive eigenvalues between $\mu_1(t)$ and $\mu_2(t)$ (with $0 < \mu_1(t) \le \mu_2(t)$)
- $\frac{\lambda_2(t)}{\lambda_n(t)} > \frac{(c-1)\mu_2(t)}{(c+1)\mu_1(t)}$

then the system in Eq. (29) synchronizes with $\epsilon \in \left(\frac{1-\frac{1}{c}}{\mu_1(t)\lambda_2(t)}, \frac{1+\frac{1}{c}}{\mu_2(t)\lambda_n(t)}\right)$.

For the special case where D is a positive multiple of the identity matrix, this implies that $\mu_1 = \mu_2$ and Corollary 8 says that the network in Eq. (29) synchronizes for some ϵ if $r = \frac{\lambda_2}{\lambda_n}$ is close enough to 1.

Section V shows that the second smallest eigenvalue λ_2 of L provides an upper bound on the amount of coupling needed to synchronize a continuous-time coupled network. Corollary 8 shows that for the discrete-time case the quantity useful for synchronization is the ratio $r = \frac{\lambda_2}{\lambda_n}$ between the eigenvalues λ_2 and λ_n of L. This ratio r is also useful for characterizing synchronization thresholds obtained using the Lyapunov Exponent method for some matrices D in the continuous time case (Eq. (12)) in the sense that the larger r is, the easier it is to synchronize the network [80], [89], [90]. The following result gives an upper and lower bound on the ratio r for an undirected graph.

Lemma 1 ([45], [62]):

$$\frac{\left(1-\cos\left(\frac{\pi}{n}\right)\right)\delta}{\Delta} \le r \le \frac{\delta}{\Delta}$$

where δ and Δ are the minimum and maximum vertex degree respectively.

Similar to Section V-A, in [61], [62] graph models were presented that have ratio r converging to 0 and r bounded away from 0 respectively as $n \to \infty$.

For an undirected graph that is not regular, r can be increased by modifying the coupling to be nonreciprocal. In particular, given an undirected graph with Laplacian matrix L and adjacency matrix A, it was argued in [91] via numerical experiments and analytically that the ratio r of $G = (L + A)^{-\beta} L$ reaches a maximum at $\beta = 1$.

X. Linear dynamics

Recently, there is considerable interest in studying coupled dynamical systems in the context of consensus or agreement protocols of independent agents [92]–[97], [41], [98].[10] Some applications include the study of flocking in animal formations and robots with limited range communications. For instance, in flocking problems, if the state of each agent contains the direction of motion, then synchronization implies that all agents are heading in the same direction and achieve flocking behavior. In many of these models, the resulting state equations are linear.

A. Continuous-time systems

For the continuous-time case, the first study in this context is made in [93]. In [93] an agreement problem among n agents is modeled by the following continuous-time autonomous linear dynamical system:

$$\dot{x} = -Lx \tag{30}$$

where $x \in \mathbb{R}^n$ and L is the Laplacian matrix of a graph with n vertices and x_i are scalars. Let us consider the more general affine case:

$$\dot{x} = -(L \otimes D(t))x + \mathbf{1} \otimes u(t) \tag{31}$$

where $D(t)$ has only positive real eigenvalues larger than $\epsilon > 0$ for all t. By using the transformation $x \to x - (\mathbf{1} \otimes \int_{-\infty}^{t} u(\tau)d\tau)$, which preserves the property of synchronization, the state equations can be written as

$$\dot{x} = -(L \otimes D(t))x \tag{32}$$

The coupling topology is expressed by a corresponding interaction digraph $\mathcal{G}$ (with no loops). It is easy to see that L is the Laplacian matrix of the reversal of $\mathcal{G}$. Since $L \in \mathcal{W}$ is a Laplacian matrix of a digraph, its eigenvalues have nonnegative real parts and $L\mathbf{1} = \mathbf{0}$. If the zero eigenvalue is simple, then the kernel of L is spanned by the vector $\mathbf{1}$ and all other eigenvalues of L have positive real parts due to Gershgorin's circle criterion. In this case, x in Eq. (32) approaches the kernel of $L \otimes D(t)$ which is equal to the synchronization manifold $\mathcal{M}$, i.e. the network synchronizes. This is also referred to in the literature as the system solving an *agreement* or *consensus* problem.

[10] In [92] these agents are termed "autonomous" since they can act on their own without centralized control. We refrain from using this term to avoid confusion with its use in circuits and systems theory to denote systems that do not receive external (or equivalently) time-varying stimuli.

Theorem 19: The multiplicity of the zero eigenvalue of L is equal to the minimum number of directed trees which forms a spanning forest in the interaction graph.

The following corollary to Theorem 19 gives a graph-theoretical characterization of when the zero eigenvalue of L is simple.

Corollary 9: The zero eigenvalue of L is simple if and only if the interaction graph has a spanning directed tree.

Theorem 19 and Corollary 9 have been found independently by several authors. For instance, Theorem 19 can be found in [99], [100], [57], [101] and Corollary 9 was proved in [95], [96]. Corollary 9 can be used to prove that:

Theorem 20: The state x in Eq. (31) approaches $\mathcal{M}$ and thus solves an agreement problem for all initial x if and only if the interaction graph of L contains a spanning directed tree.

This result is intuitive since the existence of a spanning directed tree in the interaction graph implies that there is a root vertex which influences directly or indirectly all other vertices. If no such spanning directed tree exists, then there exists two groups of vertices which do not influence each other [54] and thus cannot possibly reach an agreement for arbitrary initial disagreement. Since the agent at the root vertex influences all other agents, it can be considered a leader, which might not be unique if there are more than one spanning directed tree.

Definition 6: If $x \to x^*$ with $x_i^* = x_j^* = \frac{1}{n}\sum_i x_i(0)$, then Eq. (32) is said to solve the *average consensus* problem.

Theorem 21 ([41]): Let L be the Laplacian matrix of a strongly connected graph and let w be a positive vector such that $w^T L = 0$. Let W be a diagonal matrix with w_i on the diagonal. Then $W^{-1}\dot{x} = -Lx$ solves the average consensus problem.

The special case of vertex balanced graphs, where $W = I$, was solved in [93].

B. Rate of exponential convergence

We say that $x(t)$ converges exponentially towards $x^*(t)$ with rate k if $\|x(t) - x^*(t)\| \leq O(e^{-kt})$. Since Eq. (31) is linear, clearly x converges towards x^* with rate at least[11] $\gamma(L)$ which is positive for interaction graphs with a spanning directed tree. Since $\gamma(L) \geq a_3(L)$ by Theorem 11, we have the following result on the convergence rate with respect to the algebraic connectivity.

Theorem 22 ([41]): If the graph of L is strongly connected then Eq. (31) synchronizes with rate $a_3(L) > 0$.

The special case of Theorem 22 for vertex balanced graphs was shown directly in [93] using a quadratic Lyapunov function.

When the graph is undirected, $\gamma = \lambda_2$ is the algebraic connectivity of the graph. Thus similar to the algebraic connectivity, when the graph is undirected, adding extra undirected edges cannot decrease γ [45], [38]. However, this is not true for digraphs, using the same example as before. First note that the Laplacian matrix of an acyclic digraph can always have its rows and columns be simultaneously permuted to a upper-triangular matrix, say $\tilde{L}$. Since $\tilde{L}$ has zero row sums, $\tilde{L}_{nn} = 0$ and thus $\gamma(L) = \gamma(\tilde{L}) = \min_{i<n} \tilde{L}_{ii}$. Thus for an acyclic graph $\gamma(L) = \min_{i\neq j} L_{ii}$ where j is an index such that $L_{jj} = 0$. Since L_{ii} are the indegrees of the interaction graph, this in particular implies that $\gamma(L) = 1$ if the interaction graph of L is a tree.

For the directed path graph in Figure 3 with Laplacian matrix L, $\gamma(L) = 1$ since it is a tree and is isomorphic to its reversal. By adding one directed edge we get the directed cycle graph in Figure 4 with a circulant Laplacian matrix L and $\gamma(L) = 1 - \cos\left(\frac{2\pi}{n}\right)$ (See e.g. [24]).

Thus similar to a_4, by adding a single edge, γ changes from $\gamma = 1$ to $\gamma = 1 - \cos\left(\frac{2\pi}{n}\right)$ which decreases to 0 as $O(\frac{1}{n^2})$. Again this can be explained via the strongly connected components of these graphs. The discussion in Section V-A shows that for undirected graphs with bounded vertex degrees, if the diameter grows faster than $\ln(n)$ then $\gamma(L) \to 0$ as $n \to \infty$. In addition, we can obtain bounds on γ that depend on the algebraic connectivities of the SCC's and the number of edges between the SCC's [57], [102].

C. Dynamic coupling topology

In this case the state equations is

$$\dot{x} = -(L(t) \otimes D(t))x + (\mathbf{1} \otimes u(t))$$

where $L(t) \in \mathcal{W}$ for each time t.

Since this is a special case of Eq. (12), we obtain:

Theorem 23 ([41]): Eq. (30) solves the agreement problem with rate $\inf_t a_1(L(t))$.

The special case of strongly connected vertex balanced graphs for this problem was studied in [93].

D. Discrete-time nonautonomous linear dynamics

Consider the following nonautonomous discrete-time linear dynamical system:

$$x(t+1) = (G(t) \otimes D(t))x(t) + \mathbf{1} \otimes u(t) \qquad (33)$$

where $u(t) \in \mathbb{R}^m$. We assume that $G(t)$ is a stochastic matrix (i.e. a nonnegative matrix whose rows sum to 1) and $\|D(t)\| \leq 1$ for all t. In this case the maximal distance between the x_i's is nonincreasing.

Theorem 24 ([103]):

$$\max_{i,j} \|x_i(t+1) - x_j(t+1)\| \leq \max_{i,j} \|x_i(t) - x_j(t)\|$$

In order to achieve synchronization, we require that $\max_{i,j} \|x_i(t) - x_j(t)\| \to 0$ as $t \to \infty$. If the matrix product $G(t)G(t-1)\cdots G(1)$ converges to a rank-one matrix of the form $\mathbf{1}c^T$ for some vector c as $t \to \infty$, then this suffices to ensure synchronization. This condition has been shown in the 1960's to be related to conditions for weak ergodicity of inhomogeneous Markov chains. The first paper to exploit this relationship to demonstrate synchronization in Eq. (33) is [92] and the synchronization criterion is subsequently extended in [104], [95], [96], [41], [98].

Definition 7: A matrix A is *scrambling* if A is stochastic and for each pair of indices i, j there exist a column of A such that the i and j-th entries are both nonzero.

Definition 8 ([105]): A matrix A is stochastic, indecomposable and aperiodic (SIA) if A is stochastic and $\lim_{n\to\infty} A^n = \mathbf{1}c^T$ for some vector c.

[11] Or at least arbitrarily close to $\gamma(L)$, if L has nontrivial Jordan blocks.

Let $\mu(A) = \min_{j,k} \sum_i \min(A_{ji}, A_{ki})$ be the ergodicity coefficient of a matrix A [106]. Note that $0 \leq \mu(A) \leq 1$ for stochastic matrices with $\mu(A) > 0$ if and only if A is scrambling. For a set of matrices S, let S^m denote the set of products of matrices from S of length m. We first start with a fundamental Lemma due to Hajnal and generalized by Dobrušin and Paz and Reichaw:

Lemma 2 (Hajnal's inequality [106], [107]): If A and B are stochastic matrices, then $\delta(AB) \leq (1-\mu(A))\delta(B)$ where $\delta(A) = \frac{1}{2}\max_{i,j}\sum_k |A_{ik} - A_{jk}|$.

Using Hajnal's inequality, the following Lemma can be proved which slightly generalizes the results in [105], [108]. Although not explicitly stated in [105], it was discussed in the concluding remarks[12].

Lemma 3: Let S be a set of matrices such that products of matrices in S are SIA. If $\inf_{A\in S^m} \mu(A) > 0$ for some $m > 0$, then $A_n A_{n-1} \cdots A_1$ with $A_i \in S$ converges to a rank-one matrix of the form $\mathbf{1}c^T$ as $n \to \infty$.

The following result then follows immediately from Lemma 3.

Theorem 25: Let S be a compact set of matrices such that product of matrices in S are SIA, then $A_n A_{n-1} \cdots A_1$ with $A_i \in S$ converges to a rank-one matrix of the form $\mathbf{1}c^T$ as $n \to \infty$.

In fact, it is not necessary for every matrix A_i to be in S. It suffices that there are infinitely many long stretches of A_i in S. In particular, it was shown in [110] that products of at least $\frac{1}{2}(3^n - 2^{n+1} + 1)$ matrices in S is scrambling. Therefore we have:

Theorem 26: Let S be a compact set of matrices such that product of matrices in S are SIA. Let A_i be a set of stochastic matrices. Let s_i, t_i be two sets of increasing integers such that $s_i \leq t_i < s_{i+1} \leq t_{i+1}$ for each i. If for each i, $A_j \in S$ for all $s_i \leq j \leq t_i$ and $t_i - s_i + 1 \geq \frac{1}{2}(3^n - 2^{n+1} + 1)$ then $A_n A_{n-1} \cdots A_1$ with A_i stochastic converges to a rank-one matrix of the form $\mathbf{1}c^T$ as $n \to \infty$.

If the matrices in S have positive diagonal elements, then we can characterize S via the graphs of these matrices.

Definition 9: S_d is defined as the set of stochastic matrices with positive diagonal elements.

Theorem 27 ([96], [57]): For a matrix $A \in S_d$, A is SIA if and only if the interaction graph of A contains a spanning directed tree. If A, B are SIA matrices in S_d, then AB is SIA.

The next result shows that the bound of $\frac{1}{2}(3^n - 2^{n+1} + 1)$ in Theorem 26 can be reduced to $n-1$ if the matrices have positive diagonal elements.

Theorem 28 ([103]): Let $A_1, \ldots, A_{n-1}$ be SIA matrices in S_d. Then the matrix product $A_1 A_2 \cdots A_{n-1}$ is a scrambling matrix.

The example where

$$A_i = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & \ddots & \\ & & & \ddots & \ddots \\ & & & & 1 \end{bmatrix}$$

for all i shows that $n-1$ is the best possible bound.

Definition 10: $S_d(v)$ is defined as the set of stochastic matrices with positive diagonal elements where each nonzero element is larger than v.

Combining these results we can now prove the following synchronization result for Eq. (33).

Theorem 29 ([96], [41], [98]): Let $\mathcal{G}(k)$ be the weighted interaction digraph of $G(k)$. Suppose there exists $v > 0$, $N > 0$ and an infinite sequence $k_1 \leq k_2 \leq \cdots$ such that

1) $G(k) \in S_d(v)$ for all k;
2) $k_{i+1} - k_i \leq N$;
3) For each i, the union of the graphs $\mathcal{G}(k_i), \mathcal{G}(k_i+1), \cdots, \mathcal{G}(k_{i+1}-1)$ contains a spanning directed tree;

then Eq. (33) synchronizes.

The constant $N < \infty$ is important in Theorem 29. The example in [111] shows that if such an N does not exist, then it is possible for synchronization to fail among agents. In other words, it is not sufficient (although it is easy to see that it is necessary) in order to reach synchronization to have two sequences k_i, n_i such that the union of $\mathcal{G}(k_i)$,$\mathcal{G}(k_i+1), \ldots, \mathcal{G}(k_i+n_i)$ contains a spanning directed tree for all i. Furthermore, a modification of the example in [111] shows that the hypothesis in Theorem 29 is sufficient, but not necessary for synchronization. On the other hand, if each digraph $\mathcal{G}(k)$ is a disjoint union of strongly connected components, then the constant N is not necessary in Theorem 29, i.e. $k_{i+1} - k_i$ can be arbitrarily large:

Theorem 30 ([112]): Let $\mathcal{G}(k)$ be the weighted interaction digraph of $G(k)$. Suppose each graph $\mathcal{G}(k)$ is a disjoint union of strongly connected components. If there exists $v > 0$ and an infinite sequence $k_1 \leq k_2 \leq \cdots$ such that

1) $G(k) \in S_d(v)$ for all k,
2) For each i, the union of the graphs $\mathcal{G}(k_i), \mathcal{G}(k_i+1), \cdots, \mathcal{G}(k_{i+1}-1)$ contains a spanning directed tree,

then Eq. (33) synchronizes.

This occurs, for example, if $\mathcal{G}(k)$ (after ignoring the weights on the edges) are undirected graphs. In particular, there is no need for a uniform bound N in the results in [92].

Suppose that some of the matrices $G(k)$ are stochastic matrices that are not SIA, while the rest satisfies Theorem 29, would we still have synchronization? The answer is no, as the following example indicates. Consider the stochastic matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix $A \in S_d$ is SIA and system (33) with $G(k) = A$ synchronizes. However, alternating $G(k)$ between A and B will not result in a synchronized state since BA is a decomposable matrix and decouples the agents from interacting with each other.

On the other hand, Theorems 26 and 28 show that we can still have synchronization if the matrices that are not SIA are sparse enough among $G(k)$.

[12] See also [109].

E. Follow the leader dynamics and leadership in coordinated agents

Ref. [92] also considered a follow-the-leader configuration, where n agents are connected via an undirected connected graph. An additional agent, the leader, influences some of these n agents, but is itself not influenced by other agents. In other words, the state of the leader is constant. This case is shown in Figure 2. Since the leader vertex has indegree 0, every spanning directed tree must have the leader vertex as root. Spanning directed trees exist since the subgraph of the n agents is strongly connected.

We can generalize the concept of a leader as follows. From the strongly connected components of a graph, we create a condensation digraph [42] by associating the SCC to vertices of the condensation digraph with an edge from i to j if and only if there are some edges from the i-th SCC to the j-th SCC. The condensation digraph does not contain directed cycles and satisfies:

Lemma 4 ([41]): The condensation digraph $\mathcal{H}$ of $\mathcal{G}$ contains a spanning directed tree if and only if $\mathcal{G}$ contains a spanning directed tree.

Thus when $\mathcal{G}$ contains a spanning directed tree, the unique SCC which corresponds to the root of the condensation digraph can be considered a "leading" strongly connected component (LSCC), with the property that agents in the LSCC influencing all other agents outside the component, but not vice versa. When $\mathcal{G}$ changes with time, the LSCC also changes with time. It is clear that the roots of spanning directed trees are equal to the vertices in LSCC.

When $\mathcal{G}$ does not change with time, an alternative way of viewing the dynamics is the following. First the agents in the LSCC synchronize. Their states are then "collapsed" into a single "leader" state. The agents that the LSCC influences then synchronize to the "leader" state and are absorbed into the "leader" state etc, until finally all agents are synchronized. This reduces the problem to the case of a single leader.

In addition, we can consider a range of "leadership" in the collection of agents, with the set of root vertices of spanning directed trees as the leaders in the system. The system can be considered leaderless if the size of this set (which is equal to LSCC) approaches the number of agents.

As an application of these results, consider a network of linear discrete-time dynamical systems where the coupling topology is generated randomly. In this model, first studied in [94], the network at Eq. (33) is considered where the coupling matrix $G(t)$ is chosen at each iteration from a probability distribution. Again we assume that $G(t)$ is a stochastic matrix for each t.

Theorem 31 ([103]): If H is a compact set of stochastic scrambling matrices such that $P(H) > 0$, then the network synchronizes in probability.

Consider the following (classical) models of random graphs and random digraphs:

- $G(n,p)$: Each undirected edge is chosen with probability p. Thus a graph with m edges has probability $p^m(1-p)^{\frac{n(n-1)}{2}-m}$ where $\frac{n(n-1)}{2}$ is the total number of possible edges.
- $G(n,M)$: Each graph of n vertices and M edges is given equal probability whereas the rest of the graphs of n vertices has probability 0.
- $G_d(n,p)$: Each directed edge is chosen with probability p. A graph with m edges has probability $p^m(1-p)^{n(n-1)-m}$.
- $G_d(n,M)$: Each digraph of n vertices and M edges is given equal probability whereas the rest of the graphs of n vertices has probability 0.

We can connect these unweighted random graph models to Eq. (33) by associating a stochastic matrix A to each unweighted graph $\mathcal{G}$ such that the graph of A ignoring the weights is equal to $\mathcal{G}$. We then have the following Corollaries to Theorem 31:

Corollary 10: If the random graph model is $G(n,p)$ with $p > 0$, then Eq. (33) synchronizes in probability.

Corollary 11: If the random graph model is $G(n,M)$ with $M \geq 2n-3$, then Eq. (33) synchronizes in probability.

Corollary 12: If the random graph model is $G_d(n,p)$ with $p > 0$, then Eq. (33) synchronizes in probability.

Corollary 13: If the random graph model is $G_d(n,M)$ with $M \geq 2n-1$, then Eq. (33) synchronizes in probability.

If we allow some matrices to have positive diagonal elements, then the scrambling condition in Theorem 31 can be relaxed.

Theorem 32: Consider n_p compact sets of matrices $S_i \subset S_d$ with $Pr(S_i) > 0$ for each $i = 1,\ldots,n_p$. Suppose that if $G_i \in S_i$, then the union of the interaction graphs of G_i, $i = 1,\ldots,n_p$, contains a spanning directed tree. Then Eq. (33) synchronizes in probability.

Applying this to the random graph models where we associate a stochastic matrix in S_d to each graph, we obtain:

Corollary 14: If the random graph model is $G(n,M)$ or $G_d(n,M)$ with $M > 0$, then Eq. (33) synchronizes in probability.

F. A nonlinear approach

Recently, an nonlinear approach to consensus problem is proposed in [97]. In this framework, the system is defined by the nonlinear equation $x_i(t+1) = f_i(x_1(t), x_2(t), \ldots x_n(t))$. The continuous functions f_i are defined such that $x_i(t+1)$ is in the convex hull of $x_1(t), x_2(t), \cdots, x_n(t)$ and in the relative interior of the convex hull when this interior is nonempty. This can be recast as Eq. (33) as follows. Since $x_i(t+1)$ is in the convex hull, it can be written as a convex combination of $x_1(t), x_2(t), \ldots, x_n(t)$, i.e. $x_i(t+1) = \sum_j g_{ij} x_j(t)$ where g_{ij} are nonnegative numbers such that $\sum_j g_{ij} = 1$. These numbers g_{ij} form a stochastic matrix $G(t)$. In other words, at each time t a stochastic matrix $G(t)$ can be chosen such that $x(t+1) = (G(t) \otimes I)x(t)$, i.e. Eq. (33) for the case $D(t) = I$ and $u(t) = 0$. It is easy to show that the relative interior condition implies that $G(t)$ can be chosen to have positive diagonal elements for each t. Note that the continuity of f_i is not needed in writing the state equations as Eq. (33). Another benefit of writing the system as Eq. (33) is that a lower bound on the convergence rate can be derived using Hajnal's inequality, a feature that is absent in the analysis in [97].

XI. Concluding remarks and further reading

Identical synchronization is a very active interdisciplinary research area with participation from engineers, physicists, mathematicians and social scientists and we have only attempted to cover a small sampling of this research. As synchronization is a pervasive phenomenon, we expect this research to have far reaching applications.

There are other aspects of synchronization which we have not covered in this paper. We have already mentioned phase synchronization. Another related concept is generalized synchronization. In identical synchronization, the synchronization manifold is a linear subspace. By assuming the synchronization manifold to be more general manifolds, the concept of generalized synchronization is obtained [113], [114]. Generalized synchronization can occur when the n systems are not identical.

When the collection of coupled systems is not identically synchronized, it can still be that subsets of systems are synchronized, forming synchronized clusters. We have mentioned that if the number of eigenvalues of G in Eq. (14) is k, then one has at least k clusters. In [115], [116], this is studied using the framework of lattice dynamical systems and the possible cluster configurations are analyzed and characterized for several classes of coupling topologies.

Finally, the reader is referred to [117]–[120], [24], [121] for further reading. A comprehensive list of early papers in this area can be found at `http://www.ee.cityu.edu.hk/~gchen/chaos-bio.html`.

Acknowledgment

The author would like to thank Leon Chua, Guanrong Chen, Alan Hoffman, Ljupco Kocarev, Lou Pecora, Tamas Roska, Mike Shub, Charles Tresser and Guo-Qun Zhong for fruitful discussions and conversations over the years.

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