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Stochastic Banach Principle in Operator Algebras

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STOCHASTIC BANACH PRINCIPLE IN OPERATOR ALGEBRAS

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ABSTRACT. Classical Banach principle is an essential tool for the investigation of the ergodic properties of Česaro subsequences.

The aim of this work is to extend Banach principle to the case of the stochastic convergence in the operator algebras.

We start by establishing a sufficient condition for the stochastic convergence (stochastic Banach principle). Then we formulate stochastic convergence for the bounded Besicovitch sequences, and, as consequence for uniform subsequences.

1. Introduction and Preliminaries

In this paper we establish Stochastic Banach Principle. Banach Principle is one of the most useful tools in "classical" point-wise ergodic theory. Banach principle was used to give alternative proof of the Birkgoff- Khinchin individual ergodic theorem. Typical application of the Banach Principle are Sato's theorem for the uniform subsequences [17] and individual ergodic theorem for the Besicovitch Bounded sequences [15] . Non-commutative analogs for the (double side) almost everywhere convergence maybe found in papers [8], [2].

In this paper we establish Banach Principle for the convergence by measure (Stochastic Banach Principle, Theorem 2.3). We reformulate the theorem in the form convenient for applications (Theorem 2.4). Based on the principle we give simplified prove of the stochastic ergodic theorem (compare with [9]). We establish stochastic convergence for Sato's uniform subsequences (Theorem 3.6) and stochastic ergodic theorem for the Besicovitch Bounded sequences (Theorem 3.7).

Note that these results are new even in the commutative case.

Through the paper we denote by M von Neumann algebra with semi-finite normal faithful trace τ acting on Hilbert space $\mathfrak{H}$. Denote by $P(M)$ set of all orthogonal projections in M.

Recall following definitions (combined from the papers by Segal [16], Nelson [14], Yeadon [20], Fack and Kosaki [6]):

Definition 1.1. A densely defined closed operator x affiliated with von Neumanm algebra M is called **(τ) measurable** if for every $\epsilon > 0$ there exists projection $e \in P(M)$ with $\tau(\mathbb{I} - e) < \epsilon$ such that $e(\mathfrak{H}) \subset \mathfrak{D}(x)$, where $\mathfrak{D}(x)$ is a domain of x.

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Space of all **(τ) measurable operators affiliated with** M is denoted by $S(M)$.

For convenience for self-adjoint $x \in S(M)$ we denote by $\{x > t\}$ **spectral projection of** x **corresponding to the interval** $(t, \infty]$.

Definition 1.2. Sequence $\{x_n\}_{n=1}^{\infty}$ **converges to** 0 **in measure (stochastically)** if for every $\epsilon > 0$ and $\delta > 0$ there exists an integer N_0 and a set of projections $\{e_n\}_{n \geq N_0} \subset P(M)$ such that $\|x_n\, e_n\|_\infty < \epsilon$ and $\tau(\mathbb{I} - e_n) < \delta$ for $n \geq N_0$.

Remark 1.1. We will use terms **converges in measure** and **converges stochastically** interchangeably.

Definition 1.3. Let x be a measurable operator from $S(M)$ and $t > 0$. The **t-th singular number of** x is defined as

(1) $\quad \mu_t(x) = inf\{||xe|| \text{ where } e \text{ is a projection in } P(M) \text{ with } \tau(\mathbb{I} - e) < t\}.$

Remark 1.2. Note that **measure topology** is defined in Fack and Kosaki's [6] as linear topology with fundamental system of neighborhoods around 0 given by $V(\epsilon, \delta) = \{x \in S(M)$ such that there exists projection $e(x, \epsilon, \delta)$ with $\|xe\| < \epsilon$ and $\tau(\mathbb{I} - e) < \delta\}$.

Definition 1.4. Denote by $\lambda_t(x)$ **distribution function of** x defined as

$$\lambda_t(x) = \tau(E_{(t,\infty)}(|x|)), \quad t \geq 0, \tag{2}$$

here $E_{(t,\infty)}(|x|))$ is a spectral projection of x corresponding to interval (t, ∞).

Remark 1.3. For the measurable operator x, we have $\lambda_t(x) < \infty$ for the large enough t and $\lim_{t\to\infty} \lambda_t(x) = 0$. Moreover, the map $\mathbb{R} \ni t \to \lambda_t(x)$, is non-increasing and continuous from the right (because τ is normal and $\{|x| > t_n\} \uparrow \{|x| > t\}$ (and hence in strong operator topology) as $t_n \downarrow t$). The $\lambda_t(x)$ is a non-commutative analogue of the distribution function in classical analysis, (see. [6] p. 272 or [18]).

We would need following statement about properties of the $\mu_t(x)$ (see for example proposition 2.4 [20], or lemma 2.5 [6]):

Lemma 1.1. *Let $x, y \in S(M)$ are measurable operators.*

i) *Map $\mathbb{R} \ni t \to \mu_t(x)$ is non-decreasing and continuous from the right. Moreover $lim_{t\downarrow 0}\mu_t(x) = \|x\|_\infty \in [0, \infty]$,*
ii) $\mu_t(x) = \mu_t(|x|) = \mu_t(x^*)$ *and* $\mu_t(\alpha x) = |\alpha|\mu_t(x)$ *for* $\alpha \in \mathbb{C},\ t > 0$,
iii) $\mu_t(x) \leq \mu_t(y)$ *for* $0 \leq x \leq y,\ t > 0$,
iv) $\mu_{t+s}(x + y) \leq \mu_t(x) + \mu_s(y)$ *for* $t, s > 0$,
v) $\mu_t(yxz) \leq \|y\|_\infty \|z\|_\infty\ \mu_t(x)$, *for* $y, z \in M,\quad t > 0$,
vi) $\mu_{t+s}(yx) \leq \mu_t(x)\mu_s(y)$ *for* $t, s > 0$.

2. Stochastic Banach Principle.

We start the section with the description of some conditions equivalent to the stochastic convergence (cmp. with lemma 3.1 [6]).

Lemma 2.1. *Let M, τ be as before. Consider following conditions:*

i) *Sequence $\{x_n\}_{n=1}^{\infty}$ converges to 0 in measure,*
ii) *For every $\epsilon > 0, \delta > 0$ there exist a positive real $0 < \delta' < \delta$ and an integer N_0 such that for $n \geq N_0$*

$$\mu_{\delta'}(x_n) < \epsilon,$$

iii) *For every* $\epsilon > 0, \delta > 0, p \in P(M)$ *with* $\tau(p) < \infty$ *there exists an integer* N_0 *and a sequence of projections* $\{e'_n\}_{n \geq N_0} \subset P(M)$, $e'_n \leq p$ *such that*

$$\|x_n\, e'_n\|_\infty < \epsilon \text{ and } \tau(p - e'_n) < \delta \text{ for } n \geq N_0.$$

Following relations have place: i) $\Leftrightarrow$ *ii)* $\Rightarrow$ *iii). If* τ *is finite then iii)* $\Rightarrow$ *i).*

Proof. Implication $ii) \Rightarrow i)$ follows from the fact that condition $\mu_{\delta'}(x_n) < \epsilon$ implies for the sequence of projections $\{e_n = \{|x_n| \leq n\}\}_{n=1}^\infty$, holds $\|x_n\, e_n\| \leq \epsilon$ and $\tau(\mathbb{I} - e_n) \leq \delta'$. Implication $i) \Rightarrow ii)$ follows from the definition of measure convergence 1.2.

Implication $i), ii) \Rightarrow iii)$ follows from the inequality
$\tau(p - p \wedge q) = \tau(p \vee q - q) \leq \tau(\mathbb{I} - q)$, hence sequence $\{e'_n = \{|x_n| \leq n\} \wedge p\}_{n=1}^\infty$ satisfies iii).

Case when τ is finite follows immediately since $\tau(\mathbb{I}) < \infty$. □

We need following technical statement which is interesting by itself:

Lemma 2.2. *Let* x, y *be self-adjoint measurable operators from* $S(M)$, t, s *be positive real. Then*

$$\lambda_{t+s}(x + y) \leq \lambda_t(x) + \lambda_s(y) \tag{3}$$

Proof. Indeed,

(4)
$$\begin{aligned}
&\||(x + y)\,|(\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})\| = \\
&\|(x + y)\,(\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})\| \leq \\
&\leq \|x\,(\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})\| + \|y\,(\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})\| = \\
&= \||x|\,(\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})\| + \||y|\,(\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})\| \leq \\
&\leq \||x|\,(\mathbb{I} - \{|x| > t\})\| + \||y|\,(\mathbb{I} - \{|y| > s\})\| \leq \\
&t + s,
\end{aligned}$$

here first and second equality follows from the equality $\||z|\, u_z^* u_z\, |z|\| = \||z|^2\| = \|z^*\, z\|$, here $z \in M_h$, u_z is a partial isometry from M such that $z = u_z\, |z|$, and

$$u_z^* u_z = l(z),\ u_z u_z^* = r(z), \tag{5}$$

where $l(z)(r(z))$ is a left (right) support of z. Inequality 4 means that

$$\mu_{\lambda_t(x) + \lambda_s(y)}(x + y) \leq t + s. \tag{6}$$

Let ξ be a vector from Hilbert space $\mathfrak{H}$ and suppose that

$$\xi \in \{|x + y| > s + t\}\mathfrak{H} \cap (\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})\mathfrak{H}. \tag{7}$$

Then

$$(t + s)\|\xi\|^2 \leq ((x + y)\xi, (x + y)\xi) = (|x + y|\xi, |x + y|\xi) < (t + s)\|\xi\|^2, \tag{8}$$

here first inequality follows from inclusion $\xi \in (\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})\mathfrak{H}$, equality follows from the spectral decomposition 5, second inequality follows from inclusion $\xi \in \{|x + y| > s + t\}\mathfrak{H}$.

Inequality 8 implies that $\|\xi\| = 0$ or $\{|x+y| > s+t\} \wedge ((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) = 0$. Hence,

$$\begin{aligned}
&\{|x+y| > t+s\} = \{|x+y| > t+s\} - \\
&\{|x+y| > t+s\} \wedge ((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) \sim \\
&\sim \{|x+y| > t+s\} \vee ((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) - \\
&((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) \leq \\
&\leq \mathbb{I} - ((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) = \{|x| > t\} \vee \{|y| > s\},
\end{aligned} \tag{9}$$

here $\sim$ means projection equivalence. Since trace τ is invariant on equivalent projections,

$$\tau(\{|x+y| > t+s\}) \leq \tau(\{|x| > t\} \vee \{|y| > s\}) \leq \tau(\{|x| > t\}) + \tau(\{|y| > s\}) \tag{10}$$

or 3 have place. □

Theorem 2.3. *Let $(B, \|.\|)$ be a Banach space. Let $\Sigma = \{A_n, n \in \mathbb{N}\}$ be a set of the linear operators $A_n : B \to S(M)$.*

i) *Suppose that there exists function $C(\lambda) : \mathbb{R}_+ \to \mathbb{R}_+$ with $lim_{\lambda\to\infty} C(\lambda) = 0$, and*

$$\sup_{n\in\mathbb{N}} \tau(\{|A_n(b)| > \lambda \|b\|\}) \leq C(\lambda) \tag{11}$$

holds for every $b \in B$, $\lambda \in \mathbb{R}_+$.

Then subset $\tilde{B}$ of B where $A_n(b)$ converges in measure (stochastically) is closed in B.

ii) *In converse, if A_n is a set of continues in measure maps from B into $S(M)$ and for each $b \in B, \lambda \in R_+$*

$$\lim_{\lambda\to\infty} \sup_{n\in\mathbb{N}} \tau(\{|A_n(b)| > \lambda\}) = 0, \tag{12}$$

then there exists function $C(\lambda) : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\lambda\to\infty} C(\lambda) = 0$, and

$$\sup_{n\in\mathbb{N}} \tau(\{|A_n(b)| > \lambda \|b\|\}) \leq C(\lambda) \tag{13}$$

Part i) of the theorem 2.3 means that under the condition of the linear uniform boundness (11) set of the stochastical convergence is closed.

Part ii) of the theorem 2.3 means that if the set of uniform boundness is closed, then linear uniform boundness has place.

Proof. Part i) Let first show that condition 11 implies continuity of set Σ of operators . Let $B \supset \{b_k\}_{k=1}^{\infty}$ is a sequence in B converging to the $b \in B$. Then for $\lambda, \epsilon \in \mathbb{R}_+$ with $2\lambda \sup_{n\leq 1} \|b_k\| < \epsilon$

$$\begin{aligned}
\tau(\{|A_n(b_k) - A_n(b)| > \epsilon\}) \leq \tau(\{|A_n(b_k - b)| > \lambda \|b_k - b\|\}) \leq \\
C(\lambda \|b_k - b\|^{-1}) \xrightarrow{b_k \to b} 0,
\end{aligned} \tag{14}$$

and, hence continuous. Note that the inequality follows from the fact that right part of 11 does not depend on norm of b.

There exists subsequence b_{k_j} of b_k such that sequence $x_j = lim_{n\to\infty} A_n(b_{k_j})$ converges stochastically. Choose sequence of $\{k'_i\}_{i=1}^{\infty}$ base on the inequality 11 in such a manner that

$$\tau(\{|A_n(b_{k'_j} - b_{k'_{j+l}})| > 2^{-j}\}) \leq 2^{-j} \text{ for all n} \tag{15}$$

and

$$\tau(\{|A_n(b_{k'_j} - b)| > 2^{-j}\}) \leq 2^{-j} \text{ for all n} \tag{16}$$

This may be done since $b_n \stackrel{n\to\infty}{\longrightarrow} b$, and $C(\lambda) \stackrel{\lambda\to\infty}{\longrightarrow} 0$. It is sufficient to choose sequence of λ_j in such a way that $C(\lambda_j) < 2^{-j}$ and $\|b_{k_j} - b\| < \lambda_j^{-1} 2^{-2j}$.

Choose n_j in such a manner that for $N > n_j$ holds

$$\tau(\{|A_n(b_{k_j}) - x_j| > 2^{-j}\}) < 2^{-j}. \tag{17}$$

This is possible since $A_n(b_{k_j})$ converges stochastically to x_j.

Then for $j, i \in \mathbb{N}$

$$\begin{aligned} \tau(\{|x_j - x_{j+i}| > 3\cdot 2^{-j}\}) =& \tau(\{|(x_j - A_n(b_{k_j}))+ \\ (A_n(b_{k_j}) - A_n(b_{k_{j+i}}))+&(A_n(b_{k_{j+i}}) - x_{j+i})| > 3\cdot 2^{-j}\}) \leq \\ \tau(\{|A_n(b_{k_j}) - x_j| > 2^{-j}\})+&\tau(\{|A_n(b_{k_j}) - A_n(b_{k_{j+i}})| > 2^{-j}\})+ \\ \tau(\{|A_n(b_{k_{j+i}}) - x_{j+i}| >& 2^{-(j+i)}\}) \leq 3\cdot 2^{-j} \end{aligned} \tag{18}$$

,here the first inequality follows from the 3, the second became valid for the $n > n_{j+i}$.

Denote stochastic limit of $\{x_j\}_{j=1}^{\infty}$ by x_0. If necessary taking subsequence of $\{x_j\}$ and reindexing, we suppose that

$$\tau(\{|x_j - x_0| > 2^{-j}\}) \leq 2^{-j}. \tag{19}$$

Sequence $\{A_n(b)\}_{n=1}^{\infty}$ converges to x_0 stochastically. Indeed, for $n > n(j)$ holds following inequality

$$\begin{aligned} \tau(\{|A_n(b) - x_0| > 3\cdot 2^{-j}\}) =& \tau(\{|(A_n(b) - A_n(b_{k_j}))+ \\ (A_n(b_{k_j}) - x_j) + (x_j - x_0)| > 3\cdot 2^{-j}\}) \leq& \tau(\{|A_n(b) - A_n(b_{k_j})| > 2^{-j}\})+ \\ \tau(\{|A_n(b_{k_j}) - x_j| > 2^{-j}\})+&\tau(\{|x_j - x_0| > 2^{-j}\}) \leq 3\cdot 2^{-j} \end{aligned} \tag{20}$$

,here the first inequality follows from 3, and the second inequality follows from: the first part follows from 16, the second part follows from 17 and choice of n, and the third part follows from 19.

Part i) is established.

Part ii) Suppose that for every $b \in B$ and $\lambda \in \mathbb{R}_+$ holds

$$\sup_n \tau(\{|A_n(b)| > \lambda\}) \stackrel{\lambda\to\infty}{\longrightarrow} 0. \tag{21}$$

For fixed $\epsilon > 0$ and $\lambda \in \mathbb{N}$ define $B_\lambda = \{b \in B | sup_n \tau(\{|A_n(b)| > \lambda\}) \leq \epsilon\}$. Then from 21 it follows that

$$B = \bigcup_{\lambda\in\mathbb{N}} B_\lambda \tag{22}$$

Let $B_{\lambda,k}$ be a set defined as $\{b \in B | \sup_{n>k} \tau(\{|A_n(b)| > \lambda\}) \leq \epsilon\}$. Then

$$B_\lambda = \bigcap_{k\in\mathbb{N}} B_{\lambda,k} \tag{23}$$

Sets $B_{\lambda,k}$ are closed in measure. Indeed, let $B_{\lambda,k} \supset \{b_j\}_{j=1}^{\infty}$ converges to $b \in B$. Then

$$\begin{aligned} \tau(\{|A_n(b)| > \lambda + \gamma\}) =& \tau(\{|A_n(b_k) - (A_n(b_k) - A_n(b))| > \lambda + \gamma\}) \leq \\ \tau(\{|A_n(b_k)| > \lambda\}+&\tau(\{|(A_n(b_k) - A_n(b))| > \gamma\}) \leq \epsilon \end{aligned} \tag{24}$$

, here the first inequality follows from 3, and the first estimate follows from definition of $B_{\lambda,k}$ and the second estimate follows from free choice of b_k and continuity of A_n in measure.

Since $\lambda_t(x)$ is continuous from the right (1.3), then

$$\tau(\{|A_n(b)| > \lambda\}) = \lim_{m\to\infty} \tau(\{|A_n(b)| > \lambda + \gamma_m\}) \le \epsilon \tag{25}$$

,where $\gamma_m \overset{m\to\infty}{\longrightarrow} 0$. Hence, $b \in B_{\lambda,k}$, or $B_{\lambda,k}$ is closed. Set B_λ is closed as an intersection of closed set (23).

It follows from the Baire category principle that there exists λ such that set B_λ has non empty interior. Let $B(b_0, r) = \{b \in B | \|b - b_0\| \le r\}$ is contained in the B_λ.

Then

$$\tau(\{|A_n(b)| > \lambda\}) \le \epsilon \text{ for every } b \in B(b_0, r). \tag{26}$$

Moreover, for the $b = b_0 - r \cdot c \in B(b_0, r)$ with $c \in B,\ \|c\| \le 1$ holds

$$\begin{aligned}\tau(\{|A_n(r \cdot c)| > 2 \cdot \lambda\}) = \tau(\{|A_n(r \cdot c - b_0) + A_n(b_0)| > 2 \cdot \lambda\}) \le \\ \tau(\{|A_n(r \cdot c - b_0)| > \lambda\}) + \tau(\{|A_n(b_0)| > \lambda\}) \le 2 \cdot \epsilon.\end{aligned} \tag{27}$$

Let $\gamma \ge 2 \cdot \lambda/r$. From 27 it follows that $\tau(\{|A_n(c)| > \gamma\}) \le 2 \cdot \epsilon$, for every $c \in B,\ \|c\| \le 1$.

Let $C(\gamma) = sup_{c\in B,\ \|c\|\le 1} \tau(\{|A_n(c)| > \gamma\}) \le 2 \cdot \epsilon$. Free choice of ϵ implies that

$$\lim_{\gamma\to\infty} C(\gamma) = 0 \tag{28}$$

, hence 11 is valid. □

For the application of the theorem 2.3 it is convenient to combine both parts i) and ii) together.

Theorem 2.4. *Let $(B, \|.\|)$ be a Banach space. Let A_n is a set of continues in measure linear maps from B into $S(M)$ and for each $b \in B, \lambda \in R_+$ holds*

$$\lim_{\lambda\to\infty} \sup_{n\in\mathbb{N}} \tau(\{|A_n(b)| > \lambda\}) = 0. \tag{29}$$

Then subset $\tilde{B}$ of B where $A_n(b)$ converges in measure (stochastically) is closed in B.

Proof. Follows immediately from applying consequently Theorem 2.3 part ii) then part i). □

Let e be a projection in M, let M_e is an von Neumann algebra consisting from operators of view $exe,\ x \in M$. If τ is a semifinite normal faithful trace on M then τ_e is a semifinite (possibly finite) faithful normal trace on M_e. Indeed, tracial property, semifiniteness, normalness and faithfulness of τ_e follows directly from similar property of τ. Space $S(M_e, \tau_e)$ is isomorphic to the $S(M, \tau)_e$ since both that spaces are closure of the $(M_{\tau-finitesupport})_e = (M_e)_{\tau_e-finitesupport}$.

Proposition 2.5. *Let B_n is a sequence of continues in measure operators in $S(M, \tau)$. Let $e_i \in P(M),\ i = 1, 2,\ \mathbb{I} = e_1 + e_2$ are projections in M. Suppose that relation $e(B_n(x)) = B_n(x_e)$ holds for every $n \in \mathbb{N}$ and $x \in S(M, \tau)$, or, in other words, e commute with B_n. Suppose also following relations holds*

$$\lim_{\lambda\to\infty} \sup_{n\in\mathbb{N}} \tau(\{|B_n(x_{e_i})| > \lambda\}) = 0 \tag{30}$$

, for $i = 1,2$ and every $x \in S(M,\tau)$. Then following inequality is valid:

$$\lim_{\lambda\to\infty} \sup_{n\in\mathbb{N}} \tau(\{|B_n(x)| > \lambda\}) = 0. \tag{31}$$

Proof. The following relations is valid:

$$\tau(\{|B_n(x_{e_i})| > \lambda\}) = \tau(e_i\{|B_n(x)| > \lambda\}). \tag{32}$$

Indeed, since for $x \in S_h(M)$ ($S_h(M)$ is a set of all self-adjoint operators in $S(M)$) limit $\tau(\{|x| > \lambda\}) \overset{\lambda\to\infty}{\longrightarrow} 0$, hence for a sequence of polynomial $P_j(y)$ in $\mathbb{R}$ converging to $\chi_{\{|y|>\lambda\}}(y)$ pointwise, sequence $P_j(x)$ converges to $\chi_{\{|x|>\lambda\}}(x)$ stochastically. Then by [6] Proposition 3.2

$$\begin{aligned}\tau(\{|B_n(x_{e_i})| > \lambda\}) &= \lim_j \tau(P_j(B_n(x_{e_i}))) = \lim_j \tau(P_j(B_n(xe_i))) = \\ \lim_j \tau(P_j(e_iB_n(x)e_i)) &= \lim_j \tau(e_iP_j(B_n(x))) = \tau(e_i\{|B_n(x)| > \lambda\}).\end{aligned} \tag{33}$$

Statement 31 follows now from the fact that (it follows from B_n commute with e_i and 3)

$$\begin{aligned}\tau(\{|B_n(x)| > \lambda_1 + \lambda_2\}) =&\tau(\{|(e_1+e_2)B_n(x)(e_1+e_2)| > \lambda_1+\lambda_2\}) = \\ &\tau(\{|(e_1B_n(x)e_1 + e_2B_n(x)e_2| > \lambda_1+\lambda_2\}) \le \\ &\tau(\{|(B_n(x_{e_1})| > \lambda_1\}) + \tau(\{|B_n(x_{e_2})| > \lambda_2\}).\end{aligned} \tag{34}$$

□

Remark 2.1. Under the conditions of the stochastic ergodic theorem estimate 29 has place.

3. Stochastic ergodic theorems

In this section we establish stochastic convergence of the bounded Besicovitch sequences, and show stochastic ergodic theorem for uniform subsequences.

In this section we use following assumptions: M is a von Neumann algebra with faithful normal tracial state τ, and α is an $*$-automorphism of algebra M. Denote by $A_n(x) = \frac{1}{n}\sum_{l=1}^{n-1} \alpha^l(x)$, for $x \in M$. Define α' as a linear map on $L_1(M,\tau)$ satisfying $\tau(x \cdot \alpha(y)) = \tau(\alpha'(x)y)$ for $x \in L_1(M,\tau), y \in M$, and $A'_n(x) = \frac{1}{n}\sum_{l=1}^{n-1} {\alpha'}^l(x)$, for $x \in L_1(M,\tau)$.

Let recall some definitions from Grabarnik and Katz [9] and Chilin Litvinov and Skalski [2].

Definition 3.1. A positive operator $h \in M_+$ is called **weakly wandering** if

$$\|A_n(h)\|_\infty \overset{n\to\infty}{\longrightarrow} 0 \tag{35}$$

Following definition is due to Ryll-Nardzewski [15].

Definition 3.2. Let $\mathbb{C}_1$ denote the unit circle in $\mathbb{C}$. A **trigonometric polynomial** is a map $P_k(n) : \mathbb{N} \mapsto \mathbb{C}$, where $P_k(n) = \sum_{j=0}^{k-1} b_j \cdot \lambda_j^n$ for $\{\lambda_j\}_{j=0}^{k-1} \subset \mathbb{C}_1$.

Bounded Besicovitch sequences are bounded sequences from l_1-average closure of the trigonometric polynomials.

More precisely,

Definition 3.3. A sequence β_n of complex numbers is called a **Bounded Besicovitch sequence** (BB-sequence) if

(i) $|\beta_n| \leq C < \infty$ for every $n \in \mathbb{N}$ and
(ii) For every $\epsilon > 0$, there exists a trigonometric polynomial P_k such that

$$\limsup_n \frac{1}{n} \sum_{j=1}^{n-1} |\beta_j - P_k(j)| < \epsilon \tag{36}$$

Let μ is the normalized Lebesque measure (Radon measure) on $\mathbb{C}_1$. Let $\tilde{M}$ be the von Neumann algebra of all essentially bounded ultra-weakly measurable functions $f : (\mathbb{C}_1, \mu) \to M$. Algebra $\tilde{M}$ is isomorphic to the $L_\infty(\mathbb{C}_1, \mu)\overline{\bigotimes}M$ -which is a W^* tensor product of $L_\infty(\mathbb{C}_1, \mu)$ and M, $\tilde{M}$ is a dual to the space $L_1(\mathbb{C}_1, \mu)\overline{\bigotimes}M_*$, see for example Takesaki, [19], Theorem IV.7.17 and $L_1(\mathbb{C}_1, \mu)\overline{\bigotimes}M_*$ maybe considered as a set of L_1 functions on $(\mathbb{C}_1, \mu)$ with values in M_*. Algebra $\tilde{M}$ has a natural trace $\tilde{\tau}(f) = \int_{\mathbb{C}_1} \tau(f(z))d\mu(z)$, and $\tilde{M}_*$ is isomorphic to $L_1(\tilde{M}, \tilde{\tau})$.

Let σ be an automorphism of $(\mathbb{C}_1, \mu)$ as a Lebesque space with measure. We define automorphism $\alpha \bigotimes \sigma$ of $(\tilde{M}, \tilde{\tau})$ as a closure of linear extension of automorphism acting on $\tilde{M}, \tilde{\tau} \ni x(z)$ as $\alpha \bigotimes \sigma(x(z)) = \alpha(x(\sigma(z)))$.

Example 3.1. Example of such automorphism is $\tilde{\alpha}_\lambda(x(z)) = \alpha(x(\lambda \cdot z)))$, for the $\lambda \in \mathbb{C}_1$.

In this case

$$A_n(x) = \frac{1}{n} \sum_{l=1}^{n-1} \tilde{\alpha_\lambda}^l(x) = \frac{1}{n} \sum_{l=1}^{n-1} \alpha^l(x(\lambda^l \cdot z)). \tag{37}$$

In particularly, if $x(z) \equiv z \cdot x$ for $x \in M$ then

$$A_n(x \cdot z) = z \cdot \frac{1}{n} \sum_{l=1}^{n-1} \lambda^l \cdot \alpha^l(x). \tag{38}$$

Following lemma connects stochastic convergence in $L_1(\tilde{M}, \tilde{\tau})$ with pointwise convergence on C_1 and stochastic convergence in M (cmp. with [2]).

Lemma 3.2. i) *If $L_1(\tilde{M}, \tilde{\tau}) \ni x_n \overset{n\to\infty}{\longrightarrow} x_0 \in L_1(\tilde{M}, \tilde{\tau})$ b.a.u. , then $x_n(z) \overset{n\to\infty}{\longrightarrow} x_0(z)$ stochastically for almost every $z \in \mathbb{C}_1$*

ii) *Suppose that h is a weakly wandering operator with support $supp(h) = \mathbb{I}$ for sequence A_n. Then $A'_n(x)$ converges to 0 stochastically.*

iii) *Let algebra $\mathcal{N} = (M, \tau)\overline{\otimes}L_\infty(X, \mu)$, (here X is a separable Hausdorff compact set, and μ is Lebesque measure), and α is an automorphism of M, σ is an automorphism of $L_\infty(X, \mu)$, then $\alpha \bigotimes \sigma$ is an automorphism of $\mathcal{N}$. Suppose that h is a weakly wandering operator with support $supp(h) = \mathbb{I}$ for sequence A_n corresponding to automorphism $\alpha \bigotimes \sigma$. Then $A'_n(x(z))$ converges to 0 stochastically for almost every $z \in \mathbb{C}_1$.*

Proof. Part i) follows from the [2], Lemma 4.1 stating under condition of part i) b.a.u. convergence of $x_n(z)$ to $x_0(z)$ for almost every z in $\mathbb{C}_1$, (hence double side stochastic convergence), and the fact that double side stochastic convergence is equivalent (one sided) stochastic convergence (see [2], Theorem 2.2).

Part ii) We suppose that $x \in L_1(M, \tau)_+$ and $A'_n(x)$ is a sequence satisfying

$$\tau(A'_n(x)h) \to 0 \text{ for } n \to \infty. \tag{39}$$

Following inequality is valid:

$$ts \cdot \tau(\{A'_n(x) > t\} \wedge \{h > s\}) \leq \tau(A'_n(x)h). \tag{40}$$

Indeed, for projections $e_1, e_2 \in P(M)$, holds $e_1e_2e_1 \geq e_1 \wedge e_2$, since $e_1 \wedge e_2$ commutes with e_1, e_2, $(\mathbb{I} - e_1 \wedge e_2)e_1e_2e_1(e_1 \wedge e_2) = 0$, and, hence $e_1e_2e_1 = (\mathbb{I} - e_1 \wedge e_2)e_1e_2e_1(\mathbb{I} - e_1 \wedge e_2) + (e_1 \wedge e_2)e_1e_2e_1(e_1 \wedge e_2) = (\mathbb{I} - e_1 \wedge e_2)e_1e_2e_1(\mathbb{I} - e_1 \wedge e_2) + (e_1 \wedge e_2)$.

Then,

$$\begin{aligned} ts \cdot \tau(\{A'_n(x) > t\} \wedge \{h > s\}) \leq & t\tau(\{A'_n(x) > t\}s\{h > s\}\{A'_n(x) > t\}) \leq \\ t\tau(\{A'_n(x) > t\}h\{A'_n(x) > t\}) = & t\tau(\{A'_n(x) > t\}h) \leq \\ & \tau(A'_n(x)h), \end{aligned} \tag{41}$$

hence, 40 is valid.

Furthermore,

$$\tau(\{A'_n(x) > t\}) \leq \frac{1}{ts}\tau(A'_n(x)h) + \tau(\mathbb{I} - \{h > s\}). \tag{42}$$

The later inequality follows from 40, and the fact that $\tau(e_1) \leq \tau(e_1 \wedge e_2) + \tau(\mathbb{I} - e_2)$. Indeed,

$$\begin{aligned} \tau(e_1 - e_1 \wedge e_2) = \tau((\mathbb{I} - e_1 \wedge e_2)e_1(\mathbb{I} - e_1 \wedge e_2)) = & \tau(e_1(\mathbb{I} - e_1 \wedge e_2)e_1) \leq \\ \tau(e_1(\mathbb{I} - e_2)e_1) = & \tau(e_1(\mathbb{I} - e_2)) \leq \tau(\mathbb{I} - e_2), \end{aligned} \tag{43}$$

hence 42 is valid.

Note that inequality 42 with the fact that $\tau(A'_n(x)h) \stackrel{n\to\infty}{\longrightarrow} 0$ implies that $\sup_{n\in\mathbb{N}} \tau(\{|A_n(b)| > \lambda \|b\|\}) \leq C(\lambda)$. Indeed, sequence $\{\tau(A'_n(x)h)\}_{n=1}^{\infty}$ is bounded by constant C_0 as a converging sequence. Choose monotonically decreasing sequence of $\{s_j\}_{j=1}^{\infty}$ such that $\tau(\mathbb{I} - \{h > s_j\}) < 2^{-j}$, and $t_j = 2^j s_j^{-1}$. Then

$$\tau(\{A'_n(x) > t_j\}) \leq \frac{1}{t_j s_j}C_0 + 2^{-j} = (C_0 + 1)2^{-j}, \tag{44}$$

hence condition of the theorem 2.4 is satisfied. From the theorem it follows stochastic convergence of the $A_n(x)$ since for the dense subset in $L_1(M, \tau)$ of view $x - A'_k(x)$ for $x \in M \cap L_1(M, \tau)$ convergence is in L_1, and hence stochastically.

Part iii). Proof follows line of proof for ii). We provide only necessary modifications. Let E_1 be a conditional expectation $(M, \tau)\overline{\otimes}L_\infty(X, \mu)$ onto $(M, \tau)\overline{\otimes}C(X, \mu)$, and E_2 be a conditional expectation of $(M, \tau)\overline{\otimes}L_\infty(X, \mu)$ onto $\mathbb{C} \cdot \mathbb{I}\overline{\otimes}L_\infty(X, \mu)$, (for definition of conditional expectation and its existence see [19]). Due to the form of the $\alpha \bigotimes \sigma$, both E_j commute with A_n, for $j = 1, 2$.

Since

$$\|A_n(h)\|_\infty \geq \|E_1 A_n(h)\|_\infty = \|A_n(E_1 h)\|_\infty \tag{45}$$

,and $supp(h) \leq supp(E_1 h)$ is valid, it follows that $supp(E_1 h) = \mathbb{I}$. (Indeed, $x \geq 0$, $x \neq 0$ implies $\tau(E_1 x) = \tau(x) > 0$ hence $0 < \tau((E_1 a)h) = \tau(a(E_1 h))$ and $supp(E_1 h) = \mathbb{I}$).

Hence $E_1(h)$ is a weakly wandering operator.

For positive $x(z) \in L_1(M, \tau)\overline{\otimes}L_1(X, \mu)$ holds

$$\|x\|_1 = \int_X \|x(z)\|_1 \cdot d\mu(z) \tag{46}$$

, hence $\|x(z)\|_1$ is an $L_1(X, \mu)$ function. Applying Hopf inequality we get

$$\mu(\sup_n \{\|A'_n(x)(z)\|_1 > \lambda\}) \leq \frac{Const}{\lambda} \int_X \|x(z)\|_1 \cdot d\mu(z), \tag{47}$$

or, outside of the set $X_0 \subset X$ of small measure value of $\|A'_n(x)(z)\|_1$ is uniformly bounded. Proceeding like in the part ii) applied for every $z \in X_0$, we get stochastic converges for every $z \in X_0$. □

Theorem 3.3 (Neveu Decomposition for the special case of tensor product of von Neumann algebras)**.** *Let algebra $\mathcal{N} = (M,\tau)\overline{\otimes}L_\infty(X,\mu)$, (here X is a Hausdorff separable compact set, and μ is Lebesque measure), and α is an automorphism of M, σ is an automorphism of $L_\infty(X,\mu)$, then $\tilde{\alpha} = \alpha \otimes \sigma$ is an automorphism of $\mathcal{N}$. Suppose that in addition automorphism σ is ergodic. Then there exists an $\tilde{\alpha}$ invariant projection in $\mathcal{N}$ of view $e_1 = e_{11} \otimes \mathbb{I}$, $e2 = \mathbb{I} - e_1$ with all $e_1(z) = e_M$ for almost every $z \in X$ such that*

i) *There exists an $\tilde{\alpha}'$ normal state ρ on (N) with $supp(\rho) = e_1$ and for almost each $z \in X$, $\rho(z)$ is invariant with respect to automorphism α'.*
ii) *There exists a weakly wandering operator $h \in \mathcal{N}$ with $supp(h) = e_2$ and for almost each $z \in X$, $h(z)$ is a weakly wandering operator in M.*

Proof. Corollary 1.1 of [9] implies existence of the projection $\tilde{e_1}$ in $\mathcal{N}$ such that i) there exists $\tilde{\alpha}'$ invariant normal state ρ with support $supp(\rho) = \tilde{e_1}$ and ii) there exists a weakly wandering operator $h \in \mathcal{N}$ with support $\mathbb{I} - \tilde{e_1}$. Our goal is to show that similar statements are valid for almost every $z \in X$.

Since $\tilde{\alpha}'$ is ergodic, then for every $x \in M \otimes \mathbb{C}(X,\mu)$ (constant function on X with values in M) holds

$$(48)\quad \rho(z)(x(z)) = (\tilde{\alpha}'\rho(z))(x(z)) = \rho(z)(\alpha(x(\sigma(z)))) = \rho(z)(\alpha(x(z))) = (\alpha'(\rho(z)))(x(z))$$

,or $\rho(z)$ is α' invariant. Suppose that $\rho(z)$ is not constant or $\rho(z)$ is such there exists real $r_0 \in \mathbb{R}_+$ and $x(z) \equiv x_0 \in M_+$ with $\mu(\{z \in X|\rho(z)(x(z)) \leq r_0\}) > 0$ and $\mu(\{z \in X|\rho(z)(x(z)) < r_0\}) > 0$. Since σ is ergodic, there exists $n \in \mathbb{N}$ such that

$$(49)\quad \mu(\sigma^{-n}(\{z \in X|\rho(z)(x(z)) \leq r_0\}) \cap \{z \in X|\rho(z)(x(z)) < r_0\}) > 0.$$

Hence,

$$(50)\quad \rho(z)(x(z)) = (\tilde{\alpha}'^n\rho(z))(x(z)) = (\alpha')^n(\rho(z))(x(\sigma^n(z))) = \rho(z)(x(\sigma^n(z))) = \rho(\sigma^{-n}z)(x((z)))$$

, or $r_0 \geq \rho(z)(x_0) = \rho(\sigma^{-n}z)(x_0) < r_0$. Contradiction shows that $\rho(z)$ is constant.

This implies that $supp(\rho) = supp(\rho(z)) = \tilde{e_1}(z)$ is constant.

Part ii) follows directly arguments of proof 45. □

Theorem 3.4. *Let algebra $\mathcal{N} = (M,\tau)\overline{\otimes}L_\infty(X,\mu)$, (here X is a separable Hausdorf compact set, and μ is normalized Lebesque measure), and α is an automorphism of M, σ is an automorphism of $L_\infty(X,\mu)$, then $\tilde{\alpha} = \alpha \otimes \sigma$ is an automorphism of $\mathcal{N}$. Suppose that in addition automorphism σ is ergodic. Then for almost every $z \in X$ averages $A'_n(x(z))$ converges stochastically.*

Proof. Proof of the theorem follows directly from 3.3 and 3.2 applied the part where there exists weakly bounded operator, and from the regular individual ergodic theorem [20] applied to the part, where invariant normal state exists and Proposition 2.5. □

Now we are in a position to proof stochastic convergence of the bounded Besicovitch sequences.

Theorem 3.5 (Stochastic Ergodic Theorem for bounded Besicovitch sequences)**.** *Let $\{\beta_j\}_{j=1}^{\infty}$ is a bounded Besicovitch sequence. Let M is a von Neumann algebra with finite faithful normal tracial state τ. Let α is an automorphism of M. Then sequence*

$$\tilde{A_n}(x) = \frac{1}{n}\sum_{j=0}^{n-1}\beta_j {\alpha'}^j(x)$$

converges stochastically for $x \in L_1(M,\tau)$.

Proof. Suppose first that bounded Besicovitch sequence $\{\beta_j\}_{j=1}^{\infty}$ is a trigonometric polynomial $P_k(j)$. Then statement of the theorem is valid.

Indeed, choosing $\tilde{\alpha}$ as in example 3.1 we get from the theorem 2.4 and the fact that irrational rotation on the $\mathbb{C}_1$ is ergodic (Equidistribution Kroneker-Weyl Theorem, see for ex. [11] p. 146) that

$$A_n(x\cdot z) = z\cdot\frac{1}{n}\sum_{l=1}^{n-1}{\lambda'}^l\cdot\alpha^l(x) \tag{51}$$

,hence

$$\frac{1}{n}\sum_{l=1}^{n-1}\lambda^l\cdot\alpha'^l(x)$$

converges stochastically for irrational λ.

For the rational λ convergence follows from the fact that it is a finite combination of averages of the ${\alpha'}^m$, where m is denominator.

Taking linear combination of the 51 implies statement for the trigonometric polynomial.

Statement of the theorem is valid for the $x \in M\cap S(M)$. Indeed, using approximation of the BB sequence by trigonometric polynomial as in 36 one gets for $A_n(k,x) = \frac{1}{n}\sum_{l=1}^{n-1}P_k(l)\cdot\alpha'^l(x)$

$$\|\tilde{A_n}(x) - A_n(k,x)\|_\infty \le \frac{1}{n}(\sum_{l=0}^{n-1}|\beta_l - P_k(l)|)\cdot\|x\|_\infty \tag{52}$$

and, hence, converges stochastically.

Note also that for every $x \in L_1(M,\tau)$

$$\|\tilde{A_n}(x) - A_n(k,x)\|_1 \le \frac{1}{n}(\sum_{l=0}^{n-1}|\beta_l - P_k(l)|)\cdot\|x\|_1. \tag{53}$$

Hence by remark 2.1 averages $\tilde{A_n}(x)$ are uniformly bounded in the sense of 11.

Result of the theorem follows from the Stochastic Banach Principle, Theorem 3.2.4 and density of the $M\cap S(M)$ in $L_1(M,\tau)$. □

The following theorem is implied from stochastic ergodic theorem for bounded Besicovitch sequences. (cmp. [13])

For the following definitions see for example [12], p. 260.

Let σ be a homeomorphism of a compact metric space X with metric ϱ such that all powers of σ^l are equicontinous. Assume also that there exists $z \in X$ with dense orbit $\sigma^l(z)$ in X. Then there exists a unique (hence ergodic) σ invariant measure ν on the σ algebra of Borel sets $\mathfrak{B}$. Each non-empty open set has a positive ν measure.

A sequence u_j is called **uniform** if there exists such dynamical system $(X, \mathfrak{B}, \nu, \sigma)$ and a set $Y \in \mathfrak{B}$ with $\nu(\partial Y) = 0$ and $\nu(Y) > 0$ and point $y \in X$ with $u_j = j^{th}$ entry time of orbit of y into Y.

Theorem 3.6. *Let M, τ, α as in previous theorem, $\{u_j\}_{j\geq 0}$ is a uniform sequence. Then averages*

$$\frac{1}{n}\sum_{j=0}^{n-1} {\alpha'}^{u_j} x$$

converges stochastically for $x \in L_1(M, \tau)$.

Proof. Follows from the previous theorem and the fact by [15] that uniform sequence is a bounded Besicovitch sequence. □

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