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A New Proof of the Csima-Sawyer Theorem Concerning Ordinary Points in Arrangements

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Abstract

A new and simpler proof is provided of the 1993 Theorem of Csima and Sawyer which states that in an arrangement of n lines or pseudolines in the projective plane, not all passing through a common point, then as long as the arrangement is not the 7 line arrangement due to Kelly and Moser having just 3 ordinary points, the arrangement must have at least $\frac{6n}{13}$ ordinary points.

1 Introduction

In 1893 J.J. Sylvester posed the following famous problem [10]: Given a collection of n not all collinear points in the plane, must there be a line determined by a pair of points which does not pass through any further point? The problem was reposed by Erdős [3] in 1944 and then solved the same year, in the affirmative, by Tibor Gallai [5].

Today, the positive result to Sylvester's problem is usually referred to as Sylvester's Theorem or the Theorem of Sylvester and Gallai. Sylvester's Theorem holds equally in the Euclidean/affine and real projective planes. We write $\mathbb{RP}^2$ to denote the real projective plane.

A line which passes through precisely two points in a configuration of points is referred to as an **ordinary line**. Analogously, given an arrangement of lines, a point lying at the intersection of precisely two lines is referred to as an **ordinary point**. By projective duality, Sylvester's Theorem has an equivalent formulation in the projective plane, namely that given a collection of lines in $\mathbb{RP}^2$, not all of which pass through a common point, there must be an ordinary point. Hereafter we shall focus exclusively on this dual formulation.

Following the proof of the existence of ordinary points by Gallai, attention turned to the problem of determining the minimum number of ordinary points in arrangements of lines. In 1951 Motzkin [9] found an argument giving a lower bound of $O(\sqrt{n})$ and both he and Motzkin [9] separately conjectured that the number of ordinary points must be at least $\frac{n}{2}$. For even $n \geq 6$ there is a family of examples due to Böröczky (as cited in [1]) with n lines and precisely $\frac{n}{2}$ ordinary points. One starts with the lines determined by the edges of a regular $\frac{n}{2}$-gon and then adds the $\frac{n}{2}$ lines of symmetry. See Figure 1.

In 1957 W. Moser [8] found a clever argument to show that if n is even, then the number of ordinary points is greater than $\frac{n+11}{6}$. In 1958 Kelly and W. Moser [6] found an example of 7 lines with just 3 ordinary

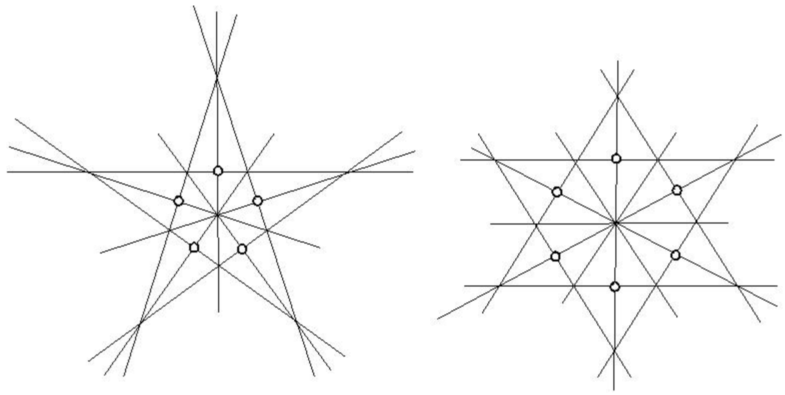

Figure 1. Böröczky's even examples with $n = 10$ and $n = 12$ lines and 6 ordinary points (hollow circles).

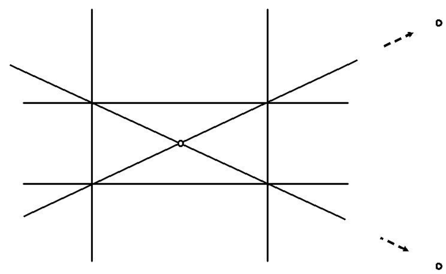

Figure 2. Kelly-Moser arrangement with 7 lines and 3 ordinary points. The seventh line and two of the ordinary points reside at infinity.

points[1] (Figure 2) and made a big push forward by bringing the lower bound on the number of ordinary points up to $\frac{3n}{7}$. In 1968 McKee [1] found the example of 13 lines with just 6 ordinary points given in Figure 3. The fact that there are just 6 ordinary points (lines) is most easily verified in the dual - where one has two regular pentagons glued together along one edge, together with the midpoint of this common edge and four points at infinity as indicated in the Figure.

In 1972 Kelly and Rottenberg [7] showed that the $\frac{3n}{7}$ bound of Kelly and Moser holds not only for arrangements of lines but for arrangements of pseudolines as well. Pseudoline arrangements are a natural generalization of the notion of arrangements of lines in that any two pseudolines cross at a single point. Pseudolines need not be straight. For a more complete treatment of pseudolines, see [4]. Finally in 1993 Csima and Sawyer [2] showed that except for the case of $n = 7$ there must be at least $\frac{6n}{13}$ ordinary points in arrangements of n lines or pseudolines.

In this paper we give a new and simpler proof of the Csima and Sawyer Theorem. The argument is free of the somewhat awkward definitions and detailed case analysis from Csima and Sawyer's now classic paper. Our essential argument (Section 3) runs just over 4 pages and is largely self-contained.

2 Preliminary Lemmas from the Literature

Our argument makes use of a definition and a number of lemmas which date back to the proof of the Kelly-Moser [6] and Kelly-Rottenberg [7] $\frac{3n}{7}$ results. Our statement of Lemma 2 parallels that in [4].

[1] It is likely that this example was known earlier.

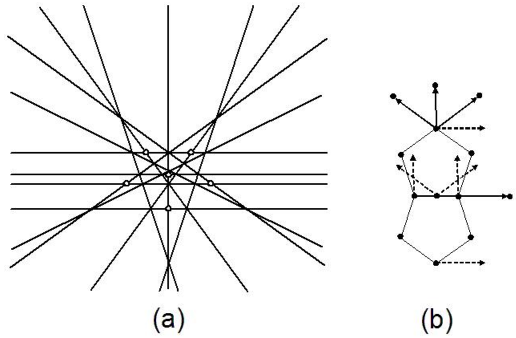

Figure 3. (a) The McKee arrangement of 13 lines and 6 ordinary points and (b) its dual configuration. In the dual configuration, the points at infinity are indicated by points at the end of solid arrows. The dashed lines are ordinary.

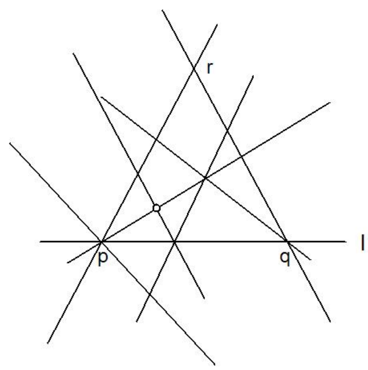

Figure 4. An example of a line l with an attached ordinary point (the hollowed circle)

Definition 1 *Say that an ordinary point p is **attached** to a (pseudo-)line l, not containing p, if l together with two (pseudo-)lines crossing at p form a (pseudo-)triangular cell of the arrangement. (See Figure 4.*

Lemma 1 ***(4-Attachment Lemma)*** *In any arrangement of (pseudo-)lines, an ordinary point can have at most* 4 *(pseudo-)lines counting that point as an attachment.*

Proof. An ordinary point is contained in 2 crossing (pseudo-)lines, and hence is a vertex of 4 faces; it can therefore be attached to at most 4 (pseudo-)lines. □

Lemma 2 *Let T be a triangle formed by three (pseudo-)lines of an arrangement $\mathcal{A}$. Let l be one of the defining (pseudo-)lines of T and $[p,q]$ the interval of intersection of l and T. If $[p,q]$ contains no ordinary points except perhaps p and (or) q, T is not a cell of the arrangement, and moreover every (pseudo-)line intersecting T also intersects $[p,q]$, then there exists an ordinary point x attached to l through some triangle T_x contained in T.*

Figure 4 illustrates the setup and conclusion of this lemma. The triangle T is the (finite) triangle $\triangle(p,q,r)$ in the Figure.

Proof. Let x be the vertex of $\mathcal{A}$ in T but not on l which has the smallest distance to l. Suppose there are three lines l_1,l_2,l_3 intersecting in x, and let v_1,v_2,v_3 be their respective intersection points with l. By assumption, all the v_i lie in $[p,q]$ and we assume v_2 lies between v_1 and v_3. Since v_2 is not ordinary there is a line $m \neq l_2$ entering T at v_2. The intersection of m and l_1 or m and l_3 is of smaller distance to l than x, a contradiction. □

Lemma 2 as stated for lines is due to Kelly and Moser [6]. The extension to pseudolines due to Kelly and Rottenberg [7] requires a reformulation of the notion of the vertex of "smallest distance" to a given pseudoline l.

Definition 2 *Suppose one is given an arrangement $\mathcal{A}$ of pseudolines and a chosen pseudoline $l \in \mathcal{A}$. Suppose v, v' are two vertices of $\mathcal{A}$ that lie on the same side of l. Write $v \prec v'$ if there is a pseudoline $k \in \mathcal{A}$ with $v, v' \in k$ and v lies between l and v' along k. In this case say that v is **closer** than v' to l. The vertex **closest** to l along k in the direction of an additional vertex v is the vertex $v' \in k$ (possibly $v' = v$) such that v, v' both lie on the same side of l and there are no additional vertices between v' and l on k.*

Lemma 3 *Given an arrangement $\mathcal{A}$ of pseudolines, and a chosen pseudoline $l \in \mathcal{A}$. Then the transitive closure of the $\prec$ relation described in Definition 2 gives rise to a partial ordering of the vertices to either side of l.*

We shall not prove this Lemma, but with it in hand, one can easily enough prove the analog of Lemma 2 by picking any closest vertex to l inside T with respect to the $\prec$ relation.

Definition 3 *A (pseudo-)line is said to be of **type** (i, j) if it contains i ordinary points and has j ordinary points attached to it.*

Using Lemma 2 and its analog for pseudolines, Kelly and Moser [6] and Kelly and Rottenberg [7] (for pseudolines) were able to establish the following:

Lemma 4 ***((0,3+) Lemma)*** *Let $\mathcal{A}$ be an arrangement of (pseudo-)lines, not all of which pass through a common point. If a (pseudo-)line $l \in \mathcal{A}$ contains no ordinary points, then there are at least 3 ordinary points attached to l.*

Lemma 5 ***((1,2+) Lemma)*** *Let $\mathcal{A}$ be an arrangement of (pseudo-)lines, not all of which pass through precisely one or the other of two points. If a (pseudo-)line $l \in \mathcal{A}$ contains a single ordinary point, then l has at least 2 ordinary points attached to it.*

Arrangements of (pseudo-)lines all of which pass through one or the other of two points will not be of interest to us since if $\mathcal{L} = \{l_i\}$ are the (pseudo-)lines passing through one of the points and $\mathcal{K} = \{k_j\}$ are the (pseudo-)lines passing through the other point, then the intersection point of any pair $l_i \cap k_j : l_i \in \mathcal{L}, k_j \in \mathcal{K}$ is ordinary, being distinct from the intersection of any other such pair. Hence such arrangements have many $(> \frac{n}{2})$ ordinary points.

Definition 4 *An arrangement of n lines is said to be **Sylvester-critical** if it has fewer than $\frac{n}{2}$ ordinary points.*

In trying to find Sylvester-critical arrangements, the following insight turns out to be pivotal (though the Theorem is not essential for what follows):

Theorem 6 *A Sylvester-critical arrangement must contain (pseudo-)lines of type* $(2,0)$.

Proof. The proof is a simple double counting argument using Lemmas 4 and 5. Suppose one had a Sylveseter-critical arrangement of n lines without lines of type $(2,0)$. Then by virtue of the $(0,3+)$ and $(1,2+)$ Lemmas, all lines are of type $(0,3+),(1,2+),(2,1+),(3,0+),(4,0+)$ and so on, where $j+$ means "j or more." Let k_0 denote the number of lines of type $(0,3+)$, k_1 the number of lines of type $(1,2+)$ and so forth. Additionally, let $3+\varepsilon_0$ denote the average number of attached points for lines of type $(0,3+)$, $2+\varepsilon_1$ denote the average number of attached points for lines of type $(1,2+)$ and so forth, where $\varepsilon_i \geq 0, \forall i \geq 0$.

Now, if we denote by K the number of ordinary points, then by virtue of the fact that we have a Sylvester-critical arrangement, we have

$$\sum_{i\geq 1} ik_i = 2K < n \tag{1}$$

since the left hand side counts each ordinary point twice, once for each line containing it. Then, since

$$\sum_{i\geq 0} k_i = n \tag{2}$$

we obtain

$$k_0 > k_2 + \sum_{i\geq 3}(i-1)k_i. \tag{3}$$

On the other hand, if we let M denote the total number of ordinary point-line attachments, and we count the second coordinates of the (i,j) tuples, then by virtue of the 4-Attachment Lemma we have

$$M = (3+\varepsilon_0)k_0 + (2+\varepsilon_1)k_1 + (1+\varepsilon_2)k_2 + \sum_{i\geq 3}\varepsilon_i k_i \leq 4K = 2\sum_{i\geq 1} ik_i \tag{4}$$

so

$$3k_0 + 2k_1 + k_2 \leq 2\sum_{i\geq 1} ik_i. \tag{5}$$

Thus

$$3k_0 \leq 3k_2 + 2\sum_{i\geq 3} ik_i \tag{6}$$

or

$$k_0 \leq k_2 + \sum_{i\geq 3}\frac{2}{3} ik_i \tag{7}$$

which contradicts Equation 3 since $i-1 \geq \frac{2}{3}i$ for $i \geq 3$. □

3 The New Argument

With these preliminary lemmas in hand, we turn to our original contributions.

Definition 5 *An ordinary point p is* ***saturated in attachments*** *if exactly* 4 *(pseudo-)lines count p as an attached point. We often just say that a point or arrangement is* ***saturated****, meaning that the point or arrangement is saturated in attachments.*

As earlier, we assume that all (pseudo-)lines contain at least 3 vertices, so e.g. we rule out the pathological case of three lines in general position and the question of whether or not that constitutes a saturated arrangement.

The following is obvious from the definition:

Lemma 7 *An ordinary point is saturated iff it is surrounded by triangular cells of the arrangement.*

Lemma 8 *The ordinary points on a line of type* $(2,0)$ *are necessarily saturated.*

Proof. Let l be a (pseudo-)line of type $(2,0)$ in an arrangement $\mathcal{A}$ and let p be one of the two ordinary points on l. Let k be the second (pseudo-)line through p and let p' be the nearest vertex to p along k on one side of l. Consider a second (pseudo-)line k' through p' as in Figure 5. By Lemma 2, since l has just 2 ordinary points and no attached points, either triangle T_1 or T_2 is a cell. Without loss of generality, suppose it is T_1. p' cannot be ordinary since otherwise it would be attached to l through T_1. Hence let k'' be a third

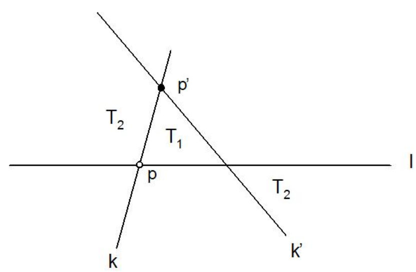

Figure 5. A line l of type $(2,0)$ with an ordinary point p. k is the second line through p and p' is the closest vertex to p in one direction.

(pseudo-)line through p' which intersects l at a vertex q. See Figure 6. Applying Lemma 2 to the $\triangle(p,p',q)$

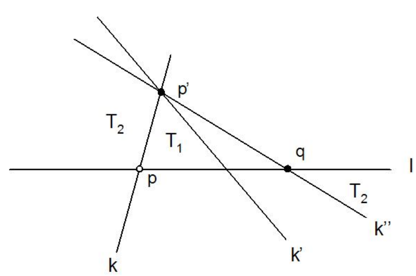

Figure 6. Having concluded that T_1 is a cell, and so p' cannot be ordinary we draw a third line k'' through p' intersecting l at the vertex q.

which contains T_1 we conclude that the segment (p,q) on this side must contain an ordinary point. On the other hand, the segment (p,q) on the other side, cannot contain an ordinary point, so the triangle $\triangle(p,p',q)$ which is labelled T_2 in the Figure (but which is smaller than the T_2 from the prior figure) must be a cell.

Thus p is surrounded by triangles "above" l. Considering the closest vertex to p along k "below" l allows one to similarly conclude that p is surrounded by triangles from below as well, and hence p is saturated. □

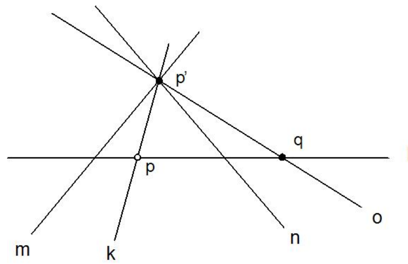

Figure 7. An ordinary point p on the $(2,0)$ line l. k is the second line through p and p' is a neighboring vertex to p on one side of l. m,n are the two lines through p' counting p as an attachment and o is a hypothetical fourth line through p'.

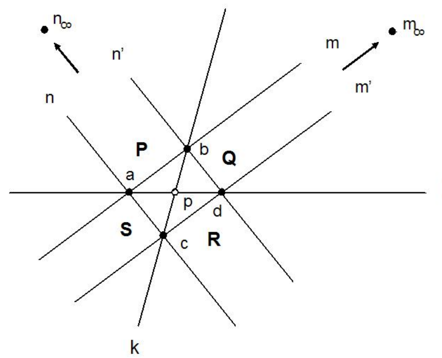

Figure 8. Two lines l,k of type $(2,0)$ meeting at an ordinary point p. By Lemma 8 p is saturated and using a projective transformation we may assume that lines m,m' and lines n,n' are parallel pairs, meeting respectively at m_∞, n_∞.

Lemma 9 *Let p be an ordinary point on (pseudo-)line l of type* $(2,0)$ *and let k be the other (pseudo-)line through p. Then the closest vertex to p along k (in either direction) is necessarily a* 3*-crossing.*

Proof. Let p' be a closest vertex to p along k as in the statement of the Lemma. Since p is saturated, let m and n be the additional (pseudo-)lines through p' counting p as an attachment, as depicted in Figure 7, and let o be a hypothetical fourth line through p' intersecting l at a vertex q. Consider the two triangles $\triangle(p,p',q)$ which have the finite $[p,p']$ as a common edge. Neither is a cell, so by Lemma 2, l must contain an ordinary point both in the finite and infinite segments (p,q). However, l has just 2 ordinary points, a contradiction. The Lemma follows. □

The key lemma which Csima and Sawyer [2] added to the mix was the following:

Lemma 10 ***(Csima-Sawyer, 1993)*** *Let $\mathcal{A}$ be an arrangement of n (pseudo-)lines such that two (pseudo-)lines of type* $(2,0)$ *intersect in an ordinary point. Then $\mathcal{A}$ is the Kelly-Moser arrangement (Figure 2).*

Thanks to Lemma 9 we are in a position to give a much simpler and shorter proof than that given by Csima and Sawyer.

Proof. Suppose we have two $(2,0)$ (pseudo-)lines l,k intersecting at a common ordinary point p. By the $(2,0)$-saturation lemma (Lemma 8), p is saturated. By Lemma 9 all neighboring vertices to p are 3-crossings. See Figure 8. By use of a projective transformation, we may suppose $m \| m', n \| n'$ (which for pseudolines just means that these pairs each cross at points on the line at infinity). Let us identify by P the cell adjacent to $\triangle(a,b,p)$ along $[a,b]$, and analogously identify cells Q,R and S. Since the vertices a,b,c,d

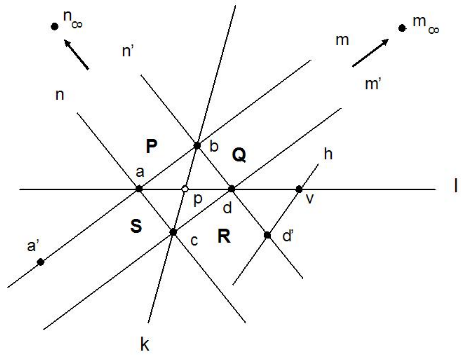

Figure 9. Two lines l,k of type $(2,0)$ meeting at the saturated ordinary point p. We consider the case where there are necessarily two adjacent $(4+)$-gons amongst P,Q,R,S, e.g. R,S in the cyclical clockwise ordering. Since R is a $(4+)$-gon there is a closest vertex d' adjacent to d along $[d,n_\infty]$ and a line meeting d' which forms an edge of R next to $[d,d']$. We call this line h. Analogously, since S is a $(4+)$-gon there is a closest vertex a' adjacent to a along $[a,m_\infty]$ and a line (not drawn) meeting a' which forms an edge of S next to $[a,a']$. We call the second line i.

are all 3-crossings either some two consecutive cells amongst P,Q,R,S in the clockwise ordering are 4-or-more-gons (henceforth written $(4+)$-gons) or two opposite cells amongst P,Q,R,S - say Q,S - are actually triangles with a common vertex at, in this case, m_∞. However, if Q,S are triangular cells with a common vertex at m_∞, then there can be no additional vertices along m other than a,b,m_∞ while all crossings at a,b have been identified. It follows that all additional (pseudo-)lines pass through m_∞ and so are either the line at infinity or (pseudo-)lines parallel to m.

However, (pseudo-)lines parallel to m can't pass through Q and if such (pseudo-)lines intersected R or P, they would each create additional ordinary points on k - hence there can be at most one such (pseudo-)line. But if the one (pseudo-)line passed through R say, it would create an ordinary point on n attached to k, which is impossible since k cannot have attached points. It follows that there is at most one additional (pseudo-)line - the line at infinity. Indeed, since we have accounted for only one of the ordinary points on l,k there *must* be this additional (pseudo-)line - giving us the Kelly-Moser arrangement.

The only remaining case is that there are two adjacent $(4+)$-gons amongst P,Q,R,S, e.g. R,S. There is then a first vertex d' adjacent to d along $[d,n_\infty]$ and a (pseudo-)line meeting d' which forms an edge of R next to $[d,d']$. Call this (pseudo-)line h and suppose h meets l at a vertex v. See Figure 9. Since d' cannot be attached to l, and a (pseudo-)line cannot cut through R since it is a cell, there is a (pseudo-)line through $[d',v)$ that cuts through the interior of $\triangle(d,d',v)$ and so we may apply Lemma 2 to conclude that l has an ordinary point in (d,v). Now consider the $(4+)$-gon S. There is a first vertex a' adjacent to a along $[a,m_\infty]$ and a (pseudo-)line meeting a' which forms an edge of S next to $[a,a']$ - call this line i. By virtue of the fact that i cannot pass through R, and hence through (d,d') and h cannot pass through S, and hence through (a,a'), i must intersect l outside (c,v) in a point v'. Otherwise, unless $i=h$ and $v=v'$ (which is possible), the (pseudo-)lines i and h would cross twice. Again the point a' cannot be attached so there is a (pseudo-)line through $[a',v')$ that cuts the interior of $\triangle(a,a',v')$ and so we may again apply Lemma 2 to conclude that l has a third ordinary point in (a,v'), contradicting the fact that l is a $(2,0)$ line. It follows that our arrangement is the Kelly-Moser. □

And finally:

Theorem 11 ***(Csima-Sawyer, 1993)*** *An arrangement of n (pseudo-)lines not all passing through a common point which is not graph theoretically the Kelly-Moser arrangement (Figure 2) must contain at least* $6n/13$

ordinary points.

Proof. We argue very much as in the proof of Theorem 6 and adopt the same notation, with the addition that we use $k_{2,0}$ to denote the number of (pseudo-)lines of type $(2,0)$ and $k_{2,1}$ to denote the number of (pseudo-)lines with 2 ordinary points and at least 1 attachment. Further we let $1+\varepsilon_{2,1}$ denote the average number of attachments of (pseudo-)lines with 2 ordinary points and at least 1 attachment.

Since each (pseudo-)line of type $(2,0)$ intersects two (pseudo-)lines of type other than $(2,0)$ in its two ordinary points we have

$$2k_{2,0} \leq k_1 + 2k_{2,1} + \sum_{i\geq 3} ik_i. \tag{8}$$

The left hand side of Equation 8 counts the number of ordinary points on (pseudo-)lines of type $(2,0)$ and the right hand side counts the number of ordinary points on all other (pseudo-)lines.

We argue by contradiction and assume we have an arrangement $\mathcal{A}$ with fewer than $6n/13$ ordinary points, such that no two lines of type $(2,0)$ intersect in an ordinary point. Then

$$k_1 + 2k_{2,0} + 2k_{2,1} + \sum_{i\geq 3} ik_i < \frac{12n}{13}, \tag{9}$$

and since

$$k_0 + k_1 + k_{2,0} + k_{2,1} + \sum_{i\geq 3} k_i \quad = \quad n, \tag{10}$$

$$k_0 + k_1 + k_{2,0} + k_{2,1} + \sum_{i\geq 3} k_i \quad > \quad \frac{13}{12}(k_1 + 2k_{2,0} + 2k_{2,1} + \sum_{i\geq 3} ik_i) \tag{11}$$

we have

$$k_0 > \frac{1}{12}k_1 + \frac{7}{6}k_{2,0} + \frac{7}{6}k_{2,1} + \sum_{i\geq 3}(\frac{13}{12}i - 1)k_i. \tag{12}$$

On the other hand, counting attachments, and applying the 4-Attachment Lemma, we obtain

$$(3+\varepsilon_0)k_0 + (2+\varepsilon_1)k_1 + (1+\varepsilon_{2,1})k_{2,1} + \sum_{i\geq 3} \varepsilon_i k_i \leq 2k_1 + 4k_{2,0} + 4k_{2,1} + \sum_{igeq3} 2ik_i \tag{13}$$

Since $\varepsilon_i \geq 0, \varepsilon_{2,1} \geq 0$ the above equation remains true of we set $\varepsilon_i = 0, \varepsilon_{2,1} = 0$, which we may rewrite in the form

$$k_0 \leq \frac{4}{3}k_{2,0} + k_{2,1} + \sum_{i\geq 3} \frac{2i}{3}k_i \tag{14}$$

Now, rewriting equation 8 in terms of k_1 gives

$$k_1 \geq 2k_{2,0} - 2k_{2,1} - \sum_{i\geq 3} ik_i \tag{15}$$

which when substituted into 12 gives:

$$k_0 > \frac{1}{12}(2k_{2,0} - 2k_{2,1} - \sum_{i\geq 3} ik_i) + \frac{7}{6}k_{2,0} + \frac{7}{6}k_{2,1} + \sum_{i\geq 3}(\frac{13}{12}i - 1)k_i, \tag{16}$$

or

$$k_0 > \frac{4}{3}k_{2,0} + k_{2,1} + \sum_{i\geq 3}(i-1)k_i. \tag{17}$$

Since $i-1 \geq \frac{2i}{3}$ for $i \geq 3$, equations 14 and 17 cannot simultaneously hold and the Theorem follows. $\square$

4 Concluding Remarks

This result is part of a larger body of work undertaken by the author to first establish that the McKee arrangement is the unique arrangement with $\frac{6n}{13}$ ordinary points and to hopefully, finally, close the gap on the $\frac{n}{2}$ conjecture.

I acknowledge Hervé Brönnimann for many valuable conversations.

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