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On Test Sets for Nonlinear Integer Maximization

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Abstract

A finite test set for an integer optimization problem enables us to verify whether a feasible point attains the global optimum. We establish in this paper several general results that apply to integer optimization problems with nonlinear objective functions.

Key words: Integer programming, test sets, certificates Hilbert basis, Gordan Lemma, superadditive

1 Introduction and related work

Given a feasible point x^* of an optimization problem P , one important concern is to establish a set of points $T = T(x^*, P)$ with which one can verify whether x^* is optimal for P . We refer to such a set T as a *test set.* Usually, for a test set T , we compare feasible elements of $\{x^* + t \ : \ t \in T\}$, via objective value, against x^* . Our goal is to establish, in various settings, the existence of a *finite* test set T . An initial feasible solution together with a description of a finite test set for any feasible point allows us to design an algorithm that iteratively builds up a sequence of better and better points.

It is preferred that a test set T does not depend on x^* , but this is not always possible. In general, it is interesting to establish a finite test set T and to understand the dependence of T on x^* and on the parameters that define P .

Before proceeding, we briefly set some nota-

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tion. $\mathbb{Z}$ (resp., $\mathbb{R}$) denotes the set of integers (resp., real numbers). $\mathbb{Z}_+$ (resp., $\mathbb{Z}_-$) denotes the set of non-negative (resp., non-positive) integers, and we use $\mathbb{Z}_{-\infty} := \mathbb{Z} \cup \{-\infty\}$, $\mathbb{Z}_{+\infty} := \mathbb{Z} \cup \{+\infty\}$ and $\mathbb{Z}_{\pm\infty} := \mathbb{Z} \cup \{\pm\infty\}$. Analogously, we use such notation for $\mathbb{R}$. For $S \subset \mathbb{Z}^n$, and simple variable bounds $l \in \mathbb{Z}^n_{-\infty}$, $u \in \mathbb{Z}^n_{+\infty}$, with $l \leq u$, let

$$F(S, l, u) := \{x \in S \ : \ l \leq x \leq u\}.$$

For a function $f : \mathbb{Z}^n \rightarrow \mathbb{R}$, we consider the optimization problem

$$P(f(x), S, l, u): \quad \max\{f(x) \ : \ x \in F(S, l, u)\}.$$

Often, we focus on the case in which S is defined by linear equations. For $A \in \mathbb{Z}^{m\times n}$ and right-hand side $b \in \mathbb{Z}^m$ (l, u as above), we sometimes consider S of the form $S := \{x \in \mathbb{Z}^n : Ax = b\}$, and we write

$$F(A, b, l, u) := \{x \in \mathbb{Z}^n \ : \ Ax = b, \ l \leq x \leq u\}$$

and

$$P(f(x), A, b, l, u): \quad \max\{f(x) \ : \ x \in F(A, b, l, u)\}.$$

Also, we write $L(A) := \{x \in \mathbb{R}^n \ : \ Ax = 0\}$. An *augmentation* for $x^* \in F(A, b, l, u)$ is a $t \in \mathbb{Z}^n$ such that $x^* + t \in F(A, b, l, u)$. Necessarily, an augmentation t is in $L(A)$. The augmentation t is *improving* if $f(x^*) < f(x^* + t)$.

We let O^n_j denote the j-th orthant of $\mathbb{R}^n$, for integers j satisfying $0 \leq j < 2^n$. Specifically, the j-th orthant of $\mathbb{R}^n$ is defined, for $0 \leq j < 2^n$, by having $x_k \geq 0$ (resp., $x_k \leq 0$) if bit k is 0 (resp., 1) in the binary representation of j . Hence $O^n_0 = \mathbb{R}^n_+$. When we need to work with an arbitrary orthant (of $\mathbb{R}^n$), we often simply use O .

While it is no surprise that for a linear *continuous* optimization problem a finite test set (appropriately defined) can be given, the situation becomes more difficult when we consider linear *integer* optimization. The following general result establishes a finite test set for linear integer optimization.

Theorem 1.1 (Graver [3]) *For all $A \in \mathbb{Z}^{m\times n}$, there exists a finite set $T(A) \subset \mathbb{Z}^n \cap L(A)$ such that for every $c \in \mathbb{R}^n$, $b \in \mathbb{Z}^m$, and $l \in \mathbb{Z}^n_{-\infty}$, $u \in \mathbb{Z}^n_{+\infty}$, with $l \leq u$, the point $x^* \in F(A, b, l, u)$ is optimal for $P(c^T x, A, b, l, u)$ if and only if $c^T(x^* + t) \leq c^T x^*$ for all $t \in T(A)$ such that $x^* + t \in F(A, b, l, u)$.*

One such set $T(A)$ is the so-called *Graver basis* $G(A)$, which will be defined and used in Section 2. It turns out that Theorem 1.1 can be extended to a broader class of functions.

Let $\mathcal{F}$ be the set of functions $f : \mathbb{Z}^n \rightarrow \mathbb{R}$ of the form $f(x) = \sum_{i=1}^r \phi_i(c_i^T x)$, where $c_i \in \mathbb{Z}^n$ and $\phi_i : \mathbb{R} \rightarrow \mathbb{R}$ is concave (univariate), for $i = 1, \ldots, r$.

Theorem 1.2 (Murota, Saito and Weismantel [5]; Hemmecke[4]) *For all $f \in \mathcal{F}$ and $A \in \mathbb{Z}^{m\times n}$, there exists a finite set $T(f(x), A) \subset \mathbb{Z}^n \cap L(A)$ such that for every $b \in \mathbb{Z}^m$, and $l \in \mathbb{Z}^n_{-\infty}$, $u \in \mathbb{Z}^n_{+\infty}$, with $l \leq u$, a point $x^* \in F(A, b, l, u)$ is optimal for $P(f(x), A, b, l, u)$ if and only if $f(x^* + t) \leq f(x^*)$ for all $t \in T(f(x), A)$ such that $x^* + t \in F(A, b, l, u)$.*

It is our intention with this note to extend both Theorem 1.1. and 1.2 to a broader class of functions with a property related to super- and subadditivity, that will be defined in Section 3.

The paper is organized as follows. In Section 2, we study the general problem $P(f(x), S, l, u)$ with varying l, u . In this case it is possible to derive, for every feasible point x^* , a finite set for verifying its optimality. In Section 3 we introduce the notion of oriented subadditive and superadditive functions and exploit their structure. In Section 4, we construct finite test sets for integer optimization problems where the objective function has certain oriented subadditive and superadditive properties on an orthant refinement. In this setting, the sets that we construct are always universal in that they do not depend on the feasible point x^* that we test for optimality.

2 Finite test sets for feasible points

Define a partial order $\sqsubseteq$ on $\mathbb{Z}^n$ that extends the coordinate-wise partial order $\leq$ on $\mathbb{Z}^n_+$ as follows: For a pair of vectors $u, v \in \mathbb{Z}^n$, we write $u \sqsubseteq v$ and say that u is *conforms* to v if $|u_i| \leq |v_i|$ and $u_i v_i \geq 0$ for $i = 1, \ldots, n$, that is, u and v lie in the same orthant of $\mathbb{Z}^n$, and each component of u is bounded by the corresponding component of v in absolute value. Points with some zero components are in multiple orthants, but it is easy to see that $\sqsubseteq$ is well defined.

Here and throughout the paper, we make heavy use of the following natural extension to $\sqsubseteq$ and $\mathbb{Z}^n$ of the well-known Gordan Lemma [2] for $\leq$ and $\mathbb{Z}^n_+$.

Lemma 2.1 (Extended Gordan Lemma) *For every set $S \subset \mathbb{Z}^n$, the set $T(S) \subset S$ of $\sqsubseteq$-minimal elements of S is finite.*

Applying Lemma 2.1 we obtain

Theorem 2.2 *For every set $S \subset \mathbb{Z}^n$, function $f : \mathbb{Z}^n \to \mathbb{R}$, and point $x^* \in S$, there is a finite set $T(x^*, f(x), S) \subset \mathbb{Z}^n$ such that, for every $l \in \mathbb{Z}^n_{-\infty}$, $u \in \mathbb{Z}^n_{+\infty}$ with $l \leq x^* \leq u$, the point x^* is optimal for $P(f(x), S, l, u)$ if and only if there is no $t \in T(x^*, f(x), S)$ with $l \leq x^* + t \leq u$.*

PROOF. Let

$$H(x^*, f(x), S) := \{h \in \mathbb{Z}^n \ : \ x^* + h \in S,\ f(x^*) < f(x^* + h)\}.$$

We claim that the set $T(x^*, f(x), S) \subset H(x^*, f(x), S)$ of $\sqsubseteq$-minimal elements, guaranteed to be finite by the Extended Gordan Lemma 2.1, is the desired set. Consider any $l \in \mathbb{Z}^n_{-\infty}$, $u \in \mathbb{Z}^n_{+\infty}$ with $l \leq x^* \leq u$. If x^* is not optimal for $P(f(x), S, l, u)$, then there is an $\bar{x} \in S$ with $l \leq \bar{x} \leq u$ and $f(x^*) < f(\bar{x})$, and hence $h := \bar{x} - x^* \in H(x^*, f(x), S)$. Therefore there is a $t \in T(x^*, f(x), S)$ with $t \sqsubseteq h$. Now $t \sqsubseteq h$ and $l \leq x^*, x^* + h = \bar{x} \leq u$ imply that $l \leq x^* + t \leq u$.

Conversely, if there is a $t \in T(x^*, f(x), S)$ with $l \leq x^* + t \leq u$, then $T(x^*, f(x), S) \subset H(x^*, f(x), S)$ implies that $x^* + t \in F(S, l, u)$ and $f(x^*) < f(x^* + t)$, and therefore x^* is not optimal for $P(f(x), S, l, u)$. This completes the proof. □

We next present a refined result for the case in which the set S is defined using linear equations. For this we need to use the Graver basis $G(A)$ of an $m \times n$ integer matrix A, which can be defined as follows: $G(A) := T(S)$ is the set of all $\sqsubseteq$-minimal elements in $S := \{x \in \mathbb{Z}^n \cap L(A) \ : \ x \neq 0\}$.

Theorem 2.3 *Let $f : \mathbb{Z}^n \to \mathbb{R}$ be a function, and let $A \in \mathbb{Z}^{m \times n}$. For every $x^* \in \mathbb{Z}^n$, there exists a finite set $T(x^*, f(x), A) \subset \mathbb{Z}^n \cap L(A)$ such that for all $l \in \mathbb{Z}^n_{-\infty}$, $u \in \mathbb{Z}^n_{+\infty}$ with $l \leq x^* \leq u$, letting $b^* := Ax^*$, $x^* \in F(A, b^*, l, u)$ is optimal for $P(f(x), A, b^*, l, u)$ if and only if there is no $t \in T(x^*, f(x), A)$ with $l \leq x^* + t \leq u$.*

PROOF. Let $G(A) = \{g_1, \ldots, g_k\}$ be the Graver basis of A . Let

$$\mathcal{A}(x^*) \ := \ \{\alpha \in \mathbb{Z}^k_+ \ : \ f(x^*) < f(x^* + \textstyle\sum_{i=1}^k \alpha_i g_i)\} \ .$$

Let $\mathcal{B}(x^*) \subset \mathcal{A}(x^*)$ be the subset of $\mathcal{A}(x^*)$ of $\leq$-minimal elements, which is finite by the (standard) Gordan Lemma. We claim that the desired test set is provided by

$$T(x^*, f(x), A) := \left\{\sum_{i=1}^k \beta_i g_i \ : \ \beta \in \mathcal{B}(x^*)\right\} .$$

First, note that $T(x^*, f(x), A) \subset L(A)$ and therefore, for all $t \in T$, we have $A(x^* + t) = Ax^* = b^*$. Also, for all $t \in T(x^*, f(x), A)$, we have $f(x^*) < f(x^* + t)$ by the construction of $T(x^*, f(x), A)$. So if there is a $t \in T(x^*, f(x), A)$ with $l \leq x^* + t \leq u$ then $x^* + t$ is a better feasible point than x^*, so x^* is not optimal. Conversely, suppose that x^* is not optimal and let x' be a better feasible point. Let $h := x' - x^*$. Then $h \in L(A)$ and therefore there is an $\alpha \in \mathbb{Z}_+^k$ providing a *conformal decomposition* of h into Graver bases elements, that is, $h = \sum_{i=1}^k \alpha_i g_i$ and $g_j \sqsubseteq h$ whenever $\alpha_j > 0$. Now $f(x^*) < f(x') = f(x^* + \sum_{i=1}^k \alpha_i g_i)$ implies that $\alpha \in \mathcal{A}(x^*)$ and hence there is a $\beta \in \mathcal{B}(x^*)$ satisfying $\beta \leq \alpha$. Consider the element $t := G\beta$ in $T(x^*, f(x), A)$. Then $h = \sum_{i=1}^k \alpha_i g_i$ being a conformal decomposition of h and $\beta \leq \alpha$ imply $t \sqsubseteq h$. Now $l \leq x^*, x^* + h = x' \leq u$ imply $l \leq x^* + t \leq u$. □

Note that, in an actual construction of the test set in the proof of Theorem 2.3, it might be useful to represent the Graver basis as the union of its intersections $\mathcal{H}_j := G(A) \cap O_j^n$ with the orthants of $\mathbb{R}^n$, for $0 \leq j < 2^n$. Then each $\mathcal{H}_j$ is the so-called *Hilbert basis* of the rational cone $O_j^n \cap L(A)$, and, using the so-called *integer Carathéodory property* (see [6]), one can restrict attention in the definition of the set $\mathcal{A}(x^*)$ in the proof of Theorem 2.3 to those $\alpha \in \mathbb{Z}_+^k$ with at most $2n - 2$ nonzero components corresponding to elements of some $\mathcal{H}_j$.

3 Oriented sub/superadditive functions

In this section we introduce the notion of oriented subadditive (and superadditive) functions and show how to manipulate these functions. The next definition makes precise what we mean by this.

Definition 3.1 Let $X, D_1, D_2 \subset \mathbb{R}^n$ be given. A function $f : \mathbb{Z}^n \rightarrow \mathbb{R}$ is (X, D_1, D_2)-*oriented superadditive* if for all integral $x \in X, \ y \in D_1, z \in D_2$, we have

$$f(x + y + z) + f(x) \ \geq \ f(x + y) + f(x + z) .$$

We note that the defining inequality is equivalent to

$$\begin{aligned} & f(x + y + z) - f(x) \\ & \quad \geq [f(x + y) - f(x)] + [f(x + z) - f(x)] , \end{aligned}$$

which is perhaps more intuitive — the incremental value of adding both y and z to x exceeds the sum of the incremental values of adding y and z individually to x.

Definition 3.2 The function f is *oriented subadditive* if $-f$ is oriented superadditive.

Note that the definitions do not depend on the order of D_1 versus D_2 . That is, f is (X, D_1, D_2)-oriented superadditive if and only if f is (X, D_2, D_1)-oriented superadditive. In the special case when $D = D_1 = D_2$, then a function is (X, D, D)-oriented superadditive if and only if the family of functions $f_x : D \cap \mathbb{Z}^n \rightarrow \mathbb{R}$ defined by

$$f_x(y) = f(x + y) - f(x)$$

is superadditive (in the ordinary sense), for all $x \in X$. That is,

$$f_x(y + z) \geq f_x(y) + f_x(z) ,$$

for all $y, z \in D$. Also note that (X, D, D)-oriented superadditivity of f implies that

$$f(x + \sum_i y^i) - f(x) \geq \sum_i [f(x + y^i) - f(x)] \ ,$$

for $x \in X$ and $y^i \in D$.

Various functions are readily seen to be (X, D_1, D_2)-oriented superadditive. Trivially, all affine functions are $(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^n)$-oriented superadditive. Any univariate convex function is $(\mathbb{R}_+, \mathbb{R}_+, \mathbb{R}_+)$-oriented superadditive as well as $(\mathbb{R}_+, \mathbb{R}_- \mathbb{R}_-)$-oriented superadditive [5].

Other superadditive functions can be easily defined. For example, the function $f : \mathbb{Z}^2 \to \mathbb{Z}$ defined by

$$f(x) = x_1 x_2$$

is $(\mathbb{R}^2, O_j^2, O_j^2)$-oriented superadditive for $j = 0, 3$ (i.e., the $(+,+)$ and $(-,-)$ orthants). However, for $j = 1, 2$ (i.e., the $(+,-)$ and $(-,+)$ orthants), this function is $(\mathbb{R}^2, O_j^2, O_j^2)$-oriented subadditive. These observations follow by calculating that

$$f(x + y + z) + f(x) - f(x + y) - f(x + z) \\ = y_1 z_2 + y_2 z_1 \ ,$$

and then just considering how for y and z both in any of the four orthants, we have control on the sign of $y_1 z_2 + y_2 z_1$.

In order to exploit the property of oriented sub/superadditivity, we need the notion of an orthant refinement of a linear space.

Definition 3.3 For a d-dimensional subspace $L \subset \mathbb{R}^n$, an *orthant refinement* of L is a finite set $\mathcal{C}$ of d-dimensional (convex) polyhedral cones such that:

(1) $L = \cup_{C \in \mathcal{C}} C$;
(2) $\text{int}(C) \cap \text{int}(D) = \emptyset$, for $C, D \in \mathcal{C}$ with $C \neq D$;
(3) For all $C \in \mathcal{C}$: $\text{int}(C) \subset O_j^n$, for some $0 \leq j < 2^n$.

Of course we trivially have that the set of orthants is an orthant refinement of $\mathbb{R}^n$.

We can perform operations on oriented sub/superadditive functions. In particular, we obtain the following result.

Theorem 3.4 *Let $\mathcal{C}$ be an orthant refinement of $\mathbb{R}^n$, and let $f : \mathbb{Z}^n \to \mathbb{R}$ be $(\mathbb{R}^n, C, C)$-oriented superadditive (subadditive) for all $C \in \mathcal{C}$. Let $W \in \mathbb{Z}^{n \times m}$, and define the linear function $w : \mathbb{Z}^m \to \mathbb{Z}^n$ by $w(x) := Wx$, for all $x \in \mathbb{R}^m$. Then there exists an orthant refinement $\widetilde{\mathcal{C}}$ of $\mathbb{R}^m$, such that the composed function*

$$f \circ w : \mathbb{Z}^m \to \mathbb{R}$$

is $(\mathbb{R}^m, \widetilde{C}, \widetilde{C})$-oriented superadditive (subadditive), for all $\widetilde{C} \in \widetilde{\mathcal{C}}$.

PROOF. Without loss of generality, we consider the case where the function f is $(\mathbb{R}^n, C, C)$-oriented superadditive. For $0 \leq j < 2^m$ and $C \in \mathcal{C}$, we define

$$\widetilde{C}_j := \{x \in O_j^m \ : \ Wx \in C\} \ ,$$

and we let

$$\widetilde{\mathcal{C}} := \{\widetilde{C}_j \ : \ 0 \leq j < 2^m \text{ and } C \in \mathcal{C}\} \ .$$

Since the family of cones $\mathcal{C}$ is an orthant refinement of $\mathbb{R}^n$, the family of cones $\widetilde{\mathcal{C}}$ is an orthant refinement of $\mathbb{R}^m$. Moreover, for all $x \in \mathbb{Z}^m$ and $y, z \in \widetilde{C}_j \cap \mathbb{Z}^m$, we have that $Wy, Wz \in C$, and hence $f(Wx+Wy+Wz)+f(Wx) \geq f(Wx+Wy)+f(Wx+Wz)$. □

Theorem 3.4 illustrates that, indeed, the family of oriented sub/superadditive functions is not pathological. In combination with an orthant refinement of $\mathbb{R}^n$, the structure of oriented sub/superadditive objective functions f allows us to establish finite universal test sets for the family of optimization problems

$P(f(x), A, b, l, u)$ with varying integral data b, l, u . This is the topic of the next section.

4 Oriented sub/superadditive integer maximization

Our goal in this section is to use properties of oriented sub/superadditivity to establish the existence of finite test sets for a broad class of nonlinear integer programming problems. In doing so we need the notion of local optimality with respect to a specific subset of $\mathbb{R}^n$.

Definition 4.1 Let O be a subset of $\mathbb{R}^n$ and $f : \mathbb{Z}^n \to \mathbb{R}$ be a function. A point $x^* \in F(A, b, l, u)$ is O-*optimal* for $P(f(x), A, b, l, u)$ if for all vectors $t \in \mathbb{Z}^n \cap L(A) \cap O$ such that $l \leq x^* + t \leq u$, we have $f(x^* + t) \leq f(x^*)$. That is, there is no $t \in O$ that is an improving augmentation for x^* .

As a first result, we consider the case where, given a set O of $\mathbb{R}^n$, a function f is $(\mathbb{R}^n_+, \mathbb{R}^n_+, O)$-oriented superadditive. Then, we can deduce that the set of all non O-optimal solutions for a certain infinite family of problems has a nice combinatorial structure: it is "closed up." More precisely, we have

Lemma 4.2 *Let $A \in \mathbb{Z}^{m \times n}$, let O denote a subset of $\mathbb{R}^n$, and let $f : \mathbb{Z}^n \to \mathbb{R}$ be $(\mathbb{R}^n_+, \mathbb{R}^n_+, O)$-oriented superadditive. Then there is a finite set $T(f(x), A) \subset \mathbb{Z}^n \cap L(A) \cap O$, such that for every $b^* \in \mathbb{Z}^m$, a point $x^* \in F(A, b^*, 0, \infty)$ is O-optimal if and only if $f(x^* + t) \leq f(x^*)$ for all $t \in T(f(x), A)$ with $0 \leq x^* + t$.*

PROOF. Note that $x^* \in F(A, b^*, 0, \infty)$ is not O-optimal for $P(f(x), A, b^*, 0, \infty)$ if and only if it belongs to

$$X(f(x), A) := \Big\{x' \in \mathbb{Z}^n_+ \ : \ \exists\, t \in \mathbb{Z}^n \cap L(A) \cap O \ , \quad x' + t \geq 0,\ f(x') < f(x' + t)\Big\} \ .$$

We claim that the set $X(f(x), A)$ is *closed up*; that is, $x' \in X(f(x), A)$ implies that $x' + h \in X(f(x), A)$, for all $h \in \mathbb{Z}^n_+$.

To see this, suppose that $x' \in X(f(x), A)$. Therefore, there is a $t \in \mathbb{Z}^n \cap L(A) \cap O$ such that $x' + t \geq 0$ and $f(x') < f(x' + t)$. We wish to show that $x' + h \in X(f(x), A)$, for all $h \in \mathbb{Z}^n_+$. To do this, we just need to show that there is a $\hat{t} \in \mathbb{Z}^n \cap L(A) \cap O$ such that $(x'+h)+\hat{t} \geq 0$ and $f(x'+h) < f((x'+h)+\hat{t})$.

We simply choose $\hat{t} := t$. Then we check

$$f((x'+h)+t) - f(x'+h) \geq f(x'+t) - f(x') > 0,$$

using $(\mathbb{R}^n_+, \mathbb{R}^n_+, O)$-oriented superadditivity of f .

By the Gordan Lemma, there exists a finite set $\widetilde{X}(f(x), A) \subset X(f(x), A)$ such that for all $x^* \in X(f(x), A)$ there exists $\tilde{x}^* \in \widetilde{X}(f(x), A)$ with $\tilde{x}^* \leq x^*$. By what we have already shown, it follows that $x^* \in X(f(x), A)$ if and only if there is a $\tilde{x}^* \in \widetilde{X}(f(x), A)$ with $\tilde{x}^* \leq x^*$.

Now we just take $T(f(x), A)$ to consist of one improving augmentation for each point in $\widetilde{X}(f(x), A)$, and the proof is complete. □

As a next step, we characterize the existence of a finite set for checking local optimality with respect to an orthant, provided that the underlying function is oriented subadditive. The proof uses the arguments from the proof of Theorem 6 of [5].

Lemma 4.3 *Let O be an orthant of $\mathbb{R}^n$, let $A \in \mathbb{Z}^{m \times n}$, and $f : \mathbb{Z}^n \to \mathbb{R}$ be $(\mathbb{R}^n, O, O)$-*

oriented subadditive. Then there is a finite set $T(f(x), A) \subset \mathbb{Z}^n \cap L(A) \cap O$, *such that for every* $b \in \mathbb{Z}^m, l \in \mathbb{Z}^n_{-\infty}$ *and* $u \in \mathbb{Z}^n_{+\infty}$, *a point* $x^* \in F(A, b, l, u)$ *is O-optimal for* $P(f(x), A, b, l, u)$ *if and only if* $f(x^* + t) \leq f(x^*)$ *for all* $t \in T(f(x), A)$ *with* $l \leq x^* + t \leq u$.

PROOF. Let $T(f(x), A) := H(A) = \{g_1, \ldots, g_k\}$ be the Hilbert basis of $O \cap L(A)$. Certainly if x^* is O-optimal, there can be no such t . So we suppose that x^* is not O-optimal. Then there exist $\alpha_1, \ldots, \alpha_k \in \mathbb{Z}_+$ such that

$$f(x^* + \textstyle\sum_{i=1}^k \alpha_i g_i) - f(x^*) > 0 \ .$$

Next, $(\mathbb{R}^n, O, O)$-oriented subadditivity implies that

$$\begin{aligned} f(x^* + \textstyle\sum_{i=1}^k \alpha_i g_i) - f(x^*) \\ \leq \textstyle\sum_{i=1}^k [f(x^* + \alpha_i g_i) - f(x^*)] \\ \leq \textstyle\sum_{i=1}^k \alpha_i [f(x^* + g_i) - f(x^*)] \end{aligned}$$

Hence, there exists an index i such that $\alpha_i > 0$ and $f(x^* + g_i) - f(x^*) > 0$. This proves the claim. $\square$

We next characterize local optimality for a family of optimization problems associated with oriented superadditive functions f in the presence of lower and upper bounds.

Lemma 4.4 *Let* $A \in \mathbb{Z}^{m\times n}$ *, let* O *be an orthant of* $\mathbb{R}^n$ *, and let* $f : \mathbb{Z}^n \to \mathbb{R}$ *be oriented* (O', O', O)*-superadditive, for all orthants* O' *. Then there exists a finite set* $T(f(x), A) \subset \mathbb{Z}^n \cap L(A) \cap O$ *such that for every* $b \in \mathbb{Z}^m, l \in \mathbb{Z}^n_{-\infty}$ *and* $u \in \mathbb{Z}^n_{+\infty}$, *a point* $x^* \in F(A, b, l, u)$ *is O-optimal for* $P(f(x), A, b, l, u)$ *if and only if* $f(x^*+t) \leq f(x^*)$ *for all* $t \in T(f(x), A)$ *with* $l \leq x^* + t \leq u$.

PROOF. Let $H(A) = \{g_1, \ldots, g_k\}$ be the Hilbert basis of $O \cap L(A)$. Let

$$\mathcal{A} := \Big\{(x, \alpha) : x \in \mathbb{Z}^n, \alpha \in \mathbb{Z}^k_+, \ f(x) < f(x + \textstyle\sum_{i=1}^k \alpha_i g_i)\Big\} \ .$$

Let $\mathcal{B} \subset \mathcal{A}$ be the subset of $\mathcal{A}$ comprising $\sqsubseteq$-minimal elements, which is finite by the Extended Gordan Lemma 2.1. We claim that the test set for local optimality is provided by

$$T := \{\textstyle\sum_{i=1}^k \beta_i g_i \ : \ \exists \ x \in \mathbb{Z}^n \ , \ (x, \beta) \in \mathcal{B}\} \ .$$

First, note that $T \subset L(A) \cap O$ and therefore, for all $t \in T$, we have $A(x^* + t) = Ax^* = b$. So if there is a $t \in T$ with $l \leq x^* + t \leq u$ and $f(x^*) < f(x^* + t)$ then x^* is not optimal.

Conversely, suppose that x^* is not O-optimal, and let x' be a better feasible point. Let $h := x' - x^* \in O$. Then $h \in L(A)$, and therefore there is an $\alpha \in \mathbb{Z}^k_+$ providing a conformal decomposition $h = \sum_{i=1}^k \alpha_i g_i$ of h into Hilbert basis elements. Now $f(x^*) < f(x') = f(x^* + \sum_{i=1}^k \alpha_i g_i)$ implies $(x^*, \alpha) \in \mathcal{A}$, and hence there is a $(y^*, \beta) \in \mathcal{B}$ satisfying $y^* \sqsubseteq x^*$ and $\beta \leq \alpha$.

Consider the element $t := \sum_{i=1}^k \beta_i g_i$ in T . Then $h = \sum_{i=1}^k \alpha_i g_i$ being a conformal decomposition and $\beta \leq \alpha$ imply $t \sqsubseteq h$. So now $l \leq x^*, x^* + h = x' \leq u$ imply $l \leq x^* + t \leq u$. So $x^* + t$ is feasible. We claim that it is also better than x^* . Let $v := x^* - y^*$. Then $y^* \sqsubseteq x^*$ implies that y^* and v lie in the same orthant. Also, $(y^*, \beta) \in \mathcal{B} \subset \mathcal{A}$ implies $f(y^*) < f(y^* + \sum_{i=1}^k \beta_i g_i) = f(y^* + t)$. Now, using the hypothesized property of f, we find that, as claimed,

$$\begin{aligned} f(x^*+t) - f(x^*) \\ = f(y^* + v + t) - f(y^* + v) \\ \geq f(y^* + t) - f(y^*) \ > \ 0 \ . \quad \square \end{aligned}$$

We now put together the pieces to obtain the main result of this section.

Theorem 4.5 *Let $A \in \mathbb{Z}^{m\times n}$, and let $\mathcal{C}$ be an orthant refinement of $L(A)$. For all $C \in \mathcal{C}$, let $f : \mathbb{Z}^n \to \mathbb{R}$ be either $(\mathbb{R}^n, C, C)$-oriented subadditive or (O, O, C)-oriented superadditive for all orthants O of $\mathbb{R}^n$. Then there is a finite set $T(f(x), A) \subset \mathbb{Z}^n \cap L(A)$, such that for every $b \in \mathbb{Z}^m, l \in \mathbb{Z}^n_{-\infty}$ and $u \in \mathbb{Z}^n_{+\infty}$, a point $x^* \in F(A, b, l, u)$ is optimal for $P(f(x), A, b, l, u)$ if and only if $f(x^* + t) \leq f(x^*)$ for all $t \in T(f(x), A)$ with $l \leq x^* + t \leq u$.*

PROOF. If $x^* \in F(A, b, l, u)$ is optimal for $P(f(x), A, b, l, u)$, then $f(x^* + t) \leq f(x^*)$ for all $t \in \mathbb{Z}^n \cap L(A)$ with $l \leq x^* + t \leq u$.

If $x^* \in F(A, b, l, u)$ is non-optimal for $P(f(x), A, b, l, u)$, then there exists $C \in \mathcal{C}$ and an improving augmentation $h \in \mathbb{Z}^n \cap L(A) \cap C$, i.e., $f(x^* + h) > f(x^*)$ and $l \leq x^* + h \leq u$. If f is $(\mathbb{R}^n, C, C)$-oriented subadditive, then by Lemma 4.3, we obtain a vector t from a finite set for improvement. If f is (O, O, C)-oriented superadditive for all orthants O of $\mathbb{R}^n$, then from Lemma 4.4 we obtain a vector t from a finite set for improvement. This gives the result. □

We remark that many functions f meet the hypotheses of Theorem 4.5. In particular, the theorem applies to the function $f(x_1, x_2) = x_1 x_2$ (see Section 3). In turn, for $c, d \in \mathbb{Z}^n$, the theorem applies to the rank-1 quadratic form $f : \mathbb{Z}^n \to \mathbb{R}$ defined by $f(x) := x^T c d^T x$, since we can define new variables x_{n+1} and x_{n+2}, and then with the further *linear* constraints $x_{n+1} - c^T x = 0$ and $x_{n+2} - d^T x = 0$, we now look to maximizing $x_{n+1} x_{n+2}$.

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