RC24313 (W0707-120) July 20, 2007
Mathematics

IBM Research Report

Some Remarks on Origami and Its Limitations

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SOME REMARKS ON ORIGAMI AND ITS LIMITATIONS

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Abstract. From a mathematical point of view the Japanese art of Origami is an art of finding isometric injections of subsets of $\mathbb{R}^2$ into $\mathbb{R}^3$. Objects obtained in this manner are developable surfaces and they are considered to be fully understood. Nevertheless, until now it was not known whether or not the local shape of the Origami model determines the maximum size and shape of the sheet of paper it can be made of. In the present paper we show that it does. We construct a set $\Omega \subset \mathbb{R}^2$ containing the point $(0, \frac{1}{2})$ and an isometry $F : \Omega \to \mathbb{R}^3$ such that for every neighborhood $\omega \subseteq \Omega$ of the point $(0, \frac{1}{2})$ and for every $\varepsilon > 0$ and $\delta > 0$, F restricted to ω cannot be extended to an isometry of the set $\{-\varepsilon < x < \varepsilon, -\delta < y < 1+\delta\}$ into $\mathbb{R}^3$. We also prove that all the singularities of an Origami model are of the same type – there can appear only cones.

1. Introduction and statement of the result

Origami is an ancient Japanese art of folding the paper. There are numerous books devoted to its various aspects, see for example [2], [3]. In its traditional form Origami requires the paper being folded to be a square, but some more modern approaches allow the use of other shapes, usually rectangles. Thus the mathematical tool for describing Origami would be a theory of isometric injections of subsets (mostly square or rectangular) of $\mathbb{R}^2$ into $\mathbb{R}^3$. Let us define more rigorously those notions.

Definition 1 (Isometric injection)**.** A mapping $F : D \to \mathbb{R}^3$ of a simply connected region $D \subset \mathbb{R}^2$ is an *isometric injection* if and only if it is an injection and for every $x, y \in \mathbb{R}^2$ there holds $|x - y| = \varrho(F(x), F(y))$. The symbol $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$, and ϱ denotes a distance between two points in the submanifold $F(D)$ of $\mathbb{R}^3$ in a Riemannian metric inherited from $\mathbb{R}^3$.

Definition 2 (Developable surface)**.** By a *developable surface* S in $\mathbb{R}^3$ we shall understand a simply connected region $D \subset \mathbb{R}^2$ and a smooth isometric injection

$$F : D \to \mathbb{R}^3$$

with isolated singularities. Whenever it does not lead to a misunderstanding we shall also use the term "surface" when speaking of the geometric object – the image of the map F itself.

Note that in the above definition we do not allow self–intersections of the surfaces. In particular every point of the surface has uniquely defined preimage.

Definition 3 (Germ of a developable surface at point p)**.** Let p be an arbitrary point in $\mathbb{R}^3$. We define an equivalence relation $\sim_p$ in the set of all developable surfaces containing the point p as follows: the developable surface $F_1 : D_1 \to \mathbb{R}^3$ is in relation $\sim_p$ with $F_2 : D_2 \to \mathbb{R}^3$ if and only if there exist an isometry $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and a neighborhood ω of the point $F_1^{-1}(p)$ such that $\phi(F_1^{-1}(p)) = F_2^{-1}(p)$ and $F_2 \circ \phi|_\omega \equiv F_1$.
A *germ of a developable surface at the point* p (or simply *a germ of a developable surface*) is an equivalence class of the relation $\sim_p$.

The natural question that arises is, if given an Origami model made of a sheet of paper, the same model can be realized as a part of a model made of a larger piece of paper. Origami allows us to create the edges when folding the paper, so in particular we can always fold a sheet of paper in two to downsize it, and thus the question has a trivial positive answer. What would happen tough if this was not allowed? In our model we consider Origami models without edges, that is developable surfaces with only isolated singularities. It turns our that under this constrain the the question becomes more interesting and has a nontrivial answer. In the present paper we construct and Origami model that cannot be extended to a model made using a larger sheet of paper. Moreover, this model contains a point such that no submodel of the original model containing this point can be extended to a model made of a large sheet of paper. Using the definitions we can formulate our main result in a rigorous way as follows:

Theorem I. *There exist germs of developable surfaces such that they determine the maximum size of a domain of the definition of every developable surface in their equivalence class.*

The second result of the paper is the classification of all germs of developable surfaces. It is done in Section 2.

In Section 3 we prove Theorem I by constructing an example of a germ of developable surface impossible to be obtained as a part of any surface which is too large.

2. Local form of a developable surface

2.1. **Non–singular points.** It is well known that each developable surface has gaussian curvature $K = \frac{b}{g}$ identically equal to zero (see for example [1]). As we look at the local situation, we may limit ourselves to considering the graphs of functions

$$z : \mathbb{R}^2 \to \mathbb{R}.$$

The condition $K \equiv 0$ means that the Hessian of z $(\det(d^2 z(x, y))$ satisfies

$$Hz \equiv 0 \tag{2.1}$$

for the non–singular points (x, y). We use it for a short proof of a well known result that in a neighborhood of its each non–singular point a developable surface is a ruled surface (see for example [1]):

Proposition 1. *Take a non–singular point $(x_0, y_0, z_0) \in S$. Then there exists an segment I in $\mathbb{R}^3$ such that*

$$(x_0, y_0, z_0) \in I \subset S.$$

Moreover, the segment I can be prolonged in both directions either to infinity, or to the boundaries of S or till it reaches a singular point of S.

Proof. After a suitable linear change of coordinates we may assume that $(x_0, y_0) = (0, 0)$, and

$$d^2 z \mid_{(x_0,y_0)} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}.$$

(where $*$'s represent terms that might be possibly zeroes). From (2.1) it follows that $z_{yy} \equiv \frac{z_{xy}^2}{z_{xx}}$, and therefore

$$\frac{\partial}{\partial y} z_{xy} = \frac{\partial}{\partial x} z_{yy} = \frac{2 z_{xy} z_{xyx} z_{xx} - z_{xy}^2 z_{xxx}}{z_{xx}^2}. \tag{2.2}$$

We define $v(t) = z_{xy}(0, t)$. Restricting the equation (2.2) to the line $(0, t) \subset \mathbb{R}^2$ we see that v satisfies the (Bernoulli) ordinary differential equation $v' = A(t)v + B(t)v^2$ with initial condition $v(0) = 0$. This equation is well defined at all non–singular points of S, i.e. in the points where z is smooth and $d^2 z \neq 0$. Therefore, by uniqueness of solutions of ordinary differential equations $v \equiv 0$, $z_{yy}(0, t) \equiv 0$ and thus $z(0, t) = \alpha t + \beta$. □

2.2. **Singular points.** Now we must classify all the possible singular points of developable surfaces S. Assume that $(0, 0)$ is a unique singular point of S in a small disc $B = B(0, \varepsilon)$.

As every isometry maps straight lines onto straight lines, the structure of our developable surface induces a division of the disc B into non–intersecting segments. Moreover their ends must lie either both on the boundary of B or one end on the boundary of B and the other one in the middle of B. In other words, it defines a foliation $\mathcal{F}$ of a punctured disc, whose leaves are segments. This means defining a smooth function $\Phi : \mathbb{R}/2k\pi \to \mathbb{R}/2k\pi$ as follows: let φ be a coordinate on ∂B, and let $\Phi(\varphi) = \alpha$ where α is the oriented angle between the radius corresponding to φ and the (unique) segment from $\mathcal{F}$ having a common endpoint with that radius. See Figure 1.

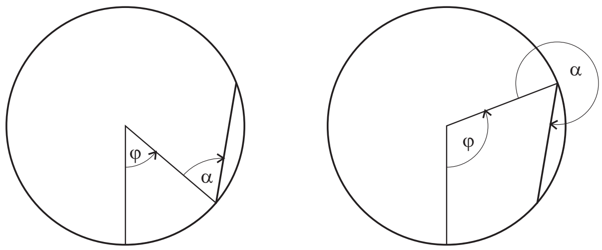

Figure 1: Angles φ and α.

One can easily verify the following

Lemma 1 (Properties of Φ)**.** *The map Φ has the following properties*

i) $\Phi(\varphi + 2\pi) = \Phi(\varphi)$
ii) $\Phi(\varphi) \neq 0 \;\Rightarrow\; \Phi(\varphi + \pi - 2\Phi(\varphi)) = -\Phi(\varphi)$ *(see Fig. 2a)*
iii) $\Phi(\varphi) = 0 = \Phi(\psi)$ *and* $\mid \varphi - \psi \mid < \pi$ *implies that for every* ξ *between* φ *and* ψ *there holds* $\Phi(\xi) = 0$ *(see Fig. 2b)*
iv) $\Phi(\varphi) = 0 = \Phi(\psi)$ *and* $0 < \mid \varphi - \psi \mid < \pi$ *implies that there exist* $\tilde{\varphi} \leq \varphi < \psi \leq \tilde{\psi}$ *such that* $\tilde{\psi} - \tilde{\varphi} = \pi$ *and for every* ξ *between* $\tilde{\varphi}$ *and* $\tilde{\psi}$ *there holds* $\Phi(\xi) = 0$ *(see Fig. 2c)*

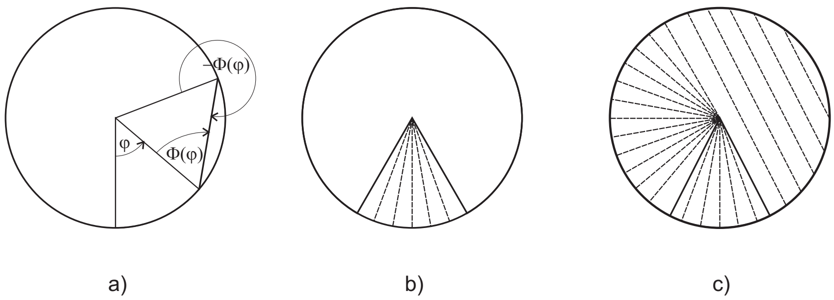

Figure 2: Properties of the function Φ.

This leaves us with three types of foliations (see Figure 3), which we call:

I the cone (umbrella) type
II the removable (pipe) type
III the mixed (sunset) type

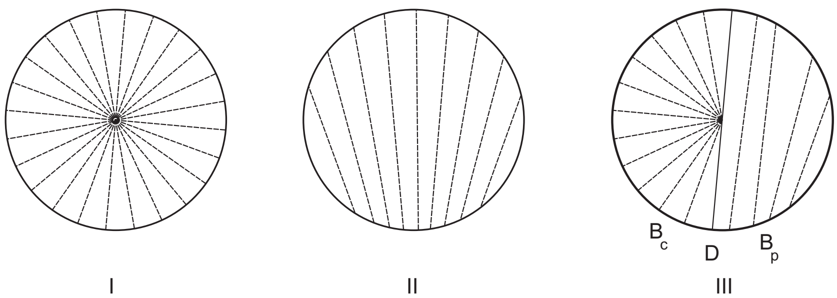

Figure 3: Three types of singularities.

Given a foliation, we shall look at the germs of surfaces possible to be obtained from it.
Let us deal with the case I (cone) first.

Let S be the image of function $F : B \to \mathbb{R}^3$. The function F maps segments of the foliation $\mathcal{F}$ onto segments in $\mathbb{R}^3$, therefore the family of germs of surfaces is equivalent to the family of length-preserving immersions from unit circle in $\mathbb{R}^2$ into a unit sphere in $\mathbb{R}^3$.

Now let us deal with the III (mixed) type points

Lemma 2 (No sunsets)**.** *Non–trivial points of the type III (mixed) do not exist on developable surfaces.*

Proof. The idea is following–"one cannot glue a half-cone to the half–pipe".
The disc B is divided by the diagonal D into two half-discs B_p and B_c (see Fig 3). The image of D is a segment in $\mathbb{R}^3$. The ends of D are antipodic points in a unit sphere, and they are connected by the image of half of the unit circle, bounding B_c. As our map is length-preserving, we immediately obtain, that this image must be a half of equator, so the image of B_c must be flat. Therefore our critical point is, in fact, a I type point.

□

The II case is in fact a regular case, but surprisingly, existence of this type of pints leads to the non–extendability of germs of developable surfaces.

3. Proof of the Main Theorem

In order to prove Theorem I we construct an example. As we have seen, all the critical isolated points of developable surfaces are of the cone type. Every such cone consists of segments "attached" to the singular point. Of course, all of the segments can be prolonged to semi–lines without any obstacles. Therefore, every possible germ of a cone type can be obtained as a fragment of a surface $F : \mathbb{R}^2 \to \mathbb{R}^3$. In the other words, all the local shapes can be obtained from an infinite sheet of paper, and therefore from a sheet of paper of any arbitrary shape.

Therefore the critical points are no problem for Origami. As we shall see, some regular points can be a problem. Below we present an example of a germ of a surface, which is made of some sheet Ω containing a square $\{| x |\leq 1,\ 0 <| y |< 1\}$, but it cannot be made with any rectangle larger than a unit square.

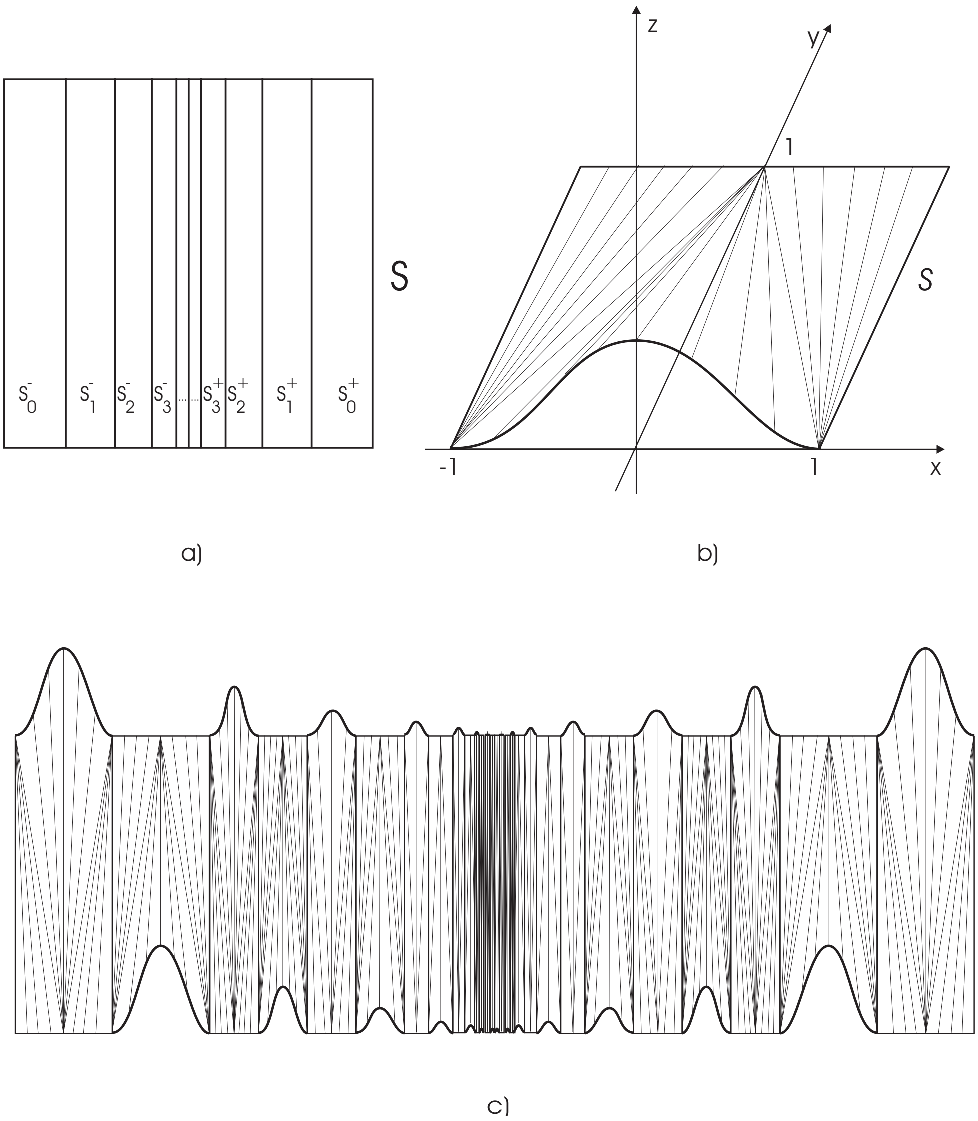

Figure 4: The example.

First let us introduce the function Ξ. It maps the rectangle $S = \{\mid x \mid \leq 1,\ 0 < \mid y \mid < 1\} \subset \mathbb{R}^2$ into the interval $[0, 1]$. We define it as follows:

$$\Xi(x,y) = \begin{cases} e^{-\frac{y^4}{(x^2-y^2)^2}} & \mid x \mid < \mid y \mid \\ 0 & \mid x \mid \geq \mid y \mid \end{cases}$$

The graph of Ξ is a kind of cone over the graph of a function $e^{-1/(x^2-1)^2}$ extending smoothly into a flat surface over a square. Now we shall construct a surface by gluing together such cones. Precisely, we introduce a division of a square $S^+ = \{0 \leq x \leq 1,\ 0 < y < 1\}$ into vertical stripes $S_n^+ = \{\frac{1}{2^{n+1}} < x \leq$

$\frac{1}{2^n}$, $0 < y < 1\}$, $n = 0, 1, \ldots$. We define a function $F : S \to [0,1]$ as

$$F(x,y) = \begin{cases} \frac{1}{e^n}\Xi(2^{n+2}(x - \frac{3}{2^{n+2}}), y) & (x,y) \in S_n^+ \quad n = 2k \\ \frac{1}{e^n}\Xi(2^{n+2}(x - \frac{3}{2^{n+2}}), 1-y) & (x,y) \in S_n^+ \quad n = 2k+1 \\ 0 & x = 0 \end{cases}$$

and finally we extend F to whole S by putting $F(x,y) = F(-x,y)$ for $x < 0$. Now the graph of F is the desired surface! It easy to check that it is smooth (but not analytic) and developable. After the unfolding we shall obtain a weird strip of paper (the set Ω), which contains a small square around the point $(0, \frac{1}{2})$. The edges of a cones form a barrier prevent us from extending the surface below $y = 0$ and above $y = 1$.

Remark 1. Note, that as our surface is defined as a graph of nontrivial $F : S \to \mathbb{R}^3$, after unfolding it does not unfold to S, so Ω is not simply the unit square.

4. Acknowledgments

The second author was partially supported by by Polish KBN Grant 2 PO3A 022 08.

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