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Supplement to "Distributions with Maximum Entropy Subject to Constraints on Their L-moments or Expected Order Statistics"

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Abstract. This report contains supplementary results to those in the paper "Distributions with maximum entropy subject to constraints on their L-moments or expected order statistics", to be published in *Journal of Statistical Planning and Inference.*

Hosking (2007) derives distributions with maximum entropy subject to constraints on their L-moments or on expectations of their order statistics. This report expands on some points that are not described in detail in the main paper. Equations and examples specific to this report are numbered (S.n). Equation and example numbers that are entirely numeric refer to the main paper.

For convenience we first restate the main results of the paper, Theorems 2.1, 2.2 and 3.1.

Theorem 2.1. Consider the problem

$$\text{Maximize} \int_0^1 \log\{Q'(u)\} du \tag{2.4}$$

$$\text{subject to} \int_0^1 K_s(u) Q'(u) du = h_s, \qquad s = 1, \ldots, S, \tag{2.5}$$

where the K_s are linearly independent polynomials, and the maximization is over functions $Q'(u)$ that are strictly positive on $(0, 1)$. If there exist constants a_s, $s = 1, \ldots, S$, that satisfy

$$\int_0^1 \frac{K_r(u) du}{\sum_{s=1}^S a_s K_s(u)} = h_r \,, \qquad r = 1, \ldots, S, \tag{2.6}$$

and

$$\sum_{s=1}^S a_s K_s(u) > 0 \,, \qquad 0 < u < 1, \tag{2.7}$$

then the problem has the solution

$$Q'(u) = Q'_0(u) \equiv 1 \Big/ \sum_{s=1}^S a_s K_s(u) \,. \tag{2.8}$$

The solution is unique up to redefinition of $Q'_0(u)$ on a set of u values that has measure zero.

Theorem 2.2. Consider the problem

$$\text{Maximize } \int_0^1 \log\{Q'(u)\}du$$
$$\text{subject to } \int_0^1 J_r(u)Q(u)du = g_r, \qquad r = 1, \dots, R,$$

where the J_r are linearly independent polynomials, and the maximization is over quantile functions Q of distributions whose cumulative distribution functions F are continuous and differentiable, with densities f that are nonzero within the range of the distribution. Suppose further that one of the following sets of additional constraints is to be satisfied:

(Case 0) $Q(0) = L$ and $Q(1) = U$;
(Case 1) $Q(0) = L$, $Q(1)$ unconstrained;
(Case 2a) no constraints on $Q(0)$ or $Q(1)$, with $\int_0^1 J_r(u)du = 0$ for all r;
(Case 2b) no constraints on $Q(0)$ or $Q(1)$, with $\int_0^1 J_r(u)du \neq 0$ for some r.

The problem is solved by the following procedures, provided that the equations (2.6) referred to below can be solved and, if applicable, that the integrals in (2.20) or (2.24) below are finite. The solution is unique except that in Case 2a the distribution is determined only up to a location shift.

Case 0:

1. Write the constraints in the form (2.5), by setting

$$K_r(u) = \int_u^1 J_r(v)dv, \qquad h_r = g_r - K_r(0)L, \qquad r = 1, \dots, R. \tag{2.19}$$

2. Add the constraint $\int_0^1 Q'(u)du = U - L$, by defining $K_{R+1}(u) = 1$, $0 \le u \le 1$, and $h_{R+1} = U - L$.
3. Set $S = R + 1$ and solve equations (2.6).
4. The maximum-entropy distribution has Q' given by (2.8) and

$$Q(u) = L + \int_0^u \frac{dv}{\sum_{s=1}^S a_s K_s(v)}\,. \tag{2.20}$$

Case 1:

1. Write the constraints in the form (2.5), via (2.19).
2. Set $S = R$ and solve equations (2.6).
3. Provided that the integral in (2.20) exists, the maximum-entropy distribution has Q' given by (2.8) and Q given by (2.20).

Case 2a:

1. Write the constraints in the form (2.5), via (2.19).
2. Set $S = R$ and solve equations (2.6).
3. The maximum-entropy distribution has Q' given by (2.8). Q is determined only up to an additive constant, by

$$Q(u) = \int^{u} \frac{dv}{\sum_{s=1}^{S} a_s K_s(v)} .$$

Case 2b:

1. Without loss of generality, suppose that $\int_0^1 J_R(u)du \neq 0$.
2. For $r = 1, \ldots, R-1$, set $J_r^*(u) = J_r(u) - \alpha_r J_R(u)$ and $g_r^* = g_r - \alpha_r g_R$, where $\alpha_r = \int_0^1 J_r(u)du / \int_0^1 J_R(u)du$. The first $R-1$ constraints are equivalent to the new constraints

$$\int_0^1 J_r^*(u)Q(u)du = g_r^*, \qquad r = 1, \ldots, R-1,$$

   for which we have $\int_0^1 J_r^*(u)du = 0$ for all r.
3. Write the new constraints in the form (2.5), by setting

$$K_r(u) = \int_u^1 J_r^*(v)dv, \qquad h_r = g_r^*, \qquad r = 1, \ldots, R-1.$$

4. Set $S = R-1$ and solve equations (2.6).
5. Provided that the integrals in (2.24) below exist, the maximum-entropy distribution has Q' given by (2.8) and quantile function given by

$$Q(u) = \frac{1}{K_R(0)} \left[g_R + \int_0^u \{K_R(0) - K_R(v)\} Q'(v)dv - \int_u^1 K_R(v)Q'(v)dv \right] , \tag{2.24}$$

   where $K_R(u) = \int_u^1 J_R(u)du$.

Theorem 3.1. The distribution that has maximum entropy given specified values of its L-moments λ_r, $r = 1, \ldots, R$, is given by the following construction, provided that the equations (3.7) below have a solution. Denote by Cases 0, 1, and 2 the instances in which the range of the distribution is constrained to be the intervals $[L, U]$, $[L, \infty)$, and $(-\infty, \infty)$, respectively. Define

(in Case 0)	$Z_0(u) = 1,$	$k_0 = U - L;$	
(in Cases 0 and 1)	$Z_1(u) = 1 - u,$	$k_1 = \lambda_1 - L;$	
(in all Cases)	$Z_r(u) = \int_u^1 P^*_{r-1}(v)\,dv\,,$	$k_r = \lambda_r,$	$r \geq 2.$

In Case m ($m = 0$, 1, or 2), the maximum-entropy distribution has quantile function $Q(u)$ with derivative given by

$$Q'(u) = 1 \Big/ \sum_{r=m}^{R} a_r Z_r(u) \tag{3.6}$$

where the a_r satisfy the equations

$$\int_0^1 \frac{Z_r(u)\,du}{\sum_{s=m}^{R} a_s Z_s(u)} = k_r\,, \qquad r = m, \ldots, R, \tag{3.7}$$

with

$$\sum_{r=m}^{R} a_r Z_r(u) > 0\,, \qquad 0 < u < 1. \tag{3.8}$$

The quantile function itself is given, in Cases 0 and 1, by

$$Q(u) = L + \int_0^u Q'(v)\,dv \tag{3.9}$$

or, in Case 2, by

$$Q(u) = \lambda_1 + \int_0^u v\,Q'(v)\,dv - \int_u^1 (1-v)\,Q'(v)\,dv \tag{3.10}$$

for any $u \in (0, 1)$.

Example 3.5, expanded.

Example 3.5 derives the maximum-entropy distribution for a distribution on the interval $[0,\infty)$ with the value of only λ_2 constrained.

This problem is excluded from the ambit of Theorem 3.1, because λ_1 is not constrained. In attempting to use Theorem 2.2 we find that equations (2.6)–(2.8) have the solution $Q'(u)=\lambda_2/\{u(1-u)\}$, but the integral in (2.20) does not exist. Thus no maximum-entropy distribution can be found by the methods of Theorems 2.2 or 3.1.

To understand why no maximum-entropy distribution can be found, consider Example 3.4, in which λ_1 is also constrained. The maximum value of the entropy is a function of the specified values of λ_1 and λ_2. We shall show that as $\lambda_1\to\infty$ with λ_2 fixed, the entropy increases monotonically and approaches a finite limit.

The maximum value of the entropy in Example 3.4 is, using (3.14),

$$\begin{aligned}\int_0^1 \log Q'(u)du &= -\int_0^1 \log(1-u)du - \int_0^1 \log(a_1+a_2u)du \\ &= 2+\frac{a_1}{a_2}\log a_1 - \frac{1}{a_2}(a_1+a_2)\log(a_1+a_2) \\ &= 2+\log\lambda_2+2\log\beta-\frac{1}{\beta}(1+\beta)\log(1+\beta)-\log\{\beta-\log(1+\beta)\}\,, \end{aligned} \tag{S.1}$$

where the last equality follows from expressing a_1 and a_2 in terms of β and λ_2, using (3.16) and (3.15). We consider the behavior of this maximized entropy as $\lambda_1\to\infty$ with λ_2 fixed, i.e. as $\beta\to\infty$ with λ_2 fixed. We write (S.1) as

$$H(\beta,\lambda_2)\equiv 2+\log\lambda_2+2\log\beta-\frac{1}{\beta}(1+\beta)\log(1+\beta)-\log\{\beta-\log(1+\beta)\}\,. \tag{S.2}$$

Differentiating, we obtain

$$\begin{aligned}\frac{\partial H}{\partial\beta} &= \frac{1}{\beta}+\frac{1}{\beta^2}\log(1+\beta)-\frac{\beta}{(1+\beta)\{\beta-\log(1+\beta)\}} \\ &= \frac{\beta^2-(1+\beta)\{\log(1+\beta)\}^2}{\beta^2(1+\beta)\{\beta-\log(1+\beta)\}}\,. \end{aligned} \tag{S.3}$$

As $\beta\to\infty$, $\log(1+\beta)=\mathrm{o}(\beta^{1/2})$, so for β sufficiently large the right side of (S.3) is positive and $H(\beta,\lambda_2)$ is an increasing function of β for fixed λ_2. The limiting value of the entropy is obtained by writing (S.2) as

$$H(\beta,\lambda_2) = 2 + \log\lambda_2 + \log\left(\frac{\beta}{1+\beta}\right) - \frac{1}{\beta}\log(1+\beta) - \log\left\{1 - \frac{1}{\beta}\log(1+\beta)\right\};$$

each of the last three terms tends to zero as $\beta \to \infty$, so the limiting value of the entropy is $2 + \log\lambda_2$. However, this limit is not attained by any distribution that has the specified value of λ_2 and a finite lower bound, so no maximum-entropy distribution exists within this class of distributions.

τ_4 for PDQ3 distribution (Example 3.7)

We have

$$\begin{aligned}
0 &= \alpha(1-\gamma\tau_3)\int_0^1 (2u-1)du \\
&= \int_0^1 (2u-1)\,.\,u(1-u)\{1-\gamma(2u-1)\}Q'(u)du \qquad \text{by (A.5)} \\
&= \int_0^1 u(1-u)\Big[2u-1-\gamma\Big\{\tfrac{1}{5}+\tfrac{4}{5}(5u^2-5u+1)\Big\}\Big]Q'(u)du \\
&= \int_0^1 u(1-u)(2u-1)Q'(u)du - \tfrac{1}{5}\gamma\int_0^1 u(1-u)Q'(u)du \\
&\qquad - \tfrac{4}{5}\gamma\int_0^1 u(1-u)(5u^2-5u+1)Q'(u)du \\
&= \lambda_3 - \tfrac{1}{5}\gamma\lambda_2 - \tfrac{4}{5}\gamma\lambda_4 \qquad \text{by (3.2)–(3.4).}
\end{aligned}$$

Dividing by λ_2, we have

$$\tau_3 - \tfrac{1}{5}\gamma - \tfrac{4}{5}\gamma\tau_4 = 0\,,$$

i.e. $\tau_4 = (5\tau_3/\gamma - 1)/4$.

Variants of Example 4.3

Example 4.3 derives the maximum-entropy distribution for a distribution on the entire real line subject to the constraint $\mathrm{E}(X_{3:4} - X_{2:4}) = \xi$. The maximum-entropy distribution has

$$Q'(u) = \xi/\{6u^2(1-u)^2\}\,, \tag{4.2}$$

and, by integration,

$$Q(u) = \tfrac{1}{6}\xi\left\{2\log\left(\frac{u}{1-u}\right) + \frac{2u-1}{u(1-u)}\right\} + c \tag{4.3}$$

where c is an undetermined constant. This quantile function corresponds to a distribution whose mean does not exist.

Two variants of this example introduce additional constraints aimed at determining the value of c in (4.3).

Example S.1. Range $(-\infty, \infty)$; constraints $\mathrm{E}(X_{3:4} - X_{2:4}) = \xi$, $\mathrm{E}\, X = \mu$.

In this variant of Example 4.3 we add a constraint on $\mathrm{E}\, X$. We obtain the same solution (4.2) for $Q'(u)$, but the integrals in (2.24) do not exist. The situation here is similar to that of Example 3.5: there are distributions that satisfy the constraints and have entropy arbitrarily close to that of the distribution (4.3), but this limit is not attained by any distribution that satisfies the constraints.

We can construct a set of distributions that contains members that approach the limit: we consider distributions with $Q'(u) = b/\{u^\alpha(1-u)^\alpha\}$. This family of distributions has been mentioned by Kamps (1991) and is related to the complementary beta distribution of Jones (2002). We consider the set of distributions with $0 < \alpha < 2$ and $b = \frac{1}{6}\,\xi\,\Gamma(6-2\alpha)/\{\Gamma(3-\alpha)\}^2$. These distributions have finite mean and satisfy $\mathrm{E}(X_{3:4} - X_{2:4}) = \xi$: we have

$$\begin{aligned}
\mathrm{E}(X_{3:4} - X_{2:4}) &= \int_0^1 6u^2(1-u)^2 Q'(u)\,du \\
&= 6b \int_0^1 u^{2-\alpha}(1-u)^{2-\alpha}\,du \\
&= 6b\,\{\Gamma(3-\alpha)\}^2/\Gamma(6-2\alpha) \\
&= \xi\,.
\end{aligned}$$

The entropy of a distribution from this set is

$$\begin{aligned}
\bar{H}(\alpha) \equiv \int_0^1 \log Q'(u)\,du &= \log b - \alpha \int_0^1 \log u\,du - \alpha \int_0^1 \log(1-u)\,du \\
&= \log b + 2\alpha \\
&= \log(\xi/6) + \log\Gamma(6-2\alpha) - 2\log\Gamma(3-\alpha) + 2\alpha\,.
\end{aligned}$$

$\bar{H}(\alpha)$ is a continuous function of $\alpha \in (0,2]$, and

$$\frac{d}{d\alpha}\bar{H}(\alpha) = -2\psi(6-2\alpha) + 2\psi(3-\alpha) + 2$$

where $\psi(x) = \frac{d}{dx}\log\Gamma(x)$ is Euler's psi function. Now for $x > 1$ we have

$$\begin{aligned}
\psi(2x) - \psi(x) &= \sum_{k=0}^{\infty}\left(\frac{1}{x+k} - \frac{1}{2x+k}\right) \qquad \text{(Gradshteyn and Ryzhik 1980, eq. 8.363.3)} \\
&= \sum_{k=0}^{\infty}\frac{x}{(x+k)(2x+k)} \\
&< \sum_{k=0}^{\infty}\frac{x}{(x+k)(x+1+k)} \\
&= 1 \qquad \text{(Gradshteyn and Ryzhik 1980, eq. 0.243.1)},
\end{aligned}$$

so for $\alpha < 2$ we have $\frac{d}{d\alpha}\bar{H}(\alpha) > 0$. Thus as $\alpha \to 2$, $\bar{H}(\alpha)$ increases towards the limiting value $\bar{H}(2) = \log(\xi/6) + 4$. However, the limiting case $\alpha = 2$ does not correspond to a distribution with finite mean.

Example S.2. Range $(-\infty, \infty)$; constraints $\mathrm{E}\, X_{2:4} = \xi_2$, $\mathrm{E}\, X_{3:4} = \xi_3$.

In this variant of Example 4.3 we specify constraints on $\mathrm{E}\, X_{2:4}$ and $\mathrm{E}\, X_{3:4}$ separately. This is equivalent to constraining $\mathrm{E}(X_{3:4} - X_{2:4})$ and $\frac{1}{2}\mathrm{E}(X_{3:4} + X_{2:4})$, the latter being a location measure that can exist even when the mean of the distribution does not.

From (1.3) the constraints are

$$\int_0^1 12u(1-u)^2 Q(u)du = \xi_2\,, \qquad \int_0^1 12u^2(1-u)Q(u)du = \xi_3\,.$$

In the notation of Theorem 2.2, both constraints have $\int_0^1 J_r(u)du \neq 0$, so we rewrite them as $\mathrm{E}(X_{3:4} - X_{2:4}) = \xi_3 - \xi_2 \equiv \xi$, $\mathrm{E}\, X_{3:4} = \xi_3$. The first constraint is now the same as in Example 4.3, and from it we obtain the same solution, (4.2), for $Q'(u)$.

To obtain $Q(u)$ we evaluate (2.24). We have $R = 2$, $g_2 = \xi_3$, $K_2(v) = \int_v^1 12u^2(1-u)du = (1-v)^2(1+2v+3v^2)$, and $Q'(u) = \frac{1}{6}(\xi_3 - \xi_2)/\{u^2(1-u)^2\}$; thus

$$
\begin{aligned}
\int_0^u \{K_2(0)-K_2(v)\}Q'(v)dv &= \int_0^u (4v^3-3v^4)\,\frac{\xi_3-\xi_2}{6v^2(1-v)^2}\,dv \\
&= \tfrac{1}{6}(\xi_3-\xi_2)\int_0^u \frac{4v-3v^2}{(1-v)^2}\,dv \\
&= \tfrac{1}{6}(\xi_3-\xi_2)\int_{1-u}^1 \frac{1+2t-3t^2}{t^2}\,dt \qquad (t=1-v) \\
&= \tfrac{1}{6}(\xi_3-\xi_2)\left[-\frac{1}{t}+2\log t-3t\right]_{1-u}^1 \\
&= \tfrac{1}{6}(\xi_3-\xi_2)\left(-1-3u-2\log(1-u)+\frac{1}{1-u}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_u^1 K_2(v)Q'(v)dv &= \int_u^1 (1-v)^2(1+2v+3v^2)\,\frac{\xi_3-\xi_2}{6v^2(1-v)^2}\,dv \\
&= \tfrac{1}{6}(\xi_3-\xi_2)\int_u^1 \frac{1+2v+3v^2}{v^2}\,dv \\
&= \tfrac{1}{6}(\xi_3-\xi_2)\left[-\frac{1}{v}+2\log v+3v\right]_u^1 \\
&= \tfrac{1}{6}(\xi_3-\xi_2)\left(2-3u-2\log(1-u)+\frac{1}{u}\right),
\end{aligned}
$$

so (2.24) gives

$$
\begin{aligned}
Q(u) &= \xi_3+\tfrac{1}{6}(\xi_3-\xi_2)\left(-3+2\log u-2\log(1-u)+\frac{1}{1-u}-\frac{1}{u}\right) \\
&= \tfrac{1}{2}(\xi_2+\xi_3)+\tfrac{1}{6}(\xi_3-\xi_2)\left\{2\log\left(\frac{u}{1-u}\right)+\frac{2u-1}{u(1-u)}\right\}.
\end{aligned}
$$

Thus the additional constraint has enabled us to determine c in (4.3). Note that, in the notation of Theorem 2.2, we have $R=2$ and $J_2(u)=12u^2(1-u)$, whence $K_2(0)-K_2(v)=v^3(4-3v)$ and $K_2(v)=(1-v)^2(1+2v+3v^2)$; these functions have high enough powers of v and $1-v$ respectively to cancel the singularities in $Q'(v)$ as $v\to 0$ or $v\to 1$, so the integrals in (2.24) exist.

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