RC24201 (W0703-003) March 1, 2007
Mathematics

IBM Research Report

On a Generalization of the Master Cyclic Group Polyhedron

Sanjeeb Dash, Ricardo Fukasawa*, Oktay Günlük
IBM Research Division
Thomas J. Watson Research Center
P.O. Box 218
Yorktown Heights, NY 10598

*Georgia Institute of Technology

On a Generalization of the Master Cyclic Group Polyhedron

Sanjeeb Dash
IBM Research

Ricardo Fukasawa
Georgia Inst. Tech.

Oktay Günlük
IBM Research

February 28, 2007

Abstract

We study the Master Equality Polyhedron (MEP) which generalizes the Master Cyclic Group Polyhedron and the Master Knapsack Polyhedron.

We present an explicit characterization of the polar of the nontrivial facet-defining inequalities for the MEP. This result generalizes similar results for the Master Cyclic Group Polyhedron by Gomory [9] and for the Master Knapsack Polyhedron by Araoz [1]. Furthermore, this characterization also gives a polynomial time algorithm for separating an arbitrary point from the MEP.

We describe how facet defining inequalities for the Master Cyclic Group Polyhedron can be lifted to obtain facet defining inequalities for the MEP, and also present facet defining inequalities for the MEP that cannot be obtained in such a way. Finally, we study the mixed-integer extension of the MEP and present an interpolation theorem that produces valid inequalities for general mixed integer programming problems using facets of the MEP.

1 Introduction

We study the Master Equality Polyhedron (MEP), which we define as:

$$K(n,r) = conv\left\{(x,y) \in \mathbb{Z}^n_+ \times \mathbb{Z}^n_+ : \sum_{i=1}^{n} ix_i - \sum_{i=1}^{n} iy_i = r\right\} \tag{1}$$

where $n, r \in \mathbb{Z}$ and $n > 0$. Without loss of generality we assume that $r \geq 0$. To the best of our knowledge, $K(n,r)$ was first defined by Uchoa [14] in a slightly different form and described as an important object for study.

Two well-known families of polyhedra can be viewed as forming lower dimensional faces of the MEP: the Master Cyclic Group Polyhedron (MCGP), which is defined as

$$P(n,r) = conv\left\{(x,y) \in \mathbb{Z}^{n-1}_+ \times \mathbb{Z}_+ : \sum_{i=1}^{n-1} ix_i - ny_n = r\right\}, \tag{2}$$

where $r, n \in \mathbb{Z}$, and $0 \leq r < n$; and the Master Knapsack Polyhedron (MKP), which is defined as

$$K(r) = conv\left\{x \in \mathbb{Z}^r_+ : \sum_{i=1}^{r} ix_i = r\right\}, \tag{3}$$

where $r \in \mathbb{Z}$ and $r > 0$.

Facets of $P(n,r)$ are a useful source of cutting planes for general MIPs. The Gomory mixed-integer cut (also known as the mixed-integer rounding (MIR) inequality) can be derived from a facet of $P(n,r)$ [10]. For work on other properties and facets of the Master Cyclic Group Polyhedron, see [2, 4, 5, 6, 7, 8, 11, 12, 13]. In particular, several relationships between facet-defining inequalities of the MCGP and facet-defining inequalities of the MKP were established in [2]. The Master Cyclic Group Polyhedron is usually presented as

$$P'(n,r) \;=\; conv\left\{x \in \mathbb{Z}_+^{n-1} : \sum_{i=1}^{n-1} i x_i \equiv r \mod n\right\}$$

which is the projection of $P(n,r)$ in the space of x variables. We use (2) as it makes the comparison to $K(n,r)$ easier and clearer.

Gomory [9] and Araoz [1] give an explicit characterization of the polar of the nontrivial facets of $P(n,r)$ and $K(r)$. In this paper, we give a similar description of the nontrivial facets of $K(n,r)$, yielding as a consequence a polynomial time algorithm to separate over it. We also analyze some structural properties of the MEP and relate it to the MCGP.

In addition, we describe how to obtain valid inequalities for general MIPs using facet defining inequalities for the MEP. Another motivation to study the MEP is that it also arises as a natural structure in a reformulation of the Fixed-Charge Network Flow problem, which has recently been used in [15] to derive strong cuts for the Capacitated Minimum Spanning Tree Problem and can also be used in other problems such as the Capacitated Vehicle Routing Problem.

In the next section, we present our characterization of the polar of the nontrivial facets of $K(n,r)$, for any $n > 0$ and any r satisfying $0 < r \leq n$. In Section 3, we study $K(n,r)$ when $r = 0$. Based on the results of these sections, we describe how to separate a point from $K(n,r)$ for arbitrary r (including the case $r > n$) in Section 4. In Section 5 and Section 6, we discuss how some, but not all, of the facets of $K(n,r)$ can be obtained by lifting facets of $P(n,r)$. In Section 7, we follow the approach of Gomory and Johnson [10] and derive valid inequalities for mixed-integer programs from facets of $K(n,r)$ via *interpolation*. We conclude in Section 8 with some remarks on directions for further research on $K(n,r)$, along the lines of the work of Gomory and Johnson [10, 11] on $P(n,r)$.

2 Polyhedral Analysis of $K(n,r)$

Throughout this section, we assume $0 < r \leq n$. The cases $r = 0$ and $r > n$ are studied in Sections 3 and 4. We start with some notation and some basic polyhedral properties of $K(n,r)$.

Let $e_i \in \mathbb{R}^{2n}$ be the unit vector with a one in the component corresponding to x_i and let $f_i \in \mathbb{R}^{2n}$ be the unit vector with a one in the component corresponding to y_i, for $i = 1, \ldots, n$.

Lemma 2.1 *$dim(K(n,r)) = 2n - 1$.*

Proof. Clearly $dim(K(n,r)) \leq 2n-1$ as all points in $K(n,r)$ satisfy $\sum_{i=1}^{n} ix_i - \sum_{i=1}^{n} iy_i = r$. Let U be the set of $2n$ points $p_1 = re_1$, $p_i = re_1 + e_i + if_1$ for $i = 2,\ldots,n$, and $q_i = (r+i)e_1 + f_i$ for $i = 1,\ldots,n$. U is an affinely independent set, as $\{u - p_1 : u \in U, u \neq p_1\}$ is a linearly independent set. As $U \subseteq K(n,r)$, $dim(K(n,r)) \geq 2n-1$. ∎

Lemma 2.2 *The nonnegativity constraints of $K(n,r)$ are facet-defining if $n \geq 2$.*

Proof. Let U be defined as in the proof of Lemma 2.1. For any $i \neq 1$, the sets $U \setminus \{p_i\}$ and $U \setminus \{q_i\}$ are affinely independent, and satisfy $x_i = 0$ and $y_i = 0$, respectively. Therefore $x_i \geq 0$ and $y_i \geq 0$ define facets of $K(n,r)$ for $i \geq 2$. To see that $y_1 \geq 0$ is facet defining, replace p_i $(2 \leq i \leq n)$ in U by $p'_i = re_1 + ne_i + if_n$ to get the affinely independent set $U' \subseteq K(n,r)$. All points in U' other than q_1 satisfy $y_1 = 0$. Finally, let V be the set of points $t_0 = e_n + (n-r)f_1$, $t_i = t_0 + ie_n + nf_i$ for $i = 1,\ldots,n$, and $s_i = t_0 + e_i + if_1$ for $i = 2,\ldots,n-1$. V is contained in $K(n,r)$, and is affinely independent as $\{v - t_0 : v \in V, v \neq t_0\}$ is linearly independent. The points in V also satisfy $x_1 = 0$. ∎

Clearly, $K(n,r)$ is an unbounded polyhedron. We next characterize all the extreme rays (unbounded one-dimensional faces) of $K(n,r)$. We represent an extreme ray $\{u+\lambda v : u, v \in \mathbb{R}^{2n}_{+}, \lambda \geq 0\}$ of $K(n,r)$ simply by the vector v. Let $r_{ij} = je_i + if_j$ for any $i,j \in \{1,\ldots,n\}$.

Lemma 2.3 *The set of extreme rays of $K(n,r)$ is given by $R = \{r_{ij} : 1 \leq i,j \leq n\}$.*

Proof. Let (c,d) be a ray of $K(n,r)$, that is, $\sum_{i=1}^{n} ic_i - \sum_{j=1}^{n} jd_j = 0$. We will show, by induction on the number of nonzero components of (c,d), that (c,d) can be written as a conic combination of rays in R.

First, we can assume that $(c,d) \neq 0$. In such a case, $c \neq 0$ and $d \neq 0$. Therefore, (c,d) has at least two nonzero components. If (c,d) has exactly two nonzero components, we have $ic_i = jd_j$ for some i,j and therefore, $(c,d) = \delta_{ij} r_{ij}$ where $\delta_{ij} = c_i/j$. On the other hand, if (c,d) has more than two nonzero components, pick any i,j such that $c_i, d_j > 0$ and let $\delta_{ij} = \min\{c_i/j, d_j/i\}$. Notice that $(c',d') = (c,d) - \delta_{ij} r_{ij}$ satisfies $\sum_{i=1}^{n} ic'_i - \sum_{j=1}^{n} jd'_j = 0$ and has fewer nonzero components than (c,d). By induction, (c',d') can be written as a conic combination of rays in R, and therefore, so can (c,d).

We have shown that R contains all extreme rays of $K(n,r)$. To complete the proof, it suffices to notice that a conic combination of 2 or more rays in R gives a ray with at least 3 nonzero entries and therefore a ray in R cannot be written as a conic combination of other rays in R. ∎

As $K(n,r)$ is not a full-dimensional polyhedron, any valid inequality $\pi x + \rho y \geq \pi_o$ for $K(n,r)$ has an equivalent representation with $\rho_n = 0$. If a valid inequality does not satisfy this condition, one can add an appropriate multiple of the equation $\sum_{i=1}^{n} ix_i - \sum_{i=1}^{n} iy_i = r$ to it. We state this formally in Observation 2.4, and subsequently assume all valid inequalities have $\rho_n = 0$.

Observation 2.4 *If $\pi x + \rho y \geq \pi_o$ defines a valid inequality for $K(n,r)$, we can assume $\rho_n = 0$ without loss of generality.*

We classify the facets of $K(n,r)$ as *trivial* and *non-trivial* facets.

Definition 2.5 *The following facet-defining inequalities of $K(n,r)$ are called trivial:*

$$x_i \geq 0, \forall i = 1,\dots,n$$

$$y_i \geq 0, \forall i = 1,\dots,n-1$$

All other facet-defining inequalities of $K(n,r)$ are called nontrivial.

According to this definition, the inequality $y_n \geq 0$ defines a non-trivial facet. With this distinction between $y_n \geq 0$ and the other trivial facets, our results are easier to state and prove. There is nothing special about the $y_n \geq 0$ inequality except that it is the only nonnegativity constraint that does not comply directly with the $\rho_n = 0$ assumption. A consequence of the above assumptions and definitions is: if $\pi x + \rho y \geq \pi_o$ defines a non-trivial facet of $K(n,r)$, then for $i = 1,\dots,n$, there exists an integral point χ^i in $K(n,r)$ lying on the facet such that $e_i^T \chi^i > 0$. A similar property holds for the components corresponding to $y_j, j = 1,\dots,n-1$.

2.1 Characterization of the non-trivial facets

Let $N = \{1,\dots,n\}$. We next state our main result:

Theorem 2.6 *The inequality $\pi x + \rho y \geq \pi_o$ defines a nontrivial facet of $K(n,r)$ if and only if it can be represented as an extreme point of $T \subseteq \mathbb{R}^{2n+1}$ where T is defined by the following linear equations and inequalities:*

$$\begin{aligned}
\pi_i + \rho_j &\geq \pi_{i-j}, && \forall i,j \in N, && i > j, && \text{(SA1)}\\
\pi_i + \pi_j &\geq \pi_{i+j}, && \forall i,j \in N, && i+j \leq n, && \text{(SA2)}\\
\rho_k + \pi_i + \pi_j &\geq \pi_{i+j-k}, && \forall i,j,k \in N, && 1 \leq i+j-k \leq n, && \text{(SA3)}\\
\pi_i + \pi_{r-i} &= \pi_o, && \forall i \in N, && i < r, && \text{(EP1)}\\
\pi_r &= \pi_o, && && && \text{(EP2)}\\
\pi_i + \rho_{i-r} &= \pi_o, && \forall i \in N && i > r, && \text{(EP3)}\\
\rho_n &= 0, && && && \text{(NC1)}\\
\pi_o &= 1. && && && \text{(NC2)}
\end{aligned}$$

We call constraints (SA1)-(SA3) *relaxed subadditivity* conditions as they are implied by the following pair-wise subadditivity conditions on the facet coefficients:

$$\begin{aligned}
\pi_i + \rho_j &\geq \pi_{i-j}, && \forall i,j \in N, && i > j, && \text{(SA1)}\\
\pi_i + \rho_j &\geq \rho_{j-i}, && \forall i,j \in N, && i < j, && \text{(SA1')}\\
\pi_i + \pi_j &\geq \pi_{i+j}, && \forall i,j \in N, && i+j \leq n, && \text{(SA2)}\\
\rho_i + \rho_j &\geq \rho_{i+j}, && \forall i,j \in N, && i+j \leq n. && \text{(SA2')}
\end{aligned}$$

As we show later, any non-trivial facet defining inequality $\pi x + \rho y \geq \pi_o$ for $K(n,r)$ satisfies (SA1)-(SA3) as well as (SA1') and (SA2'). We think it would be more natural to have a description of the coefficient polyhedron T that uses pairwise subadditivity conditions instead of the relaxed subadditivity conditions but we were not able to derive such a description.

The equations (EP1)-(EP3) essentially state that the following $n - \lfloor \frac{r-1}{2} \rfloor$ points, which we call the *elementary points* of $K(n,r)$,

$$\{e_i + e_{r-i} \,:\, 1 \leq i < r\} \cup \; e_r \; \cup \{e_i + f_{i-r}, \,:\, r < i \leq n\}$$

lie on every non-trivial facet of $K(n,r)$. In other words, $K(n,r)$ has a low dimensional face where all non-trivial facets intersect. Note that the dimension of this face is at least $n - \lfloor \frac{r-1}{2} \rfloor - 1$ as the elementary points are affinely independent.

The last two constraints (NC1) and (NC2) are normalization constraints that are necessary to have a unique representation of nontrivial facets.

Note that the definition of T in Theorem 2.6 is similar to that of a polar. However, T is not the polar of $K(n,r)$, as it does not contain extreme points of the polar that correspond to the trivial inequalities. In addition, some of the extreme rays of the polar are not present in T. It is possible to interpret T as an important subset of the polar that contains all extreme points of the polar besides the ones that lead to the trivial inequalities. In the rest of this section we develop the required analysis to prove Theorem 2.6.

2.2 Basic Properties of T

We start with a basic observation which states that any valid inequality for $K(n,r)$ has to be valid for its extreme rays and elementary points.

Observation 2.7 *Let $\pi x + \rho y \geq \pi_o$ be a valid inequality for $K(n,r)$, then the following holds:*

$$j\pi_i + i\rho_j \geq 0, \forall i,j \in N \tag{R1}$$

$$\pi_i + \pi_{r-i} \geq \pi_o, \forall i \in N, i < r, \tag{P1}$$

$$\pi_r \geq \pi_o, \tag{P2}$$

$$\pi_i + \rho_{i-r} \geq \pi_o, \forall i \in N, i > r. \tag{P3}$$

We next show that nontrivial facet-defining inequalities satisfy the relaxed subadditivity conditions and they are tight at the elementary points of $K(n,r)$.

Lemma 2.8 *Let $\pi x + \rho y \geq \pi_o$ be a nontrivial facet-defining inequality of $K(n,r)$, then it satisfies* (SA1)-(SA3) *as well as* (SA1'), (SA2') *and* (EP1)-(EP3).

Proof. (SA1): Let (x^*, y^*) be an integral point in $K(n,r)$ lying on the facet defined by $\pi x + \rho y \geq \pi_o$ such that $x^*_{i-j} > 0$. Then $(x^*, y^*) + (e_i + f_j - e_{i-j})$ is contained in $K(n,r)$. Therefore, (SA1) holds.

The proofs of (SA2), (SA3), (SA1') and (SA2') are analogous.

(EP1): Let (x', y') and (x'', y'') be integral points in $K(n, r)$ lying on the facet defined by $\pi x + \rho y \geq \pi_o$ such that $x'_i > 0$ and $x''_{r-i} > 0$. Then $(\bar{x}, \bar{y}) = (x', y') + (x'', y'') - e_i - e_{r-i} \in K(n, r)$. Therefore

$$\pi \bar{x} + \rho \bar{y} = \pi x' + \rho y' + \pi x'' + \rho y'' - \pi_i - \pi_{r-i} = 2\pi_o - \pi_i - \pi_{r-i} \geq \pi_o.$$

The last inequality above implies that $\pi_i + \pi_{r-i} \leq \pi_o$ and therefore (P1) $\Rightarrow$ (EP1).

The proofs of (EP2) and (EP3) are analogous, using (P2) and (P3) instead of (P1). ■

We next show that the normalization condition (NC2) does not eliminate any nontrivial facets.

Lemma 2.9 *Let $\pi x + \rho y \geq \pi_o$ be a nontrivial facet-defining inequality of $K(n, r)$, that satisfies $\rho_n = 0$. Then $\pi_o > 0$.*

Proof. By (R1), we have, for all $i \in N$, $n\pi_i + i\rho_n \geq 0$ and therefore $\pi_i \geq 0$. Also by (EP2), we have $\pi_o = \pi_r$ which implies that $\pi_o \geq 0$.

Assume $\pi_o = 0$. As $\pi \geq 0$, using (EP1) we have $\pi_i = 0$ for $i = 1, \ldots, r$. But then, (SA2) implies that

$$0 + \pi_{i-1} \geq \pi_i \geq 0, \text{ for } i = 2, \ldots, n.$$

Starting with $i = r + 1$, we can inductively show that $\pi_i = 0$ for all $i \in N$. This also implies that $\rho_k = 0$ for $1 \leq k \leq n - r$ by (EP3). In addition $\rho_k \geq 0$ for $n - r + 1 \leq k \leq n$ by (SA3).

Therefore, if $\pi_o = 0$, then $\pi = 0$, $\rho \geq 0$ and therefore $\pi x + \rho y \geq 0$ can be written as a conic combination of the nonnegativity facets, which is a contradiction. Thus $\pi_o > 0$. ■

Combining Lemmas 2.8 and 2.9 we have therefore established the following.

Corollary 2.10 *Let $\pi x + \rho y \geq \pi_o$ be a nontrivial facet-defining inequality of $K(n, r)$, that satisfies $\rho_n = 0$. Then $\frac{1}{\pi_o}(\pi, \rho, \pi_o) \in T$.*

In the following result, we show that a subset of the conditions presented in Theorem 2.6 suffices to ensure validity of an inequality for $K(n, r)$.

Lemma 2.11 *Let (π, ρ, π_o) satisfy* (EP2), (SA1), (SA2) *and* (SA3). *Then $\pi x + \rho y \geq \pi_o$ defines a valid inequality for $K(n, r)$.*

Proof. We will prove this by contradiction. Assume that $\pi x + \rho y \geq \pi_o$ satisfies (EP2), (SA1), (SA2) and (SA3) but $\pi x + \rho y \geq \pi_o$ does not define a valid inequality for $K(n, r)$, $r > 0$. Let (x^*, y^*) be an integer point in $K(n, r)$ that has minimum L_1 norm amongst all points violated by $\pi x + \rho y \geq \pi_o$.

If $||(x^*, y^*)||_1 = 0$ then $(x^*, y^*) = 0 \notin K(n, r)$. If $||(x^*, y^*)||_1 = 1$ then clearly $x^* = e_r$ and $y^* = 0$ but as $\pi_r = \pi_o$, (x^*, y^*) does not violate the inequality. Therefore $||(x^*, y^*)||_1 \geq 2$. We next consider three cases.

Case 1: Assume that $y^* = 0$. Then $\sum_{i=1}^n i x_i^* = r$. By successively applying (SA2), we obtain

$$\pi_o > \sum_{i=1}^n \pi_i x_i^* \geq \sum_{i=1}^n \pi_{i x_i^*} \geq \pi_{\sum_{i=1}^n i x_i^*} = \pi_r$$

which contradicts (EP2). Therefore $y^* \neq 0$.

Case 2: Assume that $x_i^* > 0$ and $y_j^* > 0$ for some $i > j$. Let $(x', y') = (x^*, y^*) + (e_{i-j} - e_i - f_j)$. Clearly, $(x', y') \in K(n, r)$, and $||(x', y')||_1 = ||(x^*, y^*)||_1 - 1$. Moreover, as $\pi x + \rho y \geq \pi_o$ satisfies (SA1), $\pi x' + \rho y' = \pi x^* + \rho y^* + \pi_{i-j} - \pi_i - \rho_j \leq \pi x^* + \rho y^* < \pi_o$, which contradicts the choice of (x^*, y^*). Therefore $i \leq j$ whenever $x_i^* > 0$ and $y_j^* > 0$.

Case 3: Assume that for any $i, j \in N$, if $x_i^* > 0$ and $y_j^* > 0$, then $i \leq j$. Assume that either there exists $i, j \in N$ such that $x_i^* > 0$, $x_j^* > 0$ or there exists $i \in N$ such that $x_i^* \geq 2$ (in which case, we let $j = i$). If $i + j \leq n$, let $(x', y') = (x^*, y^*) + (e_{i+j} - e_i - e_j)$. If $i + j > n$, as $y^* \neq 0$, there exists k such that $y_k^* > 0$ and $k \geq i$, and therefore $i + j - k \leq n$. Then let $(x', y') = (x^*, y^*) + (e_{i+j-k} - e_i - e_j - f_k)$. In either case, $(x', y') \in K(n, r)$ and $||(x', y')||_1 < ||(x^*, y^*)||_1$. Moreover, as (π, ρ, π_o) satisfy (SA2) and (SA3), in either case $\pi x' + \rho y' \leq \pi x^* + \rho y^* < \pi_o$, which contradicts the choice of (x^*, y^*). ∎

Corollary 2.12 *Let $(\pi, \rho, \pi_o) \in T$, then $\pi x + \rho y \geq \pi_o$ is a valid inequality for $K(n, r)$.*

Remark 2.13 *The proof technique in Lemma 2.11 can be used to show that the pair-wise subadditivity conditions and the condition $\pi_r = \pi_o$ imply $\pi x + \rho y \geq \pi_o$ is a valid inequality for $K(n, r)$. More precisely, an inequality satisfying (SA1), (SA2), (SA1') and $\pi_r \geq \pi_o$ is valid for $K(n, r)$.*

We next determine the extreme rays of T.

Lemma 2.14 *The extreme rays of T are $(f_k, 0) \in \mathbb{R}^{2n+1}$ for $n - r < k < n$*

Proof. First note that $(f_k, 0)$ is indeed an extreme ray of T for $n - r < k < n$.

Let (π, ρ, π_o) be an extreme ray of T that is not equivalent to f_k for some $n - r < k < n$. Clearly $\pi_o = 0$. In this case, $\pi x + \rho y \geq 0$ is a valid inequality for $K(n, r)$. Using the same arguments presented in the proof of Lemma 2.9, it is straight forward to to establish that $\pi_i = 0$ for all $i \in N$, $\rho_k = 0$ for $1 \leq k \leq n - r$ and $\rho_k \geq 0$ for $n - r + 1 \leq k \leq n$. But then, (π, ρ, π_o) can be written as a conic combination of the rays $(f_k, 0)$ for $n - r < k < n$, a contradiction. ∎

2.3 Facet characterization

Let

$$\mathcal{F} = \left\{ (\pi^k, \rho^k, \pi_o^k) \right\}_{k=1}^{M}$$

be the set of coefficients of nontrivial facets of $K(n, r)$ with $\rho_n = 0$ and $\pi_o = 1$. Note that by Lemma 2.9 these two assumptions do not eliminate any nontrivial facets. Also, as $y_n \geq 0$ is a nontrivial facet, $\mathcal{F} \neq \emptyset$. By Lemma 2.8, $\mathcal{F} \subseteq T$.

We are finally ready to prove Theorem 2.6. We do it in two steps.

Lemma 2.15 *If $(\pi, \rho, \pi_o) \in \mathcal{F}$, then (π, ρ, π_o) is an extreme point of T.*

Proof. Assume that $(\pi, \rho, \pi_o) \in \mathcal{F}$ but is not an extreme point of T, and therefore can be written as a convex combination of two distinct points in T. The normalization conditions $\rho_n = 0$ and $\pi_o = 1$ imply that any two distinct points in T represent two distinct valid inequalities for $K(n, r)$ in the sense that neither inequality can be obtained from the other by scaling or by adding multiples of the equation $\sum_{i=1}^n ix_i - \sum_{i=1}^n iy_i = r$. Therefore $\pi x + \rho y \geq \pi_o$ can be written as a combination of two distinct valid inequalities for $K(n, r)$, and therefore does not define a facet of $K(n, r)$. ■

Lemma 2.16 *If (π, ρ, π_o) is an extreme point of T, then $(\pi, \rho, \pi_o) \in \mathcal{F}$.*

Proof. Let $(\hat{\pi}, \hat{\rho}, 1)$ be an extreme point of T. By Lemma 2.11, $(\hat{\pi}, \hat{\rho}, 1)$ defines a valid inequality for $K(n, r)$ and therefore it is implied by a conic combination of facet defining inequalities plus a multiple of the equation $\sum_{i=1}^n ix_i - \sum_{i=1}^n iy_i = r$. In other words, there exists multipliers $\lambda \in \mathbb{R}^M_+$ and $\alpha \in \mathbb{R}$ such that

$$\hat{\pi}_i \geq \sum_{k=1}^M \lambda_k \pi_i^k + i\alpha, \qquad \forall\, i \in N \tag{4}$$

$$\hat{\rho}_i \geq \sum_{k=1}^M \lambda_k \rho_i^k - i\alpha, \qquad \forall\, i \in N \setminus \{n\} \tag{5}$$

$$\hat{\rho}_n = \sum_{k=1}^M \lambda_k \rho_n^k - n\alpha, \tag{6}$$

$$1 \leq \sum_{k=1}^M \lambda_k + r\alpha \tag{7}$$

hold. The inequalities in (4) and (5) correspond to the fact that $x_i \geq 0$ for $i = 1, \ldots, n$ and $y_i \geq 0$ for $i = 1, \ldots, n-1$ are not included in the non-trivial facets, whereas the equality in (6) is due to the fact that $y_n \geq 0$ is considered to be nontrivial. As $\hat{\rho}_n = 0$ and $\rho_n^k = 0$ for all nontrivial facet defining inequalities, (6) implies that $\alpha = 0$. Furthermore, $\hat{\rho}_r = 1$ and $\rho_r^k = 1$, for all k, and therfore, combining (5) and (7) we conclude that $\sum_{k=1}^M \lambda_k = 1$.

For any $i < r$, inequality (4) for i and $r - i$ combined with the equation (EP1) implies that

$$1 = \hat{\pi}_i + \hat{\pi}_{r-i} \geq \sum_{i=1}^M \lambda_k (\pi_i^k + \pi_{r-i}^k) = 1$$

which can hold only if $\hat{\pi}_i = \sum_{i=1}^M \lambda_k \pi_i^k$ for all $i < r$.

Similarly, for $i > r$, we use the equation (EP3) to observe that

$$1 = \hat{\pi}_i + \hat{\rho}_{i-r} \geq \sum_{i=1}^M \lambda_k (\pi_i^k + \rho_{i-r}^k) = 1$$

and therefore $\hat{\pi}_i = \sum_{i=1}^M \lambda_k \pi_i^k$ and $\hat{\rho}_{i-r} = \sum_{i=1}^M \lambda_k \rho_{i-r}^k$ for all $i > r$,.

Finally as $\hat{\rho}_i \geq \sum_{i=1}^{M} \lambda_k \rho_i^k$ for $i > n-r$, we can write $(\hat{\pi}, \hat{\rho}, 1)$ as a convex combination of points of $\mathcal{F}$ plus a conic combination of extreme rays of T. This can only be possible if $(\hat{\pi}, \hat{\rho}, 1) \in \mathcal{F}$. Thus, $(\hat{\pi}, \hat{\rho}, 1)$ is a nontrivial facet. ∎

As a final remark, it is interesting to note that conditions (R1) do not appear in the description of T even though they are necessary for any valid inequality. This is because conditions (R1) are implied by (SA1), (SA2) and (SA3). The proof is analogous to the proof of Lemma 2.11, so we just state it as an observation.

Observation 2.17 *Let $(\pi, \rho, \pi_o) \in T$. Then $j\pi_i + i\rho_j \geq 0, \forall i, j \in N$.*

We next show that coefficients of facet defining inequalities are bounded by small numbers.

Lemma 2.18 *Let (π, ρ, π_o) be an extreme point of T, then*

$$0 \leq \pi_k \leq \lceil k/r \rceil \quad \textit{and} \quad -\lceil k/r \rceil \leq \rho_k \leq \lceil n/r \rceil$$

for all $k \in N$.

Proof. Using Observation 2.17 with $j = n$ and the fact that $\rho_n = 0$, we have $\pi \geq 0$.

For $k < r$, combining inequality (EP1) $\pi_k + \pi_{r-k} \leq 1$ with $\pi \geq 0$ gives $\pi_k \leq \lceil k/r \rceil$. For $k > r$, let $k = \lfloor k/r \rfloor r + q$, where $0 \leq q < r$. If $q = 0$, by (SA1) we have $\pi_k \leq \lfloor k/r \rfloor \pi_r = \lceil k/r \rceil$. Similarly, if $q > 0$ we have $\pi_k \leq \lfloor k/r \rfloor \pi_r + \pi_q = \lfloor k/r \rfloor + \pi_q$, where $\pi_q \leq 1$. Therefore, $0 \leq \pi_k \leq \lceil k/r \rceil$.

The inequality (SA3) with $i = 1$ and $j = k$ implies that $\rho_k \geq -\pi_k$ and therefore $\rho_k \geq -\lceil k/r \rceil$ for all $k \in N$. If $k \leq n-r$, (EP3) implies that $\rho_k = \pi_r - \pi_{k+r} \leq 1 \leq \lceil n/k \rceil$. If $k > n-r$, then as (π, ρ, π_o) is an extreme point of T, at least one of (SA1) and (SA3) must hold with equality, in which case, $\rho_k \leq \pi_i$, for some $i \in N$. Thus $\rho_k \leq \lceil n/r \rceil$ ∎

3 The case $r = 0$

Observe that $LK(n, 0)$, the linear relaxation of $K(n, 0)$, is a cone and is pointed (as it is contained in the nonnegative orthant) and has a single extreme point $(x, y) = (0, 0)$. Therefore $LK(n, 0)$ equals its integer hull, i.e., $LK(n, 0) = K(n, 0)$. In Lemma 2.3, we characterized the extreme rays of $K(n, r)$ and thereby showed that the characteristic cone of $K(n, r)$ is generated by the vectors $\{r_{ij}\}$. But the characteristic cone of $K(n, r)$ for some $r > 0$ is just $K(n, 0)$. Therefore, $LK(n, 0)$ is generated by the vectors $\{r_{ij}\}$, and the next result follows.

Theorem 3.1 *The inequality $\pi x + \rho y \geq \pi_o$ is facet defining for $K(n, 0)$ if and only if (π, ρ, π_o) is a minimal face of*

$$T_o = \begin{cases} j\pi_i + i\rho_j \geq 0 & , \forall i, j \in N, \\ \pi_o = 0. \end{cases}$$

In his work on the MCGP, Gomory also studied the convex hull of non-zero integral solutions in $P(n, 0)$ and gave a dual characterization of its facets. We now consider a similar modification of $K(n, 0)$ and study the set:

$$\bar{K}(n,0) = conv\left\{(x,y) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n : \sum_{i=1}^{n} ix_i - \sum_{i=1}^{n} iy_i = 0, (x,y) \neq 0\right\}$$

We will next prove that all non-trivial facet defining inequalities for $\bar{K}(n, 0)$ are given by the extreme points of $\bar{T}_o$ defined below.

Definition 3.2 *Let $\bar{T}_o \subseteq \mathbb{R}^{2n+1}$ be the set of points that satisfy the following linear equalities and inequalities:*

$$\pi_i + \rho_j \geq \pi_{i-j}, \qquad \forall i,j \in N, \quad i > j, \tag{SA1}$$

$$\pi_i + \rho_j \geq \rho_{j-i}, \qquad \forall i,j \in N, \quad i < j, \tag{SA1'}$$

$$\pi_i + \rho_i = \pi_o, \qquad \forall i \in N, \tag{EP1-R0}$$

$$\pi_o = 1, \tag{N1-R0}$$

$$\rho_n = 0. \tag{N2-R0}$$

It is easy to see that the conditions (SA1), (SA1'), and (EP1-R0) are together equivalent to the conditions (SA2), (SA2') and (EP1-R0). For example, replacing π_i by $\pi_o - \rho_i$ and π_{i-j} by $\pi_o - \rho_{i-j}$ in (SA1), we get (SA2'). Therefore, a point in $\bar{T}_o$ satisfies all the pair-wise subadditivity conditions given in the previous section.

Lemma 3.3 *If $(\pi, \rho, \pi_o) \in \bar{T}_o$ then $\pi x + \rho y \geq \pi_o$ is a valid inequality for $\bar{K}(n, 0)$.*

Proof. Suppose $\pi x + \rho y \geq \pi_o$ is not valid for $\bar{K}(n, 0)$. Then, let $(x^*, y^*) \in \bar{K}(n, 0)$ be the integer point in $\bar{K}(n, 0)$ with smallest L_1 norm such that $\pi x^* + \rho y^* < \pi_o$. Note that any point in $\bar{K}(n, 0)$ has L_1 norm 2 or more.

If $||(x^*, y^*)||_1 = 2$, then $(x^*, y^*) = e_i + f_i$ for some $i \in N$, but by (EP1-R0), $\pi x^* + \rho y^* = \pi_o$, which is a contradiction. So we may assume that $||(x^*, y^*)||_1 > 2$.

As $(x^*, y^*) \in \bar{K}(n, 0)$, there exists $i, j \in N$ such that $x_i^* > 0$ and $y_j^* > 0$. Let,

$$(x', y') = (x^*, y^*) - e_i - f_j + \begin{cases} f_{j-i} & \text{if } i < j, \\ 0 & \text{if } i = j, \\ e_{i-j} & \text{if } i > j \end{cases}$$

Clearly, (x', y') is an integer point in $\bar{K}(n, 0)$ and $||(x', y')||_1 \leq ||(x^*, y^*)||_1 - 1$. Furthermore, as (π, ρ, π_o) satisfies (SA1), (SA1') and (EP1-R0), we also have $\pi x' + \rho y' \leq \pi x^* + \rho x^* < \pi_o$, which contradicts the choice of (x^*, y^*). ∎

Theorem 3.4 *The inequality $\pi x + \rho y \geq \pi_o$ defines a nontrivial facet of $\bar{K}(n, 0)$ if and only if it can be represented as an extreme point of $\bar{T}_o$.*

Proof.

($\Rightarrow$):

Let $\pi x + \rho y \geq \pi_o$ define a nontrivial facet of $\bar{K}(n,0)$. We first show that (π, ρ, π_o) satisfies (SA1), (SA1') and (EP1-R0), and can be assumed to satisfy (N1-R0) and (N2-R0).

(SA1) - (SA1'): Let i, j be indices such that $i, j \in N$ and $i > j$. Let $z = (x^*, y^*)$ be an integral point lying on the above facet such that $x^*_{i-j} > 0$. As $z + (e_i + f_j - e_{i-j})$ belongs to $\bar{K}(n,0)$, (SA1) is true. The proof of (SA1') is similar.

(EP1-R0): Let $\gamma = (\pi, \rho)$. Let $z^1 = (x^1, y^1)$ and $z^2 = (x^2, y^2)$ be integral points lying on the facet such that $x^1_i > 0$ and $y^2_i > 0$. Then $z = z^1 + z^2 - e_i - f_i \in \bar{K}(n,0)$, and therefore $\gamma z = \gamma z^1 + \gamma z^2 - \pi_i - \rho_i = 2\pi_o - \pi_i - \rho_i \geq \pi_o \Rightarrow \pi_i + \rho_i \leq \pi_o$. But as $e_i + f_i \in \bar{K}(n,0)$, $\pi_i + \rho_i \geq \pi_o$ and the result follows.

(N1-R0): Assume $\pi_o < 0$, and let (x^*, y^*) be an integral point in $\bar{K}(n,0)$ satisfying $\pi x^* + \rho y^* = \pi_o$. As $\alpha(x^*, y^*) \in \bar{K}(n,0)$ for any positive integer α, whereas $\pi \alpha x^* + \rho \alpha y^* = \alpha \pi_o < \pi_o$, we obtain a contradiction to the fact that points in $\bar{K}(n,0)$ satisfy $\pi x + \rho y \geq \pi_o$.

If $\pi_o = 0$, then (EP1-R0) implies that $\rho_i = -\pi_i$ for all $i \in N$. This fact, along with (SA1) and (SA1') implies that $\pi_i = i\pi_1$ and $\rho_i = -i\pi_1$ for all $i \in N$. But then $\pi x + \rho y \geq \pi_o$ is the same as $\pi_1(\sum_{i=1}^n ix_i - \sum_{i=1}^n iy_i) \geq 0$, and therefore cannot define a proper face of $\bar{K}(n,0)$. Therefore, for any non-trivial facet, $\pi_o > 0$ and can be assumed to be 1 by scaling.

We can assume, by subtracting appropriate multiples of $\sum_{i=1}^n ix_i - \sum_{i=1}^n iy_i = 0$ from $\pi x + \rho y \geq \pi_o$, that (N2-R0) holds.

Therefore (π, ρ, π_o) can be assumed to be contained in $\bar{T}_o$. If it is not an extreme point of $\bar{T}_o$, it can be written as a convex combination of two distinct points of $\bar{T}_o$, different from itself, each of which defines a valid inequality for $\bar{K}(n,0)$ (by Lemma 3.3). As the normalization conditions (N1-R0) and (N2-R0) mean that each non-trivial facet-defining inequality corresponds to a unique point in $\bar{T}_o$, this implies that (π, ρ, π_o) is an extreme point of $\bar{T}_o$.

($\Leftarrow$):

Let $\mathcal{F} = \{(\pi^k, \rho^k, \pi^k_o)\}_{k=1}^M$ be the set of all nontrivial facets of $\bar{K}(n,0)$ such that $\rho_n = 0$ and $\pi^k_o = 1$. Let $(\pi, \rho, 1)$ be an extreme point of $\bar{T}_o$. By Lemma 3.3, $(\pi, \rho, 1)$ defines a valid inequality for $\bar{K}(n,0)$, and therefore there exist numbers λ^k and α such that

$$
\begin{aligned}
&\alpha i + \textstyle\sum_{k=1}^M \lambda_k \pi^k_i \leq \pi_i, \forall i \in N \\
&-\alpha i + \textstyle\sum_{k=1}^M \lambda_k \rho^k_i \leq \rho_i, \forall i \in N \\
&\textstyle\sum_{k=1}^M \lambda_k \geq 1 \\
&\lambda \geq 0, \alpha \text{ free}
\end{aligned}
$$

Clearly, $\pi x + \rho y \geq \sum_{k=1}^M \lambda_k$ is a valid inequality for $\bar{K}(n,0)$. As $e_i + f_i \in \bar{K}(n,0)$ and $\pi e_i + \rho f_i = 1$, we can conclude that $\sum_{k=1}^M \lambda_k = 1$. (EP1-R0) implies that for all $i \in N$, $1 = \pi_i + \rho_i \geq \sum_{i=1}^M \lambda_k(\pi^k_i + \rho^k_i) = 1$, and therefore $\pi_i = \sum_{i=1}^M \lambda_k \pi^k_i$ and $\rho_i = \sum_{i=1}^M \lambda_k \rho^k_i$. In other words, $(\pi, \rho, 1)$ can be expressed as a convex combination of the elements of $\mathcal{F}$, each of which is contained in $\bar{T}_o$. This is possible only if $(\pi, \rho, 1)$ is itself an element of $\mathcal{F}$, i.e., it defines a nontrivial facet of $\bar{K}(n,0)$. ∎

4 Separating over $K(n,r)$

Let $LK(n,r)$ be the linear relaxation of $K(n,r)$. Define the (facet-) separation problem over $K(n,r)$ as follows: given $(x^*,y^*) \in LK(n,r)$, either verify that $(x^*,y^*) \in K(n,r)$ or find a violated (facet-defining) valid inequality for $K(n,r)$. Note that the condition that $(x^*,y^*) \in LK(n,r)$ is easy to satisfy.

The next result is an immediate consequence of Theorems 2.6, 3.1 and 3.4.

Theorem 4.1 *Given $(x^*,y^*) \in LK(n,r)$, with $0 \le r \le n$, the separation problem over $K(n,r)$ can be solved in polynomial-time using an LP with $O(n)$ variables and $O(n^3)$ constraints.*

Proof. If $r > 0$, solve

$$\min\{\pi x^* + \rho y^* : (\pi,\rho,\pi_o) \in T\}.$$

As $(x^*,y^*) \in LK(n,r)$ and $y^* \ge 0$ and the extreme rays of T are given by $(f_k, 0)$ for $n-r < k < n$, the above problem is bounded and has a solution.

If the optimal extreme point $(\pi,\rho,1) \in T$ is such that $\pi x^* + \rho y^* < 1$, then $\pi x + \rho y \ge 1$ defines a facet that separates (x^*,y^*) from $K(n,r)$. Otherwise, (x^*,y^*) satisfies all nontrivial facet-defining inequalities of $K(n,r)$, and $(x^*,y^*) \in K(n,r)$.

In a similar fashion, we can use T_o and $\bar{T}_o$ to solve the separation problem over $K(n,0)$ and $\bar{K}(n,0)$ respectively. ∎

The following theorem states that the separation problem over $K(n,r)$ can also be solved for any value of r.

Theorem 4.2 *Given $(x^*,y^*) \in LK(n,r)$, the separation problem over $K(n,r)$ can be solved in time polynomial in $\max\{n,r\}$ using an LP with $O(\max\{n,r\})$ variables and $O(\max\{n,r\}^2)$ constraints.*

Proof. The case $r = 0$ is unchanged from Theorem 4.1.

Now consider the case $0 < r \le n$. Suppose that instead of T, we have a set T' that contains all nontrivial facets and is contained in the set of valid inequalities. We then solve:

$$\min\{\pi x^* + \rho y^* - \pi_o : (\pi,\rho,\pi_o) \in T'\}.$$

If there is a solution $(\pi,\rho,\pi_o) \in T'$ such that $\pi x^* + \rho y^* - \pi_o < 0$, then $\pi x + \rho y \ge \pi_o$ defines a valid inequality that separates (x^*,y^*) from $K(n,r)$. Otherwise, (x^*,y^*) satisfies all nontrivial facet-defining inequalities of $K(n,r)$, and therefore $(x^*,y^*) \in K(n,r)$.

Consider the set T' obtained by removing (SA3) from T and adding (SA1') to it. By Remark 2.13, Corollary 2.12 is true with T replaced by T'. Lemmas 2.8 and 2.9 imply that Corollary 2.10 also holds with T replaced by T'. Therefore T' satisfies all desired properties and can be used to solve the separation problem.

Finally, if $r > n$, then define $(x',y') \in \mathbb{R}^r \times \mathbb{R}^r$ such that $x'_i = x^*_i; y'_i = y^*_i$, for $i = 1,\dots,n$ and $x'_i = y'_i = 0$, for $i = n+1,\dots,r$. The point $(x',y') \in K(r,r) \iff (x^*,y^*) \in K(n,r)$, and so the separation can be done in time polynomial in r. ∎

5 Lifting facets of $P(n,r)$

Lifting is a general principle for constructing valid (facet defining) inequalities for higher dimensional sets using valid (facet defining) inequalities for lower dimensional sets. Starting with the early work of Gomory [9], this approach was generalized by Wolsey [16], Balas and Zemel [3] and Gu et. al [17], among others.

In this section we discuss how facets of $P(n,r)$ can be lifted to obtain facets of $K(n,r)$. $P(n,r)$ can also be considered as an $n-1$ dimensional face of $K(n,r)$ obtained by setting n variables to their lower bounds. Throughout this section we assume that $n > r > 0$.

We start with a result of Gomory [9] that gives a complete characterization of the nontrivial facets (i.e., excluding the non-negativity inequalities) of $P(n,r)$.

Theorem 5.1 (Gomory [9]) *The inequality $\bar{\pi}x \geq 1$ defines a non-trivial facet of $P(n,r)$ if and only if $\bar{\pi} \in \mathbb{R}^{n-1}$ is an extreme point of*

$$Q = \begin{cases} \pi_i + \pi_j \;\; \geq \;\; \pi_{(i+j)\, mod\, n} \quad \forall i,j \in \{1,\ldots,n-1\}, \\ \pi_i + \pi_j \;\; = \;\; \pi_r \quad \forall i,j \textit{ such that } r = (i+j)\, mod\, n, \\ \pi_j \;\; \geq \;\; 0 \quad \forall j \in \{1,\ldots,n-1\}, \\ \pi_r \;\; = \;\; 1. \end{cases}$$

Given a non-trivial facet defining inequality for $P(n,r)$

$$\sum_{i=1}^{n-1} \bar{\pi}_i x_i \geq 1 \tag{8}$$

it is possible to *lift* this inequality to obtain a facet-defining inequality

$$\sum_{i=1}^{n-1} \bar{\pi}_i x_i + \pi'_n x_n + \sum_{i=1}^{n-1} \rho'_i y_i \geq 1 \tag{9}$$

for $K(n,r)$. We call inequality (9) a *lifted* inequality and note that in general for a given starting inequality there might be an exponential number of lifted inequalities, see [16].

5.1 The restricted coefficient polyhedron $T^{\bar{\pi}}$

First note that a non-trivial facet of $P(n,r)$ can only yield a non-trivial facet of $K(n,r)$. This, in turn, implies that $(\bar{\pi}, \pi'_n, \rho', 0)$ has to be an extreme point of the coefficient polyhedron T. Therefore, the lifting procedure can also be seen as a way of extending an extreme point of Q to obtain an extreme point of T.

Let $p = (\bar{\pi}, \pi'_n, \rho', 0)$ be an extreme point of T. Then, p also has to be an extreme point of the lower dimensional polyhedron

$$T^{\bar{\pi}} \;=\; T \cap \Big\{\pi_i = \bar{\pi}_i, \;\forall i \in \{1,\ldots,n-1\}\Big\}$$

obtained by fixing some of the coordinates.

Let $L = \{n-r+1,\ldots,n-1\}$.

Lemma 5.2 *If inequality (8) defines a non-trivial facet of $P(n,r)$, then $T^{\bar{\pi}} \neq \emptyset$ and it has the form*

$$T^{\bar{\pi}} = \begin{cases} \tau \;\geq\; \pi_n & \geq\; 0 \\ \rho_k & \geq\; l_k \quad \forall\, k \in L \\ \rho_k + \pi_n & \geq\; t_k \quad \forall\, k \in L \\ \rho_k - \pi_n & \geq\; f_k \quad \forall\, k \in L \\ \pi_n + \rho_{n-r} & =\; 1 \\ \rho_n & =\; 0 \\ \rho_k & =\; \bar{\pi}_{n-k} \quad \forall k \in \{1,\ldots,n-r-1\} \\ \pi_i & =\; \bar{\pi}_i \quad \forall i \in \{1,\ldots,n-1\} \end{cases}$$

where numbers l_k, t_k, f_k and τ can be computed easily using $\bar{\pi}$.

Proof. First note that $\bar{\pi} \in Q$ and therefore $\bar{\pi}$ satisfies inequality (SA2) as well as equations (EP1) and (EP2). In addition, as $\bar{\pi}_i + \bar{\pi}_j = 1$ for all i,j such that $r = (i+j) \bmod n$, equality (EP3) can be rewritten as $\rho_i = \pi_{n-i}$ for all $1 \leq i \leq n-r$. Further, as $\bar{\pi}$ is subadditive (in the modular sense), inequalities (SA1) and (SA3) are satisfied for all $k \in \{1,\ldots,n-r-1\}$. Therefore, setting

$$\pi_n = 0 \quad \text{and} \quad \rho_k = \begin{cases} \bar{\pi}_{n-k} & \text{if } k \in \{1,\ldots,n-r-1\}, \\ 1 & \text{otherwise }, \end{cases}$$

produces a feasible point for $T^{\bar{\pi}}$, establishing that the set is not empty.

We next show that the $T^{\bar{\pi}}$ has the form given in the Lemma, and also compute the values of l_k, t_k, f_k and τ.

Inequality (SA1): If $i = n$, this inequality becomes $\pi_n + \rho_k \geq \bar{\pi}_{n-k}$. If $i \neq n$, it becomes $\rho_k \geq \bar{\pi}_{i-k} - \bar{\pi}_i$, and therefore $\rho_k \geq l_k^1 = \max_{n>i>k}\{\bar{\pi}_{i-k} - \bar{\pi}_i\}$.

Inequality (SA2): The only relevant case is $i+j = n$ when the inequality becomes $\pi_n \leq \bar{\pi}_i + \bar{\pi}_{n-i}$. When combined, these inequalities simply become $\pi_n \leq \tau^1 = \min_{n>i>0}\{\bar{\pi}_i + \bar{\pi}_{n-i}\}$.

Inequality (SA3): Without loss of generality assume $i \geq j$. We consider 3 cases.

Case 1, $k = n$: In this case the inequality reduces to $\pi_i + \pi_j \geq \pi_{i+j-n}$ which is satisfied by $\bar{\pi}$ when $i,j < n$. For $i = n$, this inequality simply becomes $\pi_n \geq 0$.

Case 2, $k < n$ and $i+j-k = n$: In this case the inequality becomes $\rho_k - \pi_n \geq -\pi_i - \pi_j$. If $i,j < n$ these inequalities can be combined to obtain $\rho_k - \pi_n \geq f_k^1 = \max_{1\leq i,j<n,\; k=i+j-n}\big\{-\bar{\pi}_i - \bar{\pi}_j\big\}$. If, $i = n$, then $j = k$ and the inequality becomes $\rho_k \geq -\bar{\pi}_k$.

Case 3, $k < n$ and $i+j-k < n$: If $i,j < n$ the inequality becomes $\rho_k \geq \pi_{i+j-k} - \pi_i - \pi_j$. These inequalities can be combined to obtain $\rho_k \geq l_k^2 = \max_{1\leq i,j<n:\; k<i+j<n+k}\big\{\bar{\pi}_{i+j-k} - \bar{\pi}_i - \bar{\pi}_j\big\}$. If $i = n$ then $j < n$, so the inequality becomes $\pi_n + \rho_k \geq \pi_{n+j-k} - \pi_j$, implying $\pi_n + \rho_k \geq t_k^1 = \max_{k>j}\big\{\bar{\pi}_{n+j-k} - \bar{\pi}_j\big\}$.

Therefore, combining these observations, it is easy to see that $T^{\bar{\pi}}$ has the above form where

l_k, t_k, f_k and τ are computed as follows:

$$\begin{aligned}
l_k &= \max\left\{l_k^1, l_k^2, -\bar{\pi}_k\right\}, \\
t_k &= \max\left\{t_k^1, \bar{\pi}_{n-k}\right\} \;=\; \bar{\pi}_{n-k}, \\
f_k &= f_k^1, \\
\tau &= \min\left\{\tau^1,\ 1 - l_{n-r},\ (1 - f_{n-r})/2\right\}.
\end{aligned}$$

The second equality in the description of t_k states that $t_k = \bar{\pi}_{n-k}$ comes from the fact that $\bar{\pi}$ is sub-additive and therefore $\bar{\pi}_{n-k} + \bar{\pi}_j \geq \bar{\pi}_{n+j-k}$ for all $j < k$. The $1 - l_{n-r}$ and $(1 - f_{n-r})/2$ terms in the last equation come from using the bounds on ρ_{n-r} together with the equations and inequalities of $T^{\bar{\pi}}$ to obtain implied bounds for π_n.

∎

We next make a simple observation that will help us show that $T^{\bar{\pi}}$ has a small (polynomial) number of extreme points.

Lemma 5.3 *If $p = (\bar{\pi}, \pi'_n, \rho', 0)$ is an extreme point of $T^{\bar{\pi}}$, then*

$$\rho'_k = \max\left\{l_k, t_k - \pi'_n, f_k + \pi'_n\right\}$$

for all $k \in L$.

Proof. Assume that the claim does not hold for some $k \in L$ and let $\theta = \max\left\{l_k, t_k - \pi'_n, f_k + \pi'_n\right\}$. As $p \in T^{\bar{\pi}}$, $\rho'_k \geq \theta$ and therefore $\epsilon = \rho'_k - \theta > 0$. In this case, two distinct points in $T^{\bar{\pi}}$ can be generated by increasing and decreasing the associated coordinate of p by ϵ, establishing that p is not an extreme point, a contradiction. ∎

We next characterize the set possible values π'_n can take at an extreme point of $T^{\bar{\pi}}$.

Lemma 5.4 *Let $p = (\bar{\pi}, \pi'_n, \rho', 0)$ be an extreme point of $T^{\bar{\pi}}$, if $\pi'_n \notin \{0, \tau\}$, then*

$$\pi'_n \;\in\; \Lambda \;=\; \left(\bigcup_{k \in L_1} \left\{t_k - l_k, l_k - f_k\right\}\right) \bigcup \left(\bigcup_{k \in L_2} \left\{(t_k - f_k)/2\right\}\right)$$

where $L_1 \;=\; \{k \in L \,:\, t_k + f_k < 2l_k\}$ and $L_2 \;=\; L \setminus L_1$.

Proof. Notice that the description of $T^{\bar{\pi}}$ consists of $3(r-1)$ inequalities that involve ρ_k variables and upper and lower bound inequalities for π'_n. Being an extreme point, p has to satisfy r of these inequalities as equality. Therefore, if $\pi'_n \notin \{0, \tau\}$ then, there exists an index $k \in L$ for which at least two of the following inequalities

$$\rho_k \;\geq\; l_k \tag{a}$$

$$\rho_k + \pi_n \;\geq\; t_k \tag{b}$$

$$\rho_k - \pi_n \;\geq\; f_k \tag{c}$$

hold as equality. Clearly, this uniquely determines the value of π'_n and therefore

$$\pi'_n \;\in\; \Lambda^+ \;=\; \bigcup_{k\in L} \Big\{t_k - l_k, l_k - f_k, (t_k - f_k)/2\Big\}$$

Furthermore, for any fixed $k \in L$, adding inequalities (b) and (c) gives $2\rho_k \geq t_k + f_k$. Therefore if $t_k + f_k > 2l_k$ inequality (a) is implied by inequalities (b) and (c) and it cannot hold as equality. Similarly, if $t_k + f_k < 2l_k$, inequalities (b) and (c) cannot hold simultaneously. Finally, if $t_k + f_k = 2l_k$ then it is easy to see that $t_k - l_k = l_k - f_k = (t_k - f_k)/2$. Therefore letting

$$L_1 \;=\; \{k \in S \,:\, t_k + f_k < 2l_k\}, \qquad L_2 \;=\; L \setminus L_1$$

proves the claim. ∎

Combining the previous Lemmas, we have the following result:

Theorem 5.5 *Given a non-trivial facet defining inequality (8) for $P(n, r)$, there are at most $2r$ lifted inequalities that define facets of $K(n, r)$.*

Proof. The set L has $r - 1$ members and therefore together with 0 and τ, there are at most $2r$ possible values for π'_n in a facet defining lifted inequality (9). As the value of π'_n uniquely determines the remaining coefficients in the lifted inequality, by Lemma 5.3, the claim follows. ∎

In general, determining all possible lifted inequalities is a hard task. However, the above results show that obtaining all possible facet-defining inequalities lifted from a facet of $P(n, r)$ is straightforward and can be performed in polynomial time. We conclude this section by presenting a result from Wolsey [16] adapted to $K(n, r)$, which allows us to state a result on sequential lifting.

Lemma 5.6 (Wolsey [16]) *Given a facet defining inequality (8) for $P(n, r)$ and a lifting sequence for the variables x_n and y_i for $i = 1, \ldots, n-1$, sequential lifting procedure produces a facet defining inequality for $K(n, r)$.*

Furthermore, at each step of lifting, the variable being lifted is assigned the smallest possible facet coefficient for a lifted facet that has the same coefficients for the variables that are already lifted.

Lemma 5.7 *If variable x_n is lifted before all y_k for $k \in \{n - r, \ldots, n - 1\}$, then independent of the rest of the lifting sequence the lifted inequality is*

$$\sum_{i=1}^{n-1} \bar{\pi}_i x_i + \sum_{i=1}^{n-1} \bar{\pi}_{n-i} y_i \geq 1.$$

Proof. By Lemma 5.6, we know that variable x_n will be assigned the smallest possible facet coefficient for a lifted facet. As $\pi_n \geq 0$ in the description of $T^{\bar{\pi}}$ and as $T^{\bar{\pi}}$ does contain a point with $\pi_n = 0$ (described in the proof of Lemma 5.2), we can conclude that $\pi_n = 0$ in the lifted facet.

Therefore, by Lemma 5.3, $\rho'_k = \min\big\{l_k, \bar{\pi}_{n-k}, f_k\big\}$ and we need to show that $\bar{\pi}_{n-k} \geq l_k, t_k$ for all $k \in \{1, \ldots, n-1\}$. First, observe that $0 \geq t_k$ and therefore $\bar{\pi}_{n-k} \geq t_k$ for all $k \in \{1, \ldots, n-1\}$. Finally, recall that $\bar{\pi}$ is subadditive (in the modular sense), and therefore $\bar{\pi}_{n-k} + \bar{\pi}_i \geq \bar{\pi}_{i-k}$ for all $n > i > k$ and $\bar{\pi}_{n-k} + \bar{\pi}_i + \bar{\pi}_j \geq \bar{\pi}_{i+j-k}$ for all $i, j < n,\; k < i + j < n + k$. ∎

6 Mixed integer rounding inequalities

In this section we study MIR inequalities in the context of $K(n,r)$. Our analysis also provides an example that shows that lifting facets of $P(n,r)$ cannot give all facets of $K(n,r)$. Throughout, we will use the notation $\hat{x} := x - \lfloor x \rfloor$ and $(x)^+ = \max\{x, 0\}$. Recall that, for a general single row system of the form: $\left\{w \in \mathbb{Z}^p_+ : \sum_{i=1}^p a_i w_i = b\right\}$ where $\hat{b} > 0$, the MIR inequality is:

$$\sum_{i=1}^{p} \left(\lfloor a_i \rfloor + \min\left(\hat{a}_i/\hat{b}, 1\right)\right) w_i \geq \lceil b \rceil .$$

We define the $\frac{1}{t}$-MIR (for $t \in \mathbb{Z}_+$) to be the MIR inequality obtained from the following equivalent representation of $K(n,r) = \left\{(x,y) \in \mathbb{Z}^n_+ \times \mathbb{Z}^n_+ : \sum_{i=1}^n (i/t) x_i - \sum_{i=1}^n (i/t) y_i = r/t\right\}$.

Lemma 6.1 *Given $t \in \mathbb{Z}$ such that $2 \leq t \leq n$, the $\frac{1}{t}$-MIR inequality*

$$\sum_{i=1}^{n} \left(\left\lfloor \frac{i}{t} \right\rfloor + \min\left(\frac{i \mod t}{r \mod t}, 1\right)\right) x_i + \sum_{i=1}^{n} \left(-\left\lceil \frac{i}{t} \right\rceil + \min\left(\frac{(t-i) \mod t}{r \mod t}, 1\right)\right) y_i \geq \left\lceil \frac{r}{t} \right\rceil$$

is facet defining for $K(n,r)$ provided that $r/t \notin \mathbb{Z}$.

Proof. Let $\pi x + \rho y \geq \pi_o$ denote the $\frac{1}{t}$-MIR inequality and let F denote the set of points that are on the face defined by this inequality. Also let v denote $(r \mod t)$ and q^i denote $(i \mod t)$ and note that using this definition $r = v + \lfloor r/t \rfloor$ and $i = q^i + \lfloor i/t \rfloor$.

For $i \in N \setminus \{1, t\}$, consider the point

$$w^i = e_i + (\lfloor r/t \rfloor - \lfloor i/t \rfloor)^+ e_t + (\lfloor i/t \rfloor - \lfloor r/t \rfloor)^+ f_t + (v - q^i)^+ e_1 + (q^i - v)^+ f_1$$

and observe that $w^i \in K(n,r)$. Moreover,

$$(\pi, \rho)^T w^i = (\lfloor i/t \rfloor + \min\{q^i/v, 1\}) + (\lfloor r/t \rfloor - \lfloor i/t \rfloor) + \frac{(v - q^i)^+}{v} = \lfloor r/t \rfloor + 1 = \pi_o$$

and therefore $w^i \in F$. Similarly, let $-i = -\lceil i/t \rceil + m^i$, with $0 \leq m^i < t$ and consider the point

$$z^i = f_i + (\lfloor r/t \rfloor + \lceil i/t \rceil) e_t + (v - m^i)^+ e_1 + (m^i - v)^+ f_1$$

for $i \in N \setminus \{1, t\}$. Clearly $x^i \in K(n,r)$. Furthermore,

$$(\pi, \rho)^T z^i = (-\lceil i/t \rceil + \min\{m/v, 1\}) + (\lfloor r/t \rfloor + \lceil i/t \rceil) + \frac{(v - m)^+}{v} = \lfloor r/t \rfloor + 1 = \pi_o$$

and therefore $z^i \in F$.

Additionally the following three points are also in $K(n,r) \cap F$: $u^1 = \lfloor r/t \rfloor e_t + v e_1$, $u^2 = (\lfloor r/t \rfloor + 1) e_t + (t - v) f_1$, $u^3 = (\lfloor r/t \rfloor + 1) e_t + f_t + v e_1$. Therefore, $\{u^i\}_{i=1}^3 \cup \{w^i\}_{i \in N \setminus \{1,t\}} \cup \{z^i\}_{i \in N \setminus \{1,t\}}$ is a set of $2n - 1$ affinely independent points in F. ∎

We next show that $\frac{1}{t}$-MIR inequalities are not facet defining unless they satisfy the conditions of Theorem 6.1. First, observe that the inequality is not defined if t divides r. Next, we show that $1/n$-MIR inequality dominates all $\frac{1}{t}$-MIR inequalities with $t > n$.

Lemma 6.2 *If $t > n$, then $\frac{1}{t}$-MIR inequality is not facet defining for $K(n, r)$*

Proof. When $t > n$, $\frac{1}{t}$-MIR inequality becomes

$$\sum_{i \in N} \min\{i/r, 1\} x_i - \sum_{i: i > t-r} \left(1 - \frac{t-i}{r}\right) y_i \geq 1$$

and is dominated by the $1/n$-MIR:

$$\sum_{i \in N} \min\{i/r, 1\} x_i - \sum_{i: i > n-r} \left(1 - \frac{n-i}{r}\right) y_i \geq 1$$

∎

We conclude this section by showing that $\frac{1}{t}$-MIR inequalities give facets that cannot be obtained by lifting facets of $P(n, r)$.

Theorem 6.3 *Not all facet-defining inequalities of $K(n, r)$ can be obtained from lifting facet-defining inequalities of $P(n, r)$, for $n \geq 9$ and $0 < r \leq n - 2$.*

Proof. When $0 < r \leq n - 4$, consider the facet induced by the $\frac{1}{n-2}$-MIR inequality $\pi x + \rho y \geq \pi_o$ where

$$\rho_n = -2 + \min\left(\frac{n-4}{r}, 1\right) = -1.$$

We therefore subtract $\frac{1}{n}$ times $\sum_{i=1}^n i x_i - \sum_{i=1}^n i y_i = r$ to the inequality to obtain $\pi' x + \rho' y \geq \pi'_o$ where $\rho'_n - 0$ and therefore it satisfies the normalization condition (NC1). Notice that

$$\begin{aligned} \pi'_{r+1} + \pi'_{n-1} &= \left(1 - \frac{r+1}{n}\right) + \left(1 + \frac{1}{r} - \frac{n-1}{n}\right) \\ &= 1 - \frac{r}{n} + \frac{1}{r} \end{aligned}$$

whereas $\pi'_r = 1 - r/n < \pi'_{r+1} + \pi'_{n-1}$. This proves the claim for $0 < r \leq n - 4$ as all facet defining inequalities for $P(n, r)$ have to satisfy $\pi'_{r+1} + \pi'_{n-1} = \pi'_r$.

For $r \in \{n - 3, n - 2\}$, the $\frac{1}{r-1}$-MIR provides such an example. ∎

For $r = n - 1$, all points in T automatically satisfy all equations in Q. Therefore, any given facet-defining inequality of $K(n, r)$ can be obtained by lifting a point in Q. However, this point is not necessarily an extreme point of Q.

7 Mixed-Integer extension

Consider the mixed-integer extension of $K(n, r)$:

$$K'(n, r) = conv \left\{ (v_+, v_-, x, y) \in \mathbb{R}_+^2 \times \mathbb{Z}_+^{2n} : v_+ - v_- + \sum_{i=1}^n i x_i - \sum_{i=1}^n i y_i = r \right\}$$

where $n, r \in \mathbb{Z}$ and $n \geq r > 0$. As in the case of the mixed-integer extension of the MCGP studied by Gomory and Johnson [10], the facets of $K'(n, r)$ can easily be derived from the facets of $K(n, r)$

when r is an integer. To prove such a result, we introduce a few definitions, and also state some easy results without proof.

The dimension of $K'(n,r)$ is $2n+1$, i.e., one less than the number of variables. The inequalities $x_i \geq 0$ and $y_i \geq 0$, for $i = 1,\ldots,n$, and $v_+ \geq 0$ and $v_- \geq 0$ define facets of $K'(n,r)$. We refer to the facets above – other than $y_n \geq 0$ – as trivial facets, and refer to the remaining facets of $K'(n,r)$ as non-trivial. Finally, note that the characteristic cone of $K'(n,r)$ contains the vectors $je_i + if_j$, for any i,j satisfying $1 \leq i,j \leq n$. For $K'(n,r)$, let e_+ and e_- be the unit vectors in $\mathbb{R}^{2n+2}$ with ones in, respectively, the v_+ component, and the v_- component, and zeros elsewhere. For a vector χ in the $K'(n,r)$ space, define its restriction to the $K(n,r)$ space by removing the v_+ and v_- components, and denote it by χ_{re}.

Proposition 7.1 *All non-trivial facet defining inequalities for $K'(n,r)$ have the form*

$$\pi_1 v_+ + \rho_1 v_- + \sum_{i=1}^{n} \pi_i x_i + \sum_{i=1}^{n} \rho_i y_i \geq \pi_o. \tag{10}$$

Furthermore, inequality (10) is facet defining if and only if $\pi x + \rho y \geq \pi_o$ defines a non-trivial facet of $K(n,r)$.

Proof. Let $\pi x + \rho y \geq \pi_o$ define a non-trivial facet of $K(n,r)$. We first show that the inequality (10) is valid for $K'(n,r)$. Assume (10) is violated by some integral point $\chi \in K'(n,r)$ (the x and y components of χ are integral). Then the left hand side of (10) evaluated at χ equals a number z less than π_o. Let $v'_+ = e_+^T\chi$ and $v'_- = e_-^T\chi$. The property (R1) in Observation 2.7 implies that $\pi_1 + \rho_1 \geq 0$. Therefore, if $\min\{v'_+, v'_-\} = \epsilon > 0$, then (10) is also violated by the point $\chi - \epsilon(e_+ + e_-) \in K'(n,r)$. We can thus assume that χ satisfies $\min\{v'_+, v'_-\} = 0$. But $\min\{v'_+, v'_-\} = 0$ combined with the integrality of χ implies that v'_+ and v'_- are both integers. Therefore $\chi' = \chi_{re} + v'_+ e_1 + v'_- f_1$ is an integral point contained in $K(n,r)$, and $(\pi,\rho)^T\chi' = z < \pi_o$, which contradicts the fact that $\pi x + \rho y \geq \pi_o$ is satisfied by all points in $K(n,r)$.

To see that (10) defines a facet of $K'(n,r)$, let $\chi^1, \ldots \chi^{2n-1}$ be affinely independent integral points in $K(n,r)$ which satisfy $(\pi,\rho)^T\chi^i = \pi_o$. As the facet defined by $\pi x + \rho y \geq \pi_o$ does not equal the facet defined by either $x_1 \geq 0$ or $y_1 \geq 0$, there are indices j,k such that $e_1^T\chi^j = s > 0$ and $f_1^T\chi^k = t > 0$. Define $2n+1$ affinely independent points in $\mathbb{R}^{2n+2}$ as follows:

$$\psi^i = (0,0,\chi^i) \text{ for } i = 1,\ldots,2n-1;$$

$$\psi^+ = \psi^j + se_+ - se_1;\ \psi^- = \psi^k + te_- - tf_1.$$

These points satisfy (10) as an equation, and therefore (10) defines a facet of $K'(n,r)$.

We now show that every non-trivial facet of $K'(n,r)$ has the form in (10). Let $\eta^T(v_+, v_-, x, y) \geq \eta_o$ define a non-trivial facet F of $K'(n,r)$. Let $\eta = (\alpha_+, \alpha_-, \pi, \rho)$, where $\alpha_+, \alpha_- \in \mathbb{R}$, and $\pi, \rho \in \mathbb{R}^n$. There exists a point $\chi \in K'(n,r)$ lying on the above facet such that $\chi^T e_1 > 0$. As $\chi - e_1 + e_+ \in K'(n,r)$, we conclude that $\alpha_+ \geq \pi_1$. We can similarly conclude that $\alpha_- \geq \rho_1$ and therefore $\alpha_+ + \alpha_- \geq \pi_1 + \rho_1 \geq 0$. The last inequality is implied by the fact that $e_1 + f_1$ is contained in the characteristic cone of $K'(n,r)$. If $\alpha_+ + \alpha_- = 0$, then clearly $\alpha_+ = \pi_1$ and $\alpha_- = \rho_1$. Assume $\alpha_+ + \alpha_- > 0$. As F is not the same as the facet $v_+ \geq 0$, there exists an integral point $\chi = (v'_+, v'_-, x', y') \in K'(n,r)$ lying on F such that $v'_+ > 0$. If $v'_- > 0$, let $\min\{v'_+, v'_-\} = \epsilon > 0$. Then $\chi^1 = \chi - \epsilon(e_+ + e_-) \in K'(n,r)$, but $\eta^T\chi^1 = \eta_o - \epsilon(\alpha_+ + \alpha_-) < \eta_o$. This contradicts the fact that (η, η_o) defines a valid inequality for $K'(n,r)$. We can therefore assume that $v'_- = 0$ and $v'_+ = t$, for some positive integer t. Define χ^2 as $\chi - te_+ + te_1$. As $\chi^2 \in K'(n,r)$, it follows that $\eta^T\chi^2 \geq \eta_o \Rightarrow \alpha_+ \leq \pi_1$. We can conclude that $\alpha_+ = \pi_1$; a similar argument shows that $\alpha_- = \rho_1$.

Finally, we show that if (10) defines a facet of $K'(n,r)$, then the inequality $\pi x + \rho y \geq \pi_o$ defines a facet of $K(n,r)$. Firstly, this defines a valid inequality for $K(n,r)$ as any point in $K(n,r)$ can be mapped to a point in $K'(n,r)$ by appending zeros in the v_+ and v_- components. If it does not define a facet, then $(\pi, \rho) \geq \sum_i \lambda_i(\pi^i, \rho^i)$ and $\pi_o \leq \sum_i \lambda_i \pi_o^i$ for some non-trivial facet-defining inequalities $\pi^i x + \rho^i y \geq \pi_o^i$ of $K(n,r)$, and some numbers $\lambda_i \geq 0$. But that would imply that $(\pi_1, \rho_1, \pi, \rho) \geq \sum_i \lambda_i(\pi_1^i, \rho_1^i, \pi^i, \rho^i)$. By the first part of the proof, the inequalities $\pi_1^i v_+ + \rho_1^i v_- + \pi^i x + \rho^i y \geq \pi_o^i$ define facets of $K'(n,r)$, and this contradicts the assumption that (10) defines a facet of $K'(n,r)$. ∎

7.1 General mixed-integer sets

Gomory and Johnson used facets of $P(n,r)$ to derive valid inequalities for knapsack sets. In particular, they derived *subadditive functions* from facet coefficients via *interpolation*. We show here how to derive valid inequalities for knapsack sets from facets of $K(n,r)$ via interpolation. For a real number v, we define $\hat{v}$ as $v - \lfloor v \rfloor$.

Definition 7.2 *Given a facet defining inequality $\pi x + \rho y \geq \pi_o$ for $K(n,r)$, let $f^z : \mathbb{Z} \cap [-n,n] \rightarrow \mathbb{R}$ be defined as:*

$$f^z(s) = \begin{cases} \pi_s & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ \rho_{-s} & \text{if } s < 0 \end{cases}$$

We say that $f : [-n,n] \rightarrow \mathbb{R}$ is a facet-interpolated function *derived from (π, ρ, π_o) if*

$$f(v) = (1 - \hat{v}) f^z(\lfloor v \rfloor) + \hat{v} f^z(\lceil v \rceil)$$

The function f, as defined above, equals $f^z(v)$ when v is an integer, and therefore satisfies:

$$f(v) = (1 - \hat{v}) f(\lfloor v \rfloor) + \hat{v} f(\lceil v \rceil). \tag{11}$$

In the next result, we show that continuous functions arising via interpolation from facets of $K(n,r)$ satisfy continuous analogues of the pair-wise subadditivity conditions.

Proposition 7.3 *Let f be a facet-interpolated function associated with $K(n,r)$. Then:*

$$f(u) + f(v) \geq f(u+v) \text{ if } u, v, u+v \in [-n,n].$$

Proof. The proposition is true when u and v are integers; the condition $f(u) + f(v) \geq f(u + v)$ translates to one of (SA1), (SA2), (SA1') or (SA2'). Assume u is not an integer. As $u+v \in [-n, n]$, clearly $\lfloor u + v \rfloor$ and $\lceil u + v \rceil$ also belong to $[-n, n]$.
Case 1: $\hat{u} + \hat{v} \leq 1$. Then $\lfloor u + v \rfloor = \lfloor u \rfloor + \lfloor v \rfloor$ and $\lceil u + v \rceil = \lceil u \rceil + \lfloor v \rfloor$. We can rewrite the expression for $f(u)$ in (11) as

$$f(u) = (1 - \hat{u} - \hat{v})f(\lfloor u \rfloor) + \hat{u}f(\lceil u \rceil) + \hat{v}f(\lfloor u \rfloor). \tag{12}$$

Similarly,

$$f(v) = (1 - \hat{u} - \hat{v})f(\lfloor v \rfloor) + \hat{v}f(\lceil v \rceil) + \hat{u}f(\lfloor v \rfloor). \tag{13}$$

Adding the right-most terms in the above expressions, and using the fact that the proposition is true when u and v are integers, we obtain

$$f(u) + f(v) \geq (1 - \hat{u} - \hat{v})f(\lfloor u + v \rfloor) + \hat{u}f(\lceil u + v \rceil) + \hat{v}f(\lfloor u \rfloor + \lceil v \rceil). \tag{14}$$

If v is an integer, then $\hat{v} = 0$ and the right-hand side of (14) the above expression equals $f(u+v)$. If v is not an integer, then $\lceil u + v \rceil = \lfloor u \rfloor + \lceil v \rceil$, and again the right-hand side of (14) equals $f(u+v)$.
Case 2: $\hat{u}+\hat{v} > 1$. Then $\lfloor u + v \rfloor = \lfloor u \rfloor + \lceil v \rceil = \lceil u \rceil + \lfloor v \rfloor$ and $\lceil u + v \rceil = \lceil u \rceil + \lceil v \rceil$. We can expand $\hat{u}f(\lceil u \rceil)$ in (11) as $(\hat{u} + \hat{v} - 1)f(\lceil u \rceil) + (1 - \hat{v})f(\lceil u \rceil)$. We can similarly expand $\hat{v}f(\lceil v \rceil)$. When we add the expressions for $f(u)$ and $f(v)$ in (11) after writing the expanded terms, we get

$$f(u) + f(v) \geq (\hat{u} + \hat{v} - 1)f(\lceil u + v \rceil) + (2 - \hat{u} - \hat{v})f(\lfloor u + v \rfloor).$$

The right-hand side of the inequality above equals $f(u + v)$. ∎

We say that functions satisfying the property in Proposition 7.3 are *subadditive over the interval* $[-n, n]$. We will see how to generate valid inequalities for knapsack sets from such functions in Proposition 7.5. Also, we can obtain valid inequalities using slightly more restricted functions: we say that f is a *restricted subadditive* function if $f(u) + f(v) \geq f(u + v)$ for $u \in [-n, n]$, and $v, u+v \in [0, n]$. In the next result, we show that facet-interpolated functions satisfy the continuous analogue of the condition (SA3).

Proposition 7.4 *Let f be a facet-interpolated function associated with $K(n, r)$. Then:*

$$f(u) + f(v) + f(w) \geq f(u + v + w) \text{ if } u \in [-n, n], \text{ and } v, w, u + v + w \in [0, n].$$

Proof.(sketch) The proposition is true when u, v and w are integers; the condition $f(u) + f(v) + f(w) \geq f(u + v)$ translates to (SA3). As in the proof of (7.3), we assume either that $\hat{u} + \hat{v} + \hat{w}$ is contained in $(0, 1]$ or $(1, 2]$ or $(2, 3)$. In the first case, we expand $(1 - \hat{u})f(\lfloor u \rfloor)$ as $(1 - \hat{u} - \hat{v} - \hat{w})f(\lfloor u \rfloor) + (\hat{v} + \hat{w})f(\lfloor u \rfloor)$, and proceed similarly for the terms involving $f(\lfloor v \rfloor)$ and $f(\lfloor w \rfloor)$. In the third case, we expand $\hat{u}f(\lceil u \rceil)$ as $(\hat{u}+\hat{v}+\hat{w}-2)f(\lceil u \rceil)+(2-\hat{v}-\hat{w})f(\lceil u \rceil)$, and proceed similarly for the terms involving $f(\lceil v \rceil)$ and $f(\lceil w \rceil)$. The second case has a number of sub-cases. For example, in expanding the terms in the definition of $f(u)$ in (11), we need to consider the value of $\hat{v}+\hat{w}$ with respect to 1. If $\hat{v} + \hat{w} \leq 1$, then we write $\hat{u} f(\lceil u \rceil)$ as $(\hat{u} + \hat{v} + \hat{w} - 1) f(\lceil u \rceil) + (1 - \hat{v} - \hat{w}) f(\lceil u \rceil)$. On the other hand, if $\hat{u} + \hat{v} > 1$, we expand $(1 - \hat{u}) f(\lfloor u \rfloor)$ as $((2 - \hat{u} - \hat{v} - \hat{w}) + (\hat{v} + \hat{w} - 1)) f(\lfloor u \rfloor)$. ■

It is well-known that subadditive functions yield valid inequalities for knapsack sets; the point we emphasize in the next result is that one does not need subadditivity over the entire real line.

Proposition 7.5 *Consider the set $K = \{w \in \mathbb{Z}^p : \sum_{i=1}^p a_i w_i = b\}$, where the coefficients a_i and b are rational numbers. Let t be a number such that $ta_i, tb \in [-n, n]$ and $tb > 0$. If a function f is (i) subadditive over the interval $[-n, n]$ or (ii) satisfies restricted subadditivity and the condition in Proposition 7.4, then*

$$\sum_{i=1}^{p} f(ta_i) w_i \geq f(tb)$$

is a valid inequality for K.

Proof. By scaling, we can assume the coefficients a_i and b in the constraint defining K are all integers contained in the interval $[-m, m]$. Let $t = n/m$. Define the function $g : [-m, m] \to \mathbb{R}$ by $g(w) = f(tw)$. In case (i), g is subadditive over the domain $[-m, m]$. Therefore the vector $\tilde{g} = (g(-m), g(-m+1), \ldots, g(1), \ldots, g(m))$ satisfies (SA1), (SA2), (SA1'), and (SA2') with respect to $K(m, b)$, and (by Remark 2.13)

$$\sum_{i=1}^{p} g(a_i) w_i \geq g(b)$$

is a valid inequality for K. In case (ii), $\tilde{g}$ satisfies (SA1), (SA2) and (SA3) with respect to $K(m, b)$ and by Lemma 2.11 the inequality above is valid for K. ■

We can now give the mixed-integer extension of the previous result.

Theorem 7.6 *Let f be a facet-interpolated function derived from a facet of $K(n, r)$. Consider the set*

$$Q = \left\{ (s, w) \in \mathbb{R}_+^q \times \mathbb{Z}_+^p : \sum_{i=1}^{q} c_i s_i + \sum_{i=1}^{p} a_i w_i = b \right\},$$

where the coefficients of the knapsack constraint defining Q are rational numbers. Let t be such that $ta_i, tb \in [-n, n]$ and $tb > 0$. Then the inequality

$$f(1) \sum_{i=1}^{q} (tc_i)^+ s_i + f(-1) \sum_{i=1}^{q} (-tc_i)^+ s_i + \sum_{i=1}^{p} f(ta_i) w_i \geq f(tb),$$

where $(\alpha)^+ = \max(\alpha, 0)$, is valid for Q.

8 Conclusion

We studied a generalization of the Master Cyclic Group Polyhedron and presented an explicit characterization of the polar of its nontrivial facet-defining inequalities. We also showed that one can obtain valid inequalities for a general MIP that cannot be obtained from facets of the Master

Cyclic Group Polyhedron. In addition, for mixed-integer knapsack sets with rational data and nonnegative variables without upper bounds, our results yield a pseudo-polynomial time algorithm to separate and therefore optimize over their convex hull. This can be done by scaling their data and aggregating variables to fit into the Master Equality Polyhedron framework. Our characterization of the MEP can also be used to find violated Homogeneous Extended Capacity Cuts efficiently. These cuts were proposed in [15] for solving Capacitated Minimum Spanning Tree problems and Capacitated Vehicle Routing problems.

An interesting topic for further study is the derivation of "interesting" classes of facets for the MEP, i.e., facets which cannot be derived trivially from facets of the MCGP or as rank one mixed-integer rounding inequalities.

References

[1] J. Araoz. *Polyhedral Neopolarities.* Phd thesis, University of Waterloo, Department of Computer Sciences, 1974.

[2] J. Araoz, L. Evans, R. E. Gomory, and E. Johnson. Cyclic group and knapsack facets. *Mathematical Programming Ser. B*, 96(2):377–408, 2003.

[3] E. Balas and E. Zemel. Facets of the knapsack polytope from minimal covers. *SIAM Journal of Applied Mathematics*, 34:119–148, 1978.

[4] S. Dash and O. Günlük. On the strength of gomory mixed-integer cuts as group cuts. Technical Report RC23967, IBM Research Division, Yorktown Heights, NY 10598, 2006.

[5] S. Dash and O. Günlük. Valid inequalities based on simple mixed-integer sets. *Mathematical Programming*, 105:29–53, 2006.

[6] S. Dash and O. Günlük. Valid inequalities based on the interpolation procedure. *Mathematical Programming*, 106:111–136, 2006.

[7] M. Fischetti and M. Monaci. How tight is the corner relaxation ? *Discrete Optimization*, 2007. To appear.

[8] M. Fischetti and C. Saturni. Mixed-integer cuts from cyclic groups. *Mathematical Programming A*, 109(1):27–53, 2007.

[9] R. Gomory. Some polyhedra related to combinatorial problems. *Journal of Linear Algebra and its Applications*, 2:451–558, 1969.

[10] R. Gomory and E. Johnson. Some continuous functions related to corner polyhedra i. *Mathematical Programming*, 3:23–85, 1972.

[11] R. Gomory and E. Johnson. Some continuous functions related to corner polyhedra ii. *Mathematical Programming*, 3:359–389, 1972.

[12] R. Gomory and E. Johnson. T-space and cutting planes. *Mathematical Programming*, 96:341–375, 2003.

[13] R. Gomory, E. Johnson, and L. Evans. Cyclic group and knapsack facets. *Mathematical Programming*, 96:321–339, 2003.

[14] E. Uchoa. Robust branch-and-cut-and-price for the cmst problem and extended capacity cuts. Presentation in the MIP 2005 Workshop, Minneapolis (2005). Available at `http://www.ima.umn.edu/matter/W7.25-29.05/activities/Uchoa-Eduardo/cmst-ecc-IMA.pdf`.

[15] E. Uchoa, R. Fukasawa, J. Lysgaard, A. Pessoa, M. Poggi de Aragão, and D. Andrade. Robust branch-cut-and-price for the capacitated minimum spanning tree problem over a large extended formulation. *Mathematical Programming*, To appear.

[16] L. Wolsey. Facets and strong valid inequalities for integer programs. *Oper. Res.*, 24:367–372, 1976.

[17] G. Nemhauser Z. Gu and M. Savelsbergh. Sequence independent lifting in mixed integer programming. *J. Comb. Optim.*, 4:109–129, 2000.