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On the Complexity of Cutting Plane Proofs Using Split Cuts

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Abstract

We prove that cutting-plane proofs which use split cuts have exponential length in the worst case. Split cuts, defined by Cook, Kannan, Schrijver (1993), are known to be equivalent to a number of other classes of cuts, namely mixed-integer rounding (MIR) cuts, Gomory mixed-integer cuts, and disjunctive cuts. Our result thus implies the exponential worst-case complexity of cutting-plane proofs which use the above cuts.

Key words. cutting-plane proof, split cut, mixed-integer rounding, disjunctive cut, effective interpolation, monotone ciruits.

1 Introduction

The complexity of different types of cutting-plane proofs has been a much studied topic in recent years. Some well-known classes of cutting planes (linear inequalities satisfied by integral points in polyhedra) are Gomory-Chvátal cuts [10], split cuts [7], mixed-integer rounding (MIR) cuts [17, 16], and lift-and-project cuts [2]. Let $Ax \leq b$ stand for a system of linear inequalities. A *Gomory-Chvátal* cutting plane (or cut) for $Ax \leq b$ is a linear inequality $c^T x \leq \lfloor d \rfloor$ where c is integral, and $c^T x \leq d$ is satisfied by solutions of $Ax \leq b$. For a class of cuts $\mathcal{S}$, an $\mathcal{S}$ cutting-plane proof is a way of certifying that a linear inequality is satisfied by all integral solutions of $Ax \leq b$ via cuts from $\mathcal{S}$. For example, a *Gomory-Chvátal (GC) cutting-plane proof* of $c^T x \leq d$ with *length* M is a sequence of M inequalities

$$a_i^T x \leq d_i \ (i = 1, \ldots, M)$$

such that the last inequality in the sequence is $c^T x \leq d$, and for each $i \in \{1, \ldots, M\}$ the inequality $a_i^T x \leq d_i$ is a Gomory-Chvátal cut derived from the previous inequalities in the sequence and the inequalities in $Ax \leq b$. The notion of a cutting-plane proof was introduced in [5]).

Any inequality satisfied by integral solutions of $Ax \leq b$ has a Gomory-Chvátal cutting plane proof. This follows from the work of Gomory [10] and Chvátal [5]. In this paper, we focus on cutting-plane proofs of inequalities satisfied by 0-1 solutions. In an important paper, Pudlák [18] proved that Gomory-Chvátal cutting plane proofs of inequalities satisfied by 0-1 points in polyhedra have exponential length in the worst case. Dash [8] proved a similar result for lift-and-project cutting plane proofs. An inequality $c^T x \leq d$ is a *lift-and-project* cut for $P = \{x \mid Ax \leq b\}$, if for some index j, $c^T x \leq d$ is satsfied by points in $P \cap \{x_j = 0\}$ and $P \cap \{x_j = 1\}$. Lift-and-project cuts polynomially simulate the "non-commutative" matrix cuts of Lovász and Schrijver (1991), or cuts arising from the N_0 operator, and therefore, N_0-cutting plane proofs have exponential worst-case complexity [8]. A question left open in [8] is whether cutting-plane proofs using split cuts (or *split-cut proofs*) have exponential worst-case complexity. An inequality $c^T x \leq d$ is a *split cut* for $P = \{x \in R^n \mid Ax \leq b\}$ with respect to x if $c^T x \leq d$ is satisfied by points in $P \cap \{\alpha^T x \leq \beta\}$ and $P \cap \{\alpha^T x \geq \beta\}$, where α, β are integral. We say that $c^T x \leq d$ is derived from the *disjunction* $\alpha^T x \leq \beta$ and $\alpha^T x \geq \beta + 1$. All points in $P \cap \{x \in Z^n\}$ satisfy any split cut for P. Lift-and-project cuts and Gomory-Chvátal cuts are special cases of split cuts.

Here we show that split-cut proofs have exponential worst-case complexity, thus generalizing the results of Dash and Pudlák cited above. The proof technique, and the worst-case inequality systems, are essentially the same as those in Pudlák [18]. An important component of our proof is the equivalence between split cuts and mixed-integer rounding cuts proved in [17], which we discuss in the next section. As split cuts are also equivalent to the Gomory mixed-integer cuts [11] and disjunctive cuts [1], cutting-plane proofs with these cuts also have exponential worst-case complexity.

Krajíček [14] gave an exponential lower bound on the complexity of branch-and-cut proofs which use restricted Gomory-Chvátal cuts, and branching on inequalities (see also [12]). In his result, the cuts have polynomially bounded coefficient values, but branching on an arbitrary inequality $a^T x \leq b$ and its disjunction $a^T x \geq b + 1$, where a and b, are integral is allowed. The results in this paper, combined with Lemma 5.7 in Dash [8], imply that branch-and-cut proofs which use split cuts but branch only on the inequalities $x_i \leq 0$ and $x_i \geq 1$ for 0-1 variables x_i, have exponential worst-case complexity.

In the next section, we present MIR cuts in a form given in [9], and discuss their equivalence with split cuts. In Section 3 we discuss some well-known complexity results for boolean monotone circuits, and give our exponential lower bound results in Section 4.

2 Notation and Definitions

For a number v and an integer t, define $\hat{v} = v - \lfloor v \rfloor$. Define

$$Q^1 = \{v \in R,\ z \in Z\ :\ v + z\ \geq\ b,\ \ v \geq 0\}.$$

Lemma 1 *[22] All points in Q^1 satisfy the* basic mixed-integer inequality

$$v + \hat{b}z \geq \hat{b}\lceil b \rceil. \tag{1}$$

If $\hat{b} = 0$, then $v + \hat{b}z \geq \hat{b}\lceil b \rceil$ becomes $v \geq 0$. The basic mixed-integer inequality is a split cut for Q^1 with respect to z derived from the disjunction $z \leq \bar{b}$ and $z \geq \bar{b} + 1$. See the proof of Lemma 1 in [22]. In Figure 1(a), we depict the points in Q^1 by horizotal lines. In Figure 1(b), the dashed line represents (1), and is satisfied by points in the shaded regions which are $Q^1 \cap \{z \leq \lfloor b \rfloor\}$ and $Q^1 \cap \{z \geq \lceil b \rceil\}$.

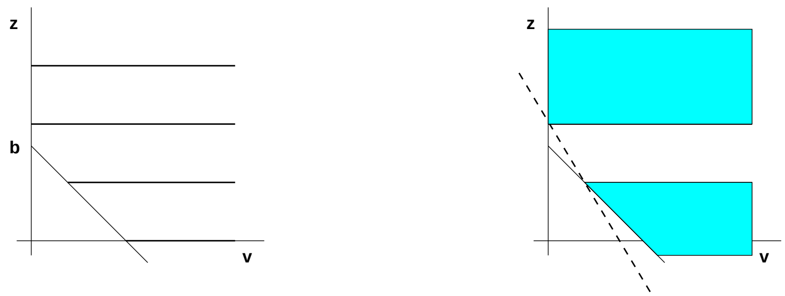

Figure 1: The basic mixed-integer inequality

Lemma 2 *$conv(Q^1) = \{v, z \in R\ :\ v + z\ \geq\ b,\ \ v + \hat{b}z \geq \hat{b}\lceil b \rceil,\ \ v \geq 0\}$.*

Proof Let Q' be the set on the right-hand side of the equation above. If $\hat{b} = 0$, Q' has only one extreme point $(0, b)$, and this is contained in Q^1. If $\hat{b} \neq 0$, the extreme points of Q' are $(0, \lceil b \rceil)$ and $(\hat{b}, \lfloor b \rfloor)$, given by the intersections of the first and third inequalities with the second inequality. Both these points lie in Q^1, and thus $Q' \subseteq conv(Q^1)$. ∎

Any linear inequality satisfied by points in Q^1 is therefore implied by a non-negative linear combination of the inequalities $v \geq 0$, $v + z \geq b$ and the basic mixed-integer inequality. See Marchand and Wolsey [15] and Wolsey [22] for ways of using the basic mixed-integer inequality to derived valid inequalities for mixed-integer sets.

Let $P = \{x \in R^n\ :\ Ax \leq b\}$. We assume that A and b have integral coefficients. Define $s = b - Ax$, and $\bar{P} = \{s \in R^m, x \in R^n\ :\ Ax + Is = b,\ s \geq 0\}$. Let $\lambda \in R^m$ be a row vector such that λA is integral. Define $\bar{\alpha}, \beta$ and $\bar{\beta}$ as

$$\bar{\alpha} = \lambda A,\ \ \beta = \lambda b,\ \ \bar{\beta} = \lfloor \beta \rfloor .$$

By definition, $\hat{\beta} = \beta - \bar{\beta}$. Let λ^+ satisfy $\lambda_i^+ \geq \max\{\lambda_i, 0\}$. Then $\lambda s + \bar{\alpha} x = \lambda b$ is a valid inequality for $\bar{P}$ and so is

$$\lambda^+ s + \bar{\alpha} x \geq \beta. \tag{2}$$

For all points in P or $\bar{P}$, $\lambda^+ s$ is non-negative and $\bar{\alpha} x$ is integral. Lemma 1 implies that

$$\lambda^+ s + \hat{\beta}\bar{\alpha} x \geq \hat{\beta}(\bar{\beta} + 1) \tag{3}$$

is a valid inequality for $\bar{P} \cap \{x \in Z^n\}$; we call this inequality a *mixed-integer rounding cut (MIR)* for $\bar{P}$. We say that (3) is derived using the multipliers λ. Substituting out the slacks in (3), we get a valid inequality for $P \cap \{x \in Z^{n_2}\}$ which we call an MIR cut for P. Any valid inequality for P is trivially an MIR cut for P. This definition is different from, but equivalent to, the definition of MIR cuts in [17]. See [9] for a detailed discussion of different forms of MIR cuts.

Every MIR cut for P is a split cut for P derived from the disjunction $\bar{\alpha} x \leq \bar{\beta}$ and $\bar{\alpha} x \geq \bar{\beta} + 1$. This is implied by the fact that (1) is a split cut for Q^1 with respect to z derived from the disjunction $z \leq \bar{b}$ and $z \geq \bar{b} + 1$. It follows from the work of Nemhauser and Wolsey [17] that every split cut for P is also an MIR cut for P.

Theorem 3 *[17] Every split cut for P is an MIR cut for P.*

Proof Let $c^T x \leq d$ be a split cut for P. Then $c^T x \leq d$ is valid for $P_1 = P \cap \{\bar{\alpha} x \leq \bar{\beta}\}$ and $P_2 = P \cap \{\bar{\alpha} x \geq \bar{\beta} + 1\}$, for some integral row vector $\bar{\alpha}$ and integer $\bar{\beta}$. There are multipliers $\lambda_1, \lambda_2 \in R^m$ and $\mu_1, \mu_2 \in R$ with $\lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0$ such that

$$\begin{aligned} c^T &= \lambda_1 A x + \mu_1 \bar{\alpha}, & c^T &= \lambda_2 A x - \mu_2 \bar{\alpha}, \\ d &\geq \lambda_1 b + \mu_1 \bar{\beta}, & d &\geq \lambda_2 b - \mu_2(\bar{\beta} + 1). \end{aligned}$$

Assume (without loss of generality) that $\lambda_2 b - \mu_2(\bar{\beta} + 1) \geq \lambda_1 b + \mu_1 \bar{\beta}$ and let d' stand for the first number. This implies that the following inequalities are valid for $\bar{P}$:

$$c^T x - \mu_1(\bar{\alpha} x - \bar{\beta}) + \lambda_1 s \leq d' \tag{4}$$

$$c^T x + \mu_2(\bar{\alpha} x - (\bar{\beta} + 1)) + \lambda_2 s = d'. \tag{5}$$

For $c^T x \leq d$ to be a non-trivial split cut (i.e., not valid for P), $\mu_1 + \mu_2$ has to be positive, and we assume this to be the case. Subtracting (4) from (5) and dividing by $\mu_1 + \mu_2$, we obtain

$$\frac{\lambda_2 s - \lambda_1 s}{\mu_1 + \mu_2} + \bar{\alpha} x \geq \bar{\beta} + \frac{\mu_2}{\mu_1 + \mu_2}$$

as a valid inequality for $\bar{P}$. Let $\hat{\beta} = \mu_2/(\mu_1 + \mu_2)$. It follows that

$$\frac{\lambda_2 s}{\mu_1 + \mu_2} + \hat{\beta}\bar{\alpha}x \geq \hat{\beta}(\bar{\beta} + 1) \tag{6}$$

is an MIR cut for $\bar{P}$ ($\lambda_1 s \geq 0$ and $\lambda_1 s \geq \lambda_1 s - \lambda_2 s$). Using equation (5) to substitute out $\lambda_2 s$, and then multiplying (6) by $\mu_1 + \mu_2$, we obtain $c^T x \leq d$ as an MIR for P. ∎

Consider the multipliers $\lambda_1, \lambda_2 \geq 0$ defined in the above proof. It is clear $c^T x \leq d$ can be viewed as split cut derived from the inequalities $\lambda_1 Ax \leq \lambda_1 b$ and $\lambda_2 Ax \leq \lambda_2 b$. We call these inequalities the *base inequalities* for $c^T x \leq d$.

Definition 4 *The split closure of a polyhedron P, denoted by $sc(P)$, is the set of points satisfying all split cuts for the polyhedron.*

It is well-known that $sc(P) \subseteq P$. The MIR closure is defined similarly in terms of MIR cuts. The split closure of a polyhedron is therefore the same as its MIR closure.

3 Boolean circuits

A boolean circuit can be thought of as a description of the elementary steps in an algorithm via a directed acyclic graph with two types of nodes: input nodes – nodes with no incoming arcs, and computation nodes (or gates), each of which is labelled by one of the boolean functions $\wedge, \vee$, and $\neg$. One of the computation nodes has no outgoing arcs is designated as the output node. For nodes i and j, an arc ij means that the value computed at i is used as an input to the gate at node j. A computation is represented by placing 0-1 values on the input gates, and then recursively applying the gates to inputs on incoming arcs, till the function at the output node is evaluated.

A *monotone function* is a real-valued non-decreasing function $f : R^n \rightarrow R$, that is, if $x \leq y$ with x, y in R^n, then $f(x) \leq f(y)$. Here, $x \leq y$ means $x_i \leq y_i$ for $i = 1, \ldots, n$. Some monotone unary and binary functions (we call these *monotone operations*) are

$$tx, \quad r + x, \quad x + y, \quad \lfloor x \rfloor, \quad thr(x, 0)$$

where t is a non-negative constant, x and y are real variables, and r is a real constant; $thr(x, 0)$ is a *threshold function* which returns 0, if $x < 0$, and 1 otherwise. The functions $\wedge$ and $\vee$ are monotone operations over the domain $\{0, 1\}$. On the other hand $f(x, y) = x - y$, where $x, y \in R$, is not a monotone function. A *monotone boolean circuit* is one which uses only $\wedge$ gates and $\vee$ gates; a *monotone real circuit* is one with arbitary monotone operations as gates. We will only consider monotone circuits with 0-1 inputs and outputs.

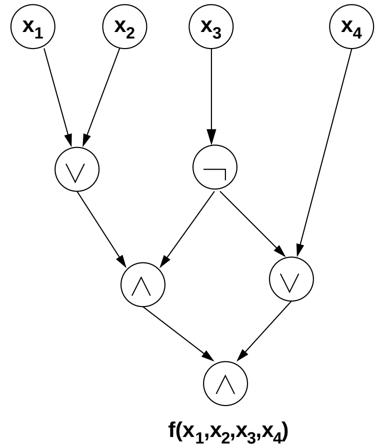

Figure 2: A boolean circuit

Consider $CLIQUE_{k,n}$ (say k is a function of n), the function which takes as input n-node graphs (represented by incidence vectors of their edges) and returns 1 if the graph has a clique of size k or more, and 0 otherwise. This is monotone, as adding edges to a graph (changing some zeros to ones in the incidence vector) causes the maximum clique size to increase. Every monotone boolean function can be computed by a monotone boolean circuit. Razborov [20] established super-polynomial lower bounds on the sizes of monotone boolean circuits, and Alon and Boppana [3] strengthened his result.

Theorem 5 *(Razborov [20], Alon and Boppana [3]) Let C_n be a monotone boolean circuit which takes as input graphs on n nodes (given as incidence vectors of edges), and returns 1 if the input graph contains a clique of size $k = \lfloor n^{2/3} \rfloor$, and 0 if the graph contains a coloring of size $k-1$ (and returns 0 or 1 for all other graphs). Then*

$$|C_n| \geq 2^{\Omega((n/\log n)^{1/3})}.$$

Thus any monotone boolean circuit computing $CLIQUE_{k,n}$, with k given as above, has exponentially many gates. This result is essentially true for monotone real circuits as well (Cook and Haken proved a slightly different statement).

Theorem 6 *(Pudlák [18], Cook and Haken [6]) Let D_n be a monotone real circuit which has the same inputs and outputs as in Theorem 5. Then D_n must have exponentially many gates (the lower bound for $|C_n|$ given in Theorem 5 is also valid for $|D_n|$).*

To use Theorem 6, we encode (in a sense) the problem of Theorem 5 using a system of inequalities. Let $k = \lfloor n^{2/3} \rfloor$. Consider the set of nodes $N = \{1, \ldots, n\}$. Let z be a vector of $n(n-1)/2$ 0-1 variables, such that every 0-1 assignment to z corresponds to the incidence vector of a graph on n nodes. Let x be the 0-1 vector of variables $(x_i \,|\, i = 1, \ldots, n)$ and let y be the 0-1 vector of variables $(y_{ij} \,|\, i = 1, \ldots, n, j = 1, \ldots, k-1)$. We want to impose the conditions:

$$\text{the set of nodes } \{i \,|\, x_i = 1\} \text{ forms a clique of size } \geq k,$$

$$\text{for all } j \in \{1, \ldots, k-1\}, \text{ the set } \{i \,|\, y_{ij} = 1\} \text{ is a stable set.}$$

Thus, the variables y_{ij} define a mapping of nodes in a graph to $k-1$ colors in a proper colouring. To this end, add the inequalities

$$\sum_i x_i \geq k, \tag{7}$$

$$x_i + x_j \leq 1 + z_{ij}, \quad \forall\, i, j \in N, \text{ with } i < j, \tag{8}$$

$$\sum_{j=1}^{k-1} y_{ij} = 1, \quad \forall i \in N, \tag{9}$$

$$y_{is} + y_{js} \leq 2 - z_{ij}, \quad \forall\, i, j \in N \text{ with } i < j, \text{ and } \forall s \in \{1, \ldots, k-1\}. \tag{10}$$

Let $Ax + Cz \leq e$ stand for the inequalities (7) and (8), along with the bounds on $0 \leq x \leq 1$. Let $By + Dz \leq f$ stand for the inequalities (9) and (10), along with the bounds $0 \leq y \leq 1$ and $0 \leq z \leq 1$. Then any 0-1 solution of $Ax + Cz \leq e$ and $By + Dz \leq f$ corresponds to a graph which has both a clique of size k, and a coloring of size $k-1$. Clearly, no such 0-1 solution exists. Note that the above inequalities have $O(n^3)$ variables and constraints; for technical reasons we will also need the fact that $C \leq 0$. Because of Theorem 6, every monotone real circuit which takes graphs on n nodes as input (in the form of a 0-1 vector z') and has the property that output 0 implies $Ax \leq e - Cz'$ has no 0-1 solution and output 1 implies $By \leq f - Dz'$ has no 0-1 solution, has exponential size.

4 Complexity of split-cut proofs

For $i = 1, 2$, let

$$P_i = \{v_i, z_i : v_i + z_i \geq b_i,\ v_i \geq 0,\ z_i \in Z\}, \tag{11}$$

$$P_3 = \{v_1, v_2, z_1, z_2 : v_1 + v_2 + z_1 + z_2 \geq b_1 + b_2,\ v_1 + v_2 \geq 0,\ z_1 + z_2 \in Z\} \tag{12}$$

Lemma 7 *Let P_1, P_2 and P_3 be defined as above. Let $b_3 = b_1 + b_2$. Then the inequality $(v_1 + v_2) + \hat{b}_3(z_1 + z_2) \geq \hat{b}_3 \lceil b_3 \rceil$ valid for P_3 is implied by a non-negative combination of the inequalities defining P_1 and P_2 and the basic mixed-integer inequalities for P_1 and P_2.*

Proof By Lemma 2, for $i = 1, 2$,

$$conv(P_i) = \{v_i, z_i : v_i + z_i \geq b_i,\ v_i + \hat{b}_i z_i \geq \hat{b}_i \lceil b_i \rceil,\ v_i \geq 0\}.$$

By applying the basic mixed-integer inequality, we can see that

$$v_1 + v_2 + \hat{b}_3(z_1 + z_2) \geq \hat{b}_3 \lceil b_3 \rceil \tag{13}$$

is valid for P_3. Therefore

$$\begin{aligned} \hat{b}_3 \lceil b_3 \rceil &\leq \min\{v_1 + v_2 + \hat{b}_3(z_1 + z_2) \,|\, (v_1, v_2, z_1, z_2) \in P_3\} \\ &\leq \min\{v_1 + \hat{b}_3 z_1 \,|\, (v_1, z_1) \in P_1\} + \min\{v_2 + \hat{b}_3 z_2 \,|\, (v_2, z_2) \in P_2\}. \end{aligned}$$

as optimal solutions (v_1^*, z_1^*) and (v_2^*, z_2^*) of the last two minimization problems with optimal values μ_1 and μ_2, respectively, yield a feasible solution $(v_1^*, v_2^*, z_1^*, z_2^*)$ of the first minimization problem with objective value $\mu_1 + \mu_2$. As

$$v_1 + \hat{b}_3 z_1 \geq \mu_1 \text{ and } v_2 + \hat{b}_3 z_2 \geq \mu_2$$

are valid inequalities for $conv(P_1)$ and $conv(P_2)$, respectively, (13) is implied by a nonnegative combination of inequalities defining $conv(P_1)$ and $conv(P_2)$. ■

Lemma 8 *Let x, y be vectors of integer variables, with no common components. Let*

$$Q_1 = \{x : a_1 x \leq e_1, a_2 x \leq e_2\},$$

$$Q_2 = \{y : c_1 y \leq f_1, c_2 y \leq f_2\},$$

$$Q_3 = \{(x, y) : x \in Q_1, y \in Q_2\},$$

where a_1, a_2, c_1, c_2 are vectors with appropriate dimensions. Let $a_3 x + c_3 y \leq d$ be a split cut for Q_3 derived from $a_1 x + c_1 y \leq e_1 + f_1$ and $a_2 x + c_2 y \leq e_2 + f_2$. Let $g = \max\{a_3 x : x \in sc(Q_1)\}$ and $h = \max\{c_3 y : y \in sc(Q_2)\}$. Then $g + h \leq d$.

Proof Let $a_3 x + c_3 y \leq d$ be a split cut for Q_3 derived as described above. It is also an MIR cut derived from the system

$$a_1 x + c_1 y + s_1 = e_1 + f_1$$

$$a_2 x + c_2 y + s_2 = e_2 + f_2$$

$$s_1, s_2 \geq 0$$

More precisely, there are real numbers λ_1 and λ_2 such that the split cut above equals

$$\begin{aligned} \lambda_1^+ s_1 + \lambda_2^+ s_2 + \quad & \hat{\beta}\big((\lambda_1 a_1 + \lambda_2 a_2)x + (\lambda_1 c_1 + \lambda_2 c_2)y\big) \geq \hat{\beta} \lceil \beta \rceil, \\ \text{where} \quad & \beta = \lambda_1(e_1 + f_1) + \lambda_2(e_2 + f_2), \\ & \lambda_1 a_1 + \lambda_2 a_2 \text{ and } \lambda_1 c_1 + \lambda_2 c_2 \text{ are integral}, \\ & \lambda_1^+ \geq \max\{\lambda_1, 0\},\ \lambda_2^+ \geq \max\{\lambda_2, 0\}. \end{aligned} \tag{14}$$

Let s_1' and s_2' be slacks for the constraints defining Q_1, i.e.,

$$a_1 x + s_1' = e_1,\ a_2 x + s_2' = e_2.$$

Similarly, let s_1'' and s_2'' be slacks for constraints defining Q_2. Define

$$\begin{aligned} v_1 &= \lambda_1^+ s_1' + \lambda_2^+ s_2', \qquad & z_1 &= \lambda_1 a_1 + \lambda_2 a_2,\ b_1 = \lambda_1 e_1 + \lambda_2 e_2, \\ v_2 &= \lambda_1^+ s_1'' + \lambda_2^+ s_2'', \qquad & z_2 &= \lambda_1 c_1 + \lambda_2 c_2,\ b_2 = \lambda_1 f_1 + \lambda_2 f_2. \end{aligned}$$

Clearly

$$s_1 = s_1' + s_1'',\ s_2 = s_2' + s_2'', \text{ and } \beta = b_1 + b_2.$$

Defining P_1, P_2 and P_3 in Lemma 7 in terms of the variables v_1, v_2, z_1, z_2 with $b_3 = \beta$, we see that (14) can be written as

$$v_1 + v_2 + \hat{b}_3(z_1 + z_2) \geq \hat{b}_3 \lceil b_3 \rceil.$$

By the proof of Lemma 7, there are numbers μ_1 and μ_2 such that $\mu_1 + \mu_2 \geq \hat{b}_3 \lceil b_3 \rceil$ and for $i = 1, 2$, $v_i + \hat{b}_3 z_i \geq \mu_i$ is a non-negative combination of the inequalities defining P_i and the basic mixed-integer inequality for P_i. Consider the case $i = 1$. Substituting out the slacks s_1' and s_2', we see that $v_1 \geq 0$ and $v_1 + z_1 \geq b_1$ are both implied by non-negative combinations of the inequalities defining Q_1, and $v_1 + \hat{b}_1 z_1 \geq \hat{b}_1 \lceil b_1 \rceil$ defines an MIR cut for Q_1. Finally, $v_1 + \hat{b}_3 z_1 \geq \mu_1$ becomes $a_3 x \leq g'$ for some g'.

Therefore, substituting out the slacks s_i, s_i', s_i'' for $i = 1, 2$, we conclude that there are numbers g' and h' such that

(i) $g' + h' \leq d$,

(ii) $a_3 x \leq g'$ is a non-negative linear combination of the inequalities defining Q_1 and some MIR cut for Q_1,

(iii) $c_3 x \leq h'$ is a non-negative linear combination of the inequalities defining Q_2 and some MIR cut for Q_2.

If we define

$$g = \max\{a_3x : x \in sc(Q_1)\}, \quad h = \max\{c_3y : y \in sc(Q_2)\},$$

$$\text{then } g \le g', \quad h \le h' \text{ and } g + h \le d.$$

■

Definition 9 *A split cut proof of an inequality is a* simplified *split cut proof if every inequality is either a non-negative linear combination of previous inequalities or a split cut derived from two previous inequalities.*

It is clear that given a split cut proof of length L of some inequality with length L, there is a *simplified* split cut proof of the same inequality with length $3L$. The simplified proof can be obtained by replacing each split cut in the proof with three inequalities: the two base inequalities for the split cut followed by the cut itself.

Theorem 10 *Let $\mathcal{R}$ be a simplified split cut proof of $0^Tx+0^Ty+0^Tz \le -1$ from $Ax+Cz \le e$ and $By + Dz \le f$ of length L. Then there exists a monotone real circuit of size $2Ln^3$ solving $CLIQUE_{k,n}$.*

Proof Let $a_i^Tx+b_i^Ty+c_i^Tz \le d_i$ be the ith inequality in $\mathcal{R}$ and call this $\mathcal{R}_i$. Let $\mathcal{R}_1, \dots, \mathcal{R}_m$ be just the inequalities in $Ax+Cz \le e$ and $By+Dz \le f$. For some k, let $\mathcal{R}_k$ be precisely $0^Tx + 0^Ty + 0^Tz \le -1$. We can assume – by scaling inequalities and multipliers – that $\mathcal{R}$ has integral inequalities. Let I_i stand for $\{1, \dots, i-1\}$. By definition, for $i > m$ $\mathcal{R}_i$ is either a non-negative linear combination of the inequalities $\mathcal{R}_1, \dots, \mathcal{R}_{i-1}$ with the multipliers $\lambda_{ij} \ge 0\,(j \in I_i)$, or a split cut derived from $\mathcal{R}_k$ and $\mathcal{R}_l$ for some $k, l \le i-1$.

Let z' stand for some 0-1 assignment to z. The sequence of inequalities $\mathcal{R}'$, where

$$\mathcal{R}'_i \text{ is } a_i^Tx + b_i^Ty \le d_i - c_i^Tz',$$

is a simplified split cut proof of infeasibility of $Ax \le e - Cz'$ and $By \le f - Dz'$, with the same length as $\mathcal{R}$. Further,

$$\mathcal{R}'_i \text{ is derived with the same multipliers as } \mathcal{R}_i.$$

Define $d'_i = d_i - c_i^Tz'$. We construct a sequence of inequalities $\mathcal{S}$ involving only x, and another sequence $\mathcal{T}$, involving only y, such that

$$\mathcal{S}_i \equiv a_i^Tx \le g_i, \quad \mathcal{T}_i \equiv c_i^Ty \le h_i, \text{ and } g_i + h_i \le d'_i,$$
$$\mathcal{S}_i, \mathcal{T}_i \text{ are valid for integral solutions of } Ax \le e - Cz' \text{ and } By \le f - Dz'. \quad (15)$$

Thus $\mathcal{S}_i + \mathcal{T}_i$ has the same left hand side as $\mathcal{R}'_i$, but an equal or smaller right-hand side.

For $i = 1, \ldots, m$, if $\mathcal{R}'_i$ involves only x, then set $\mathcal{S}_i$ to $\mathcal{R}'_i$ and $\mathcal{T}_i$ to $0^T y \leq 0$, otherwise set $\mathcal{S}_i$ to $0^T x \leq 0$ and $\mathcal{T}_i$ to $\mathcal{R}'_i$. Define subsequent terms of $\mathcal{S}$ and $\mathcal{T}$ as follows. For $i = m+1, \ldots, k$, if $\mathcal{R}'_i$ is a non-negative linear combination of inequalities $\mathcal{R}'_j\,(j \in I'_i \subseteq I_i)$ with the multipliers $\lambda_{ij} > 0\,(j \in I'_i)$, then let $\mathcal{S}_i$ and $\mathcal{T}_i$ be non-negative linear combinations of $\mathcal{S}_j\,(j \in I'_i)$ and $\mathcal{T}_j\,(j \in I'_i)$, respectively, with the same multipliers $\lambda_{ij}(j \in I'_i)$. If $\mathcal{R}'_i$ is a split cut derived from $\mathcal{R}'_k$ and $\mathcal{R}'_l$ for some $k, l \leq i-1$, then define

$$Q_1 = \{x : a_k^T x \leq g_k, a_l^T x \leq g_l\},\ Q_2 = \{y : c_k^T y \leq h_k, c_l^T y \leq h_l\}.$$

It follows from Lemma 8 that $\mathcal{R}'_i$ is implied by the inequalities defining Q_1 and Q_2 and some split cuts for these sets. More precisely, if we define

$$g_i = \max\{a_i^T x : x \in sc(Q_1)\},\ h_i = \max\{c_i^T y : y \in sc(Q_2)\},\ \text{then } g_i + h_i \leq d'_i. \tag{16}$$

We then define $\mathcal{S}_i$ to be $a_i^T x \leq g_i$, and $\mathcal{T}_i$ to be $c_i^T y \leq h_i$.

Observe that the inequality $g_i + h_i \leq d'_i$ in (15) and (16) is by definition true for $i = 1, \ldots, m$; either $g_i = d'_i$ and $h_i = 0$, or $h_i = d'_i$ and $g_i = 0$. Let $i > m$, and assume by induction that (15) is true for smaller values of i. If $\mathcal{R}_i$ is a non-negative combination of inequalities, then (15) is clearly true. If $\mathcal{R}_i$ is a split cut derived from two previous inequalities, then again (15) is true because of Lemma 8: $\mathcal{S}_i$ is valid for the split closure of $\mathcal{S}_j (j \in I_i)$, and $\mathcal{T}_i$ is valid for the split closure of $\mathcal{T}_j (j \in I_i)$. Therefore the last inequalities in $\mathcal{S}$ and $\mathcal{T}$ are, respectively, $0^T x \leq g_k$ and $0^T y \leq h_k$. As $d'_k = d_k = -1$, one of g_k and h_k is less than 0, and we have a proof of infeasibility of either $Ax \leq e - Cz'$ or $By \leq f - Dz'$.

We now define a monotone circuit $\mathcal{C}$ as follows. It takes as input the vector z' and first computes $e - Cz'$ by monotone operations (recall $C \leq 0$). It then computes $g_1, g_2, \ldots, g_k$ by monotone operations as follows. First, $g_1, \ldots, g_m$ are trivially obtained from $d'_1, \ldots, d'_m$: either $g_i = d'_i$ if the ith inequality is from $Ax + Cz \leq e$, or 0 otherwise. For $i > m$, if $\mathcal{R}_i = \sum_{j \in I'_i} \lambda_{ij} R_j$, then $g_i = \sum_{j \in I'_i} \lambda_{ij} g_j$. We can assume that $|I'_i| \leq n^3$. Therefore we can assume $\mathcal{C}$ computes g_i using at most $2n^3$ monotone operations from (3) (the λ_{ij}s are fixed as $\mathcal{R}$ is fixed; they are also non-negative). If $\mathcal{R}_i$ is a split cut derived from two previous inequalities, then $\mathcal{C}$ computes g_i as in (16). Note that only g_k and g_l are variable in this computation, and thus the computation of g_i is a monotone operation. Finally, the circuit returns $thr(g_k, 0)$, which is a monotone operation. Therefore, if the circuit returns 0, then $g_k < 0$ and $Ax \leq e - Cz'$ has no integral solutions. If the output is 1, we know that $h_k < 0$ and $By \leq f - Dz'$ has no 0-1 solutions. ∎

Corollary 11 *Every split cut proof of* $0^T x + 0^T y + 0^T z \leq -1$ *from* $Ax + Cz \leq e$ *and* $By + Dz \leq f$ *has exponential length.*

Dash [8, Lemma 5.7] proved that a branch-and-cut proof of (integer) infeasibility $\mathcal{R}$ using lift-and-project cuts and Gomory-Chvátal cuts and branching on 0-1 variables can be transformed into a cutting plane proof of infeasibility $\mathcal{S}$ with length $s+t$, where s and t are the number of cuts and branching decisions in $\mathcal{R}$, respectively. In this proof, every branching decision is replaced by a lift-and-project cut. One can easily obtain the following result using the proof technique for the result above.

Theorem 12 *Let $\mathcal{R}$ be a branch-and-cut proof of the fact that a polyhedron P has no integral solutions using s split cuts, and branching on 0-1 variables with t branching decisions. There is a split-cut proof showing P has no integral solutions with length $s+t$.*

Corollary 13 *Every branch-and-cut proof of (integer) infeasibility of $Ax + Cz \leq e$ and $By + Dz \leq f$ of the type described in Theorem 12 has exponential size.*

The technique of deriving a polynomial size circuit from a proof of infeasibility as in the theorem above is called *effective interpolation*, and *monotone interpolation* if the circuit only uses monotone operations. It was proposed by Krajíček [13, 14] to establish lower bounds on the lengths of proofs in different proof systems. Razborov [21], and Bonet, Pitassi, and Raz [4], first used this idea to prove exponential lower bounds for some proof systems.

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