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Mathematics

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An Improved Bound for the Affine Sylvester Problem

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Abstract

In 2006, Lenchner and Bronnimann showed that in the affine plane, given n lines, not all parallel and not all passing through a common point, there had to be at least $\frac{n}{6}$ ordinary points. The present paper improves on this result to show that there must be at least $\frac{2n-3}{7}$ ordinary points, except for a single arrangement of 6 lines with one ordinary point.

1 Introduction

In 1893 J. J. Sylvester posed the following celebrated problem [9]: Given a finite collection of points in the plane, not all lying on a line, show that there exists a line which passes through precisely two of the points. Sylvester's problem was reposed by Erdős in 1944 [3] and solved the same year by Gallai [5]. In its dual formulation, Gallai's result states that given an arrangement of n lines in the real projective plane, not all passing through a common point, there must be a point of intersection of just two lines. Such a point of intersection is commonly referred to as an **ordinary point**.

In 1958 Kelly and Moser [6] showed that an arrangement of n lines as in the statement of Sylvester's problem, must contain at least $3n/7$ ordinary points. They also gave an example of 7 lines with exactly 3 ordinary points. In 1993 Csima and Sawyer [2] showed that except for the case of $n = 7$ there must be at least $6n/13$ ordinary lines in a configuration of n not all collinear points.

Sylvester's problem can also be considered in the affine plane. Given an arrangement of n lines in the affine plane, not all of which are parallel and not all of which pass through a common point, must there always be an ordinary point? In fact there must be, as first pointed out by Lenchner in [7]. The existence of such ordinary points was used recently by Ackerman, et al [1], in their resolution of Murty's Magic Configuration Problem. Until now, the best known lower bound for the number of such affine ordinary points has been $n/6$, as given by Lenchner and Bronnimann [8]. In the present paper, we improve this result to $(2n-3)/7$ as long as the arrangement is not the arrangement of 6 lines with one ordinary point in Figure 1.

Although our problem concerns the affine plane we shall do most of our reasoning in the usual $2D$ model of the projective plane. A **projective ordinary point** is an ordinary point that may lie on the line at infinity. An **affine** or **finite ordinary point** is an ordinary point which necessarily resides off the line at infinity. Unless otherwise qualified, ordinary points in projective arrangements are assumed to be projective and ordinary points in affine arrangements are assumed to be affine.

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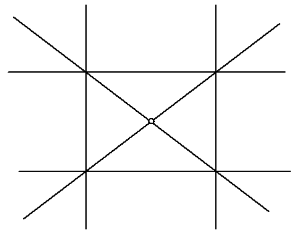

Figure 1: An affine arrangement with six lines and a single (finite) ordinary point.

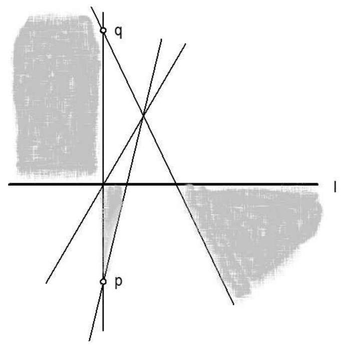

Figure 2: An example of a line l with two ordinary points, p and q, attached through respective shaded triangles. q is attached to l through an infinite triangle.

2 Results from Prior Work

The projective versions of the following definitions and lemmas are due to Kelly and Moser [6]. See Felsner [4] for a contemporary treatment.

Definition 1 *Say that an ordinary point p is* ***attached*** *to a line l, not containing p, if l together with two lines crossing at p form a (possibly infinite) triangular cell of the arrangement. See Figure 2.*

Lemma 1 ***Four Attachment Lemma:*** *In any arrangement of lines, an ordinary point can have at most 4 lines counting that point as an attachment.*

Proof. An ordinary point is contained in 2 crossing lines, and hence is a vertex of at most 4 faces. It can therefore be attached to at most 4 lines. □

The following Lemma actually applies equally to projective or affine arrangements, though we consider it first as a lemma in the projective plane.

Lemma 2 ***Three Clause Lemma:*** *Let T be a triangle formed by three lines of an arrangement $\mathcal{A}$. Let l be one of the defining lines of T, $[p, q]$ the interval of intersection of l and T, and (p, q) the subinterval of $[p, q]$ not containing p, q. If*

(i) T is not a cell of the arrangement,

(ii) (p, q) contains no ordinary points, and

(iii) every line intersecting the interior of T also intersects $[p, q]$

then there exists an ordinary point x attached to l through some triangle contained in T.

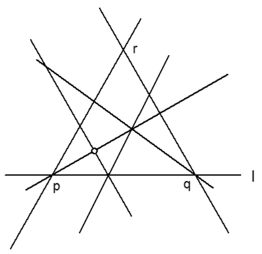

Figure 3: An illustration of the setup and conclusion of the Three Clause Lemma. The triangle T in the Lemma is the finite triangle $\triangle(p, q, r)$ in the figure. The hollow vertex is the ordinary point attached to l in T guaranteed by the Lemma.

Proof. If necessary, rotate the arrangement so that T is a finite triangle. Let x be the vertex of $\mathcal{A}$ in T, not on l, which has the smallest distance to l. If x is ordinary then by the assumptions of the Lemma it is attached to l via a triangle in T and we are done. On the other hand suppose there are three lines l_1, l_2, l_3 intersecting in x, and let v_1, v_2, v_3 be their respective intersection points with l. By assumption, all the v_i lie in $[p, q]$ and we assume v_2 lies between v_1 and v_3. Since v_2 is not ordinary there is a line $m \neq l_2$ entering T at v_2. The intersection of m and l_1 or m and l_3 is of smaller distance to l than x, a contradiction. □

We observe that the Lemma holds in the affine plane so long as the triangle T is a finite triangle, the "finite" triangle possibly containing two parallel lines for edges (the important factor being that the line at infinity should not divide T into two distinct components).

Definition 2 *A line is said to be of* ***projective type*** *(i, j) if it contains i projective ordinary points and has j projective ordinary points attached to it.*

In what follows we assume that we have an arrangement in which all lines contain at least three projective vertices. Arrangements having a line with just two vertices are easily seen to have at least $n - 2$ projective ordinary points and at least $\frac{n-3}{2}$ affine ordinary points. The Three Clause Lemma (Lemma 2) can then be used to prove the following two lemmas:

Lemma 3 ***Projective (0,3+)-Lemma:*** *If a line l of an arrangement $\mathcal{A}$ contains no projective ordinary points, then there are at least 3 projective ordinary points attached to l.*

Lemma 4 ***Projective (1,2+)-Lemma:*** *If a line l of an arrangement $\mathcal{A}$ contains a single projective ordinary point, then the line l has at least 2 projective ordinary points attached to it.*

The following are from Lenchner and Bronnimann [8]:

Definition 3 *A line is said to be of* ***affine type*** *(i, j) if it contains i affine ordinary points and has j affine ordinary points attached to it.*

Lemma 5 ***Affine (0,1+) Lemma:*** *Let $\mathcal{A}$ be a nontrivial affine arrangement of n lines. If a line $l \in \mathcal{A}$ contains no finite ordinary point, then it must have at least one finite ordinary point attached to it.*

Proof. If all the finite vertices are on a single line, then all but that line must be parallel, and all vertices are ordinary. Thus, in this case, there is no line without finite ordinary points.

Let $l \in \mathcal{A}$ be a line without finite ordinary points and let x be the closest (finite) vertex on one side of l, and rightmost if there are several such vertices. Arguing as in the proof of the Three Clause Lemma (Lemma 2) we find that x is ordinary and attached to l. □

3 New Contributions

Lemma 6 ***Sharp Affine (0,1+) Lemma:*** *Let $\mathcal{A}$ be an affine arrangement which is not the six line arrangement with one finite ordinary point in Figure 1, then at most three lines of $\mathcal{A}$ can have affine type $(0, 1)$. All other lines without finite ordinary points must have two or more finite attached points.*

Proof. (Sketch) Consider any line $l \in \mathcal{A}$ with no ordinary points. If l has only one attached point then the vertices of $\mathcal{A}$ not contained in l are either all to one side of l or the other - otherwise we could argue as in the proof of the Affine $(0, 1+)$ Lemma (Lemma 5) to conclude that a closest vertex to l on either side is an attached point. Call this property of having all vertices to one side or another of a given line, the **All to One Side Property**. If we have exactly four lines with the All to One Side Property, then a simple case analysis shows that the arrangement must be one of those in Figure 4.

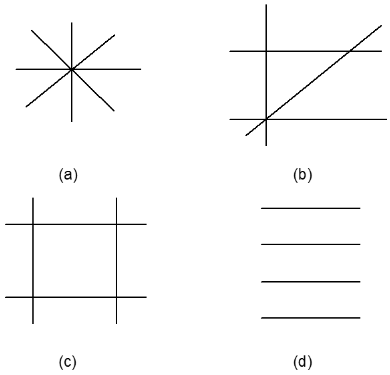

Figure 4: The four four line examples with each line having the All to One Side Property.

If we had four $(0, 1)$ lines these lines would then necessarily form one of the four subarrangements in Figure 4. It is easy to see that adding an additional line not through the common intersection point in arrangement (a) forces one of the pre-existing lines to lose the All to One Side Property, and analogously for arrangement (d) if we add a line not parallel to the existing lines. Only cases (b) and (c) are interesting, but in these cases it is impossible to add lines, keeping the All to One Side Property for each line while also limiting each line to no ordinary points unless we have the arrangement in Figure 1. □

Lemma 7 *In an affine arrangement, two lines of affine type* $(1, 0)$ *cannot intersect in their finite ordinary point unless the arrangement consists just of two intersecting lines or is the six line arrangement in Figure 1.*

Proof. Suppose two $(1, 0)$ lines l, k intersect at their ordinary point p. Under the assumption that there are more than just these two lines, there is another vertex, q, on one of the lines, say on k. q is not ordinary so there are at least two lines passing through q in addition to k. By the Three Clause Lemma (Lemma 2) the two lines must intersect l at the closest (finite) vertices to p on either side. Applying the same reasoning to these two closest vertices to p on l, we see that p is surrounded by finite triangular cells. An additional application of the Three Clause Lemma allows us to conclude that each vertex around p is a 3-crossing.

Now consider the projective cells surrounding the triangular cells about p. Ordering these cells clockwise, either there are two opposite pairs of projective triangular cells or two consecutive projective (4+)-gons. If there are opposite pairs of projective triangular cells, then the triangles meet at a possibly infinite vertex z. Any additional line would have to pass through z (in the projective sense if z is infinite) and so create an additional finite ordinary point on k and l, which is impossible. Hence we have just a six line arrangement. If z were finite it would be attached to both k and l which is impossible since both lines are of affine type $(1, 0)$. Hence z must be at infinity. If l_1, l_2 are the two lines meeting at z and l_3, l_4 are the other line contributing edges to the triangular cells about p, then since there are no additional lines, l_3, l_4 similarly cannot intersect at a finite point and so we have the six line arrangement in Figure 1.

On the other hand, suppose that there are two consecutive projective (4+)-gons surrounding triangular cells about p; see Figure 5. There are two cases: either (1)

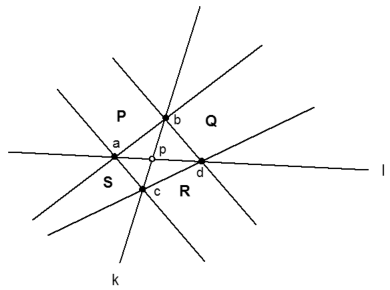

Figure 5: A hypothetical arrangement with two lines, l, k of affine type $(1, 0)$ with common ordinary point. We consider the case where two consecutive projective cells surrounding the triangular cells about p, Q and R say, are (4+)-gons.

there is a line crossing into and forming an edge of Q in the northeast quadrant, or there is a line crossing into and forming an edge of R in the southeast quadrant or (2) there are no such lines. In case (1) suppose such a crossing line and edge exists for Q. Then if $e_1 = [b, d]$, there are two successive edges, going either clockwise around Q with e_2 lying on $l(a, b)$ and the line extending e_3 intersecting k at a finite point north of b, or going counterclockwise around Q with e_2 lying on $l(c, d)$ and the line extending e_3 intersecting l at a finite point east of d. The argument is the same in either case, so suppose we have the latter situation and the line extending e_3 intersects l at a finite point g as depicted in Figure 6. Then, since a line cannot pass through f into Q, we can apply the Three Clause Lemma to conclude that l must have a finite ordinary or attached point in $[d, g]$, contrary to assumption.

With case (1) handled, let us revert back to considering Figure 5 and consideration of case (2), where there are no lines crossing into and creating an edge of Q in the

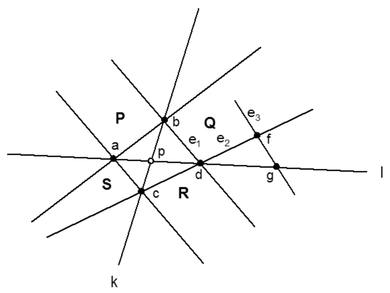

Figure 6: The hypothetical arrangement with two lines, l, k of affine type $(1, 0)$ sharing an ordinary point, two consecutive projective cells surrounding the triangular cells about p, Q and R which are (4+)-gons, and a line extending the third edge (e_3) in counter-clockwise order around Q from $e_1 = [b, d]$.

northeast quadrant, and also no lines crossing into and forming an edge of R in the southeast quadrant. Note that a line of affine type $(1, 0)$ must either have an infinite ordinary point or two infinite attached points by the Projective $(1, 2+)$ Lemma (Lemma 4). By our assumption about Q and R, there cannot be a line parallel to l, so l cannot contain an infinite ordinary point. We show that l cannot even have a single infinite attached point. Again by the assumptions on Q and R, there can be no lines in the arrangement intersecting l to the east of d. Thus any pair of parallel lines giving rise to an infinite ordinary point attached to l must intersect l at or to the west of a. However any pair of parallel lines forming an attached point to the north of l must not be cut to the north by either $l(a, c)$ or $l(b, d)$. But unless the parallel lines are parallel to or identical with $l(a, c)$ and $l(b, d)$, they will cut into R, which is impossible. They cannot be parallel with $l(a, c)$ and $l(b, d)$ since then they would not meet at an ordinary point. Further, if the parallel lines were actually equal to $l(a, c)$ and $l(b, d)$ we would have our familiar six line arrangement in Figure 1 - but actually in this case Q and R would not be (4+)-gons. The argument is clearly the same for a pair of parallel lines forming an attached point to the south of l. It follows that in case (2), l cannot have either an infinite ordinary or attached point, and so this case is ruled out and the Lemma is established. $\square$

Theorem 8 *Let $\mathcal{A}$ be an affine arrangement of n not all parallel lines, which in addition do not all pass through a common point. If $\mathcal{A}$ is not the arrangement in Figure 1 then $\mathcal{A}$ has at least $\frac{2n-3}{7}$ (finite) ordinary points.*

Proof. Let p be the fraction of lines with one ordinary point and no attachments and m the number of (finite) ordinary points. Then, by virtue of Lemma 7, we have

$$m \geq pn, \tag{1}$$

and, under the assumption that $n \neq 6$, counting either ordinary point-line or attached point-line associations, and applying the Sharp Affine (0,1+) Lemma (Lemma 6) together with the Four Attachment Lemma (Lemma 1) we have

$$6m \geq 3 + pn + 2(n - (pn + 3)), \tag{2}$$

where 3 is the maximum number of $(0, 1)$ lines, pn is the number of $(1, 0)$ lines and $n - (pn + 3)$ is the number of lines with at least a total of two ordinary plus attached points. Hence

$$m \geq \frac{(2-p)n}{6} - \frac{1}{2}. \tag{3}$$

By (1), $-pn \geq -m$, so substituting this into 3 gives

$$m \quad \geq \quad \frac{2n-3}{7}$$

$\square$

4 Conclusion

Lemma 7 is a generalization of the main lemma of Csima and Sawyer [2] which states that in a projective arrangement no two lines of (projective) type $(2, 0)$ can intersect in an ordinary point.

We conjecture that the asymptotic best lower bounds for the affine and projective variants of the Sylvester problem differ just by a constant. However, at present, this conjecture seems very hard to prove.

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