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## A Sensor Placement Algorithm for Redundant Covering Based on Riesz Energy Minimization

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# A sensor placement algorithm for redundant covering based on Riesz energy minimization

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**Abstract**—We present an algorithm for sensor placement with redundancy where each point in a 2-dimensional space is covered by at least  $k$  sensors under the constraint that all the sensors are located away from each other. We reduce the problem to distributing points evenly on the surface of a torus manifold and solve it computationally by minimizing the Riesz energy. We also study the case where the coverings are incrementally constructed. We illustrate our approach with numerical results and compare it to similar approaches in dispersed dither mask halftoning.

## I. INTRODUCTION

Ad-hoc sensor networks is an important area of study and has generated many challenging problems. One such problem is to determine how sensors are positioned to maximize coverage [1]–[4]. In [4] the problem is addressed by using a geometric approach. The basic framework is the following. Given a 2-dimensional region  $R$ , sensors are placed within this region. In the sequel, we choose  $R$  to be the 2-dimensional plane. Each sensor  $a$  has a coverage region  $R_c(a)$  which is a circular region with radius  $r_c$  and centered at the sensor. Without loss of generality we assume that  $r_c = 1$ . The goal is to place sensors in the 2-dimensional region (i.e. find a covering) satisfying the following objectives:

- 1) The coverage regions of all the sensors cover the entire region  $R$ , i.e.  $R \subset \cup_a R_c(a)$ .
- 2) Each point in  $R$  is within the coverage region of at least  $k$  sensors. This is called a  $k$ -covering of the region  $R$ .
- 3) All the sensors should be placed as much away from each other as possible.

The first objective ensures that the entire region can be sensed by the sensor network, i.e. each point is in the coverage region of some sensor. The second objective provides redundancy and data fusion capabilities by allowing each point in  $R$  to be probed by multiple sensors (of possibly different modalities). The third objective provides robustness in the sense that if the sensors are located close to each other, then a localized disruption can disable multiple sensors.

The optimal placement of the sensors for a 1-covering on the plane is well known; it is the hexagonal lattice shown in Fig. 1 [5].

In [4] a 2-covering is proposed by concatenating 2 optimal 1-coverings where there is a translation between the 2 1-coverings (Fig. 2). This approach satisfies the third objective listed above well. The question was raised how  $k$ -coverings for higher  $k$  can be achieved that also satisfy this last objective well.

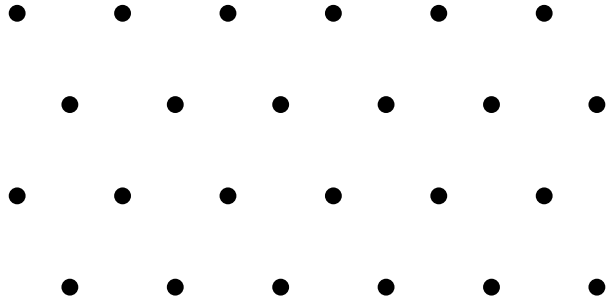


Fig. 1. The hexagonal lattice is the optimal placement for a 1-covering on the plane.

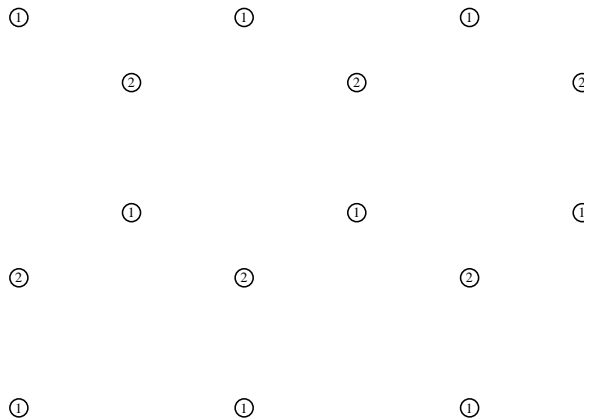


Fig. 2. A 2-covering of the plane. Sensors corresponding to the same 1-covering are labeled with the same number.

## II. MINIMIZING RIESZ ENERGY

The purpose of this paper is to propose a solution for  $k$ -coverings for  $k > 2$ . As in [4], we construct a  $k$ -covering by concatenating  $k$  1-coverings that are translations from each other. Because of the lattice structure of the optimal 1-covering, each sensor within the 1-covering can be assumed to lie on the surface of a torus, i.e. the 1-covering is reduced to a single sensor located on the surface of a torus. Since this lattice is the same for all the 1-coverings, the tori for the different 1-coverings can be thought of as the same object. In this case, to satisfy objective 3, this problem is reduced to dispersing  $k$  points on a torus. This problem has been studied

using the approach of Riesz energy minimization [6]. The  $s$ -Riesz energy of a set of points  $\{a_i\}$  is defined as

$$E_s = \sum_{i \neq j} \frac{1}{d(a_i, a_j)^s}$$

for a distance metric  $d$ . For  $s = 0$ , the Riesz energy is defined as

$$E_0 = \sum_{i \neq j} \log \frac{1}{d(a_i, a_j)}$$

By minimizing  $E_s$  using optimizing algorithms, a distribution of points is found which appear to satisfy criterion 3. In particular, for  $s \rightarrow \infty$ , minimizing  $E_s$  solves the best packing problem, i.e. the set of points such that  $d_{\min} = \min_{i \neq j} d(a_i, a_j)$  is maximized. Algorithms have been developed for cases where the number of points is in the thousands [6].

The distance metric  $d$  on the torus is induced by the distance metric  $d_R$  in the region  $R$ . Since each point on the torus corresponds to a lattice in the plane, when unwrapped onto  $R$ ,  $d$  is the Hausdorff distance between two lattices, i.e. given two hexagonal lattices  $\{a_i\}, \{b_j\}$ ,  $d = \min_{i,j} d_R(a_i, b_j)$ .

### III. EXPERIMENTAL RESULTS

By minimizing the 10-Riesz energy  $E_{10}$ , we obtain configurations of  $k$ -coverings for various values of  $k$ . In our experiments, we choose  $d_R(a_i, b_j) = \|a_i - b_j\|_2$  to be the Euclidean distance on the plane. For 2-coverings, we obtain the same result as reported in [4], i.e. Fig. 2. The results for  $k$ -coverings for various values of  $k > 2$  are shown in Figs. 3-6. In these figures, sensors labeled with the same number belong to the same hexagonal 1-covering. Note that the union of all the sensors can form a regular geometric structure, and in some instances a hexagonal lattice. For instance, for 3-covering, 4-covering and 7-covering, the union of all the sensors form a hexagonal lattice, the optimal 1-covering. More generally, there exists a set  $Q = \{3, 4, 7, 13, 19, 25, \dots\}$  of integers such that a hexagonal lattice can be partitioned into  $q$  disjoint hexagonal lattices for each  $q \in Q$ .

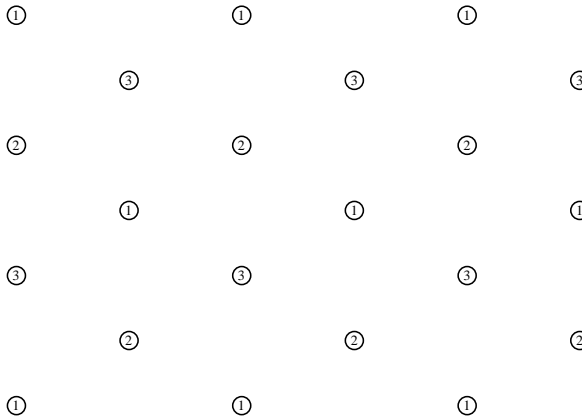


Fig. 3. A 3-covering of the plane with  $d_{\min} = \frac{1}{\sqrt{3}}$ . Note that the union of 3 hexagonal 1-coverings forms a hexagonal covering.

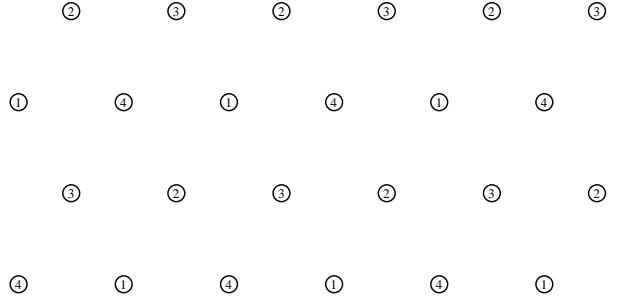


Fig. 4. A 4-covering of the plane with  $d_{\min} = \frac{1}{2}$ . Note that the union of the 4 hexagonal 1-coverings is again a hexagonal covering.

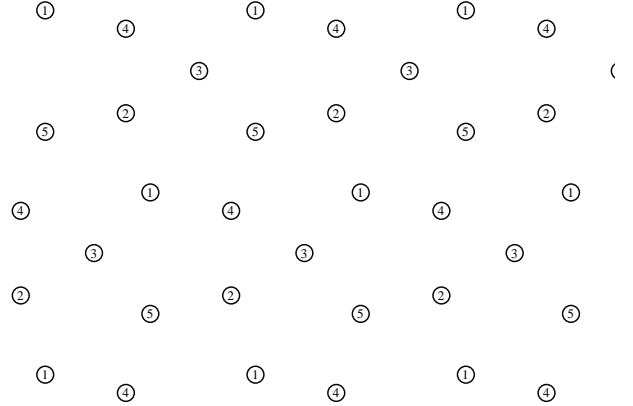


Fig. 5. A 5-covering of the plane with  $d_{\min} = 0.3937$ .

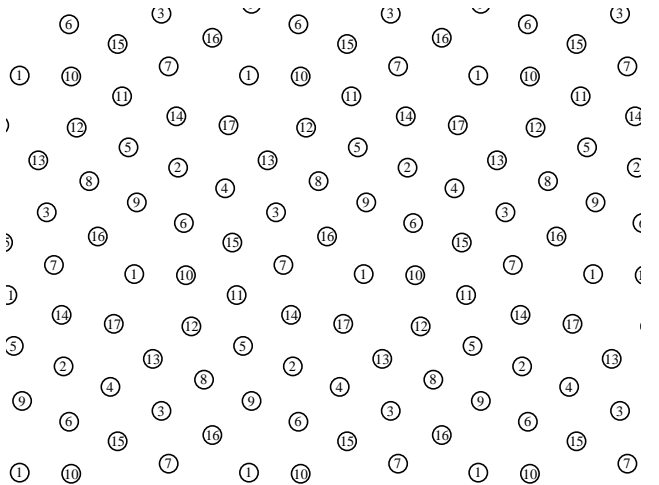


Fig. 6. A 17-covering of the plane with  $d_{\min} = 0.2198$ .

#### IV. SUBLATTICES OF THE HEXAGONAL LATTICE

As discussed in Section III, there exists a set  $Q$  of integers such that a hexagonal lattice can be partitioned into  $q$  disjoint hexagonal sublattices for each  $q \in Q$ . Thus for  $q \in Q$ , the hexagonal 1-covering can be partitioned into a  $q$ -covering after rescaling the distances by a factor of  $\sqrt{q}$ . By recursively partitioning this way (and scaling the sensor locations appropriately), this shows that for  $k$ -coverings, where  $k$  is of the form  $k = \prod_{q_i \in Q} q_i^{m_i}$ ,  $m_i \in \mathbb{Z}_0^+$ , there is an optimal configuration where the union of all the sensors form a hexagonal lattice. In this case,  $d_{\min} = \frac{1}{\sqrt{k}}$ . For any  $k'$ , if  $k = \prod_{q_i \in Q} q_i^{m_i}$  is the smallest integer of this form such that  $k' \leq k$ , then a subset of the optimal  $k$ -covering described above can serve as a  $k'$ -covering and  $\frac{1}{\sqrt{k}}$  is a lower bound for the best  $d_{\min}$  for a  $k'$ -covering. On the other hand, because of the optimality of the hexagonal lattice,  $\frac{1}{\sqrt{k'}}$  is an upper bound for  $d_{\min}$  for a  $k'$ -covering.

Similarly, given a  $k'$ -covering, we can extend it to a  $k' \prod_{q_i \in Q} q_i^{m_i}$ -covering by replacing recursively 1-coverings with optimal  $q_i$ -coverings. In general, this covering is suboptimal.

Let  $Q' = \{\prod_{q_i \in Q, m_i \geq 0} q_i^{m_i}\}$ . Thus  $Q'$  is the set of integers  $k'$  for which there is an optimal  $k'$ -covering such that the union of sensors form a hexagonal lattice. The question of determining  $Q'$  is equivalent to determining the indices for which there is a sublattice of a hexagonal lattice which is also a hexagonal lattice. This question has been solved [7] and  $Q' = \{a^2 + ab + b^2 | a, b \in \mathbb{Z}^+\}$ . Another way to characterize  $Q'$  is that for  $k' \in Q'$ , all prime factors of  $k'$  which are of the form  $3m+2$  have even exponents (<http://www.research.att.com/~njas/sequences/A003136>). We have chosen  $Q$  such that each element in  $Q$  is not a product of elements of  $Q$  and in this case  $Q$  is the set of the norms of Eisenstein-Jacobi primes (<http://www.research.att.com/~njas/sequences/A055664>).

#### V. CHOOSING THE EXPONENT $s$

How large should the value of the exponent  $s$  be when computing the Riesz energy  $E_s$ ? In order to approximate the energy function that solves the best packing problem,  $s$  should be chosen to be large. However, a large  $s$  leads to numerical problems. On the other hand, in our experiments we found that choosing small values of  $s$  can produce results which are less optimal. For instance, we show in Fig. 7 the optimal 9-covering optimized using the Riesz energy for  $s = 1$ , and in Fig. 8 the 9-covering for  $s = 2$ . According to the discussion above, Fig. 8 is the optimal configuration which maximizes  $d_{\min}$ . Numerical experiments show that there is a transition from one configuration to the other configuration around  $s = 1.2$ .

#### VI. INCREMENTAL $k$ -COVERINGS AND OTHER BASE COVERINGS

Consider the scenario where a  $k$ -covering is in place and we want to expand the sensor network to a  $k'$ -covering ( $k' > k$ ) by adding more sensors and the question is where the new sensors should go. Again the same optimization approach can

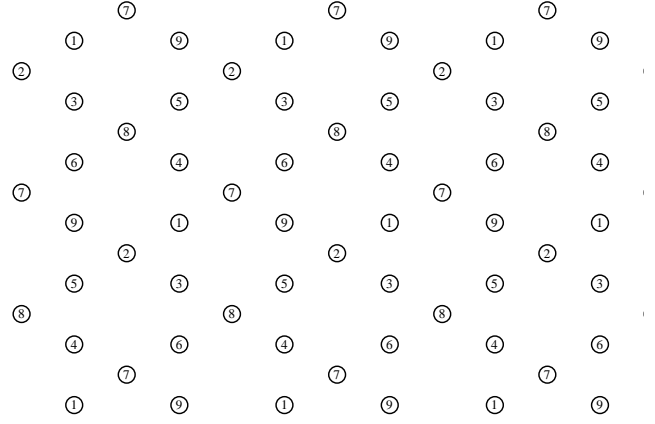


Fig. 7. A 9-covering of the plane optimized using  $s = 1$ .  $d_{\min} = \frac{1}{2\sqrt{3}}$ .

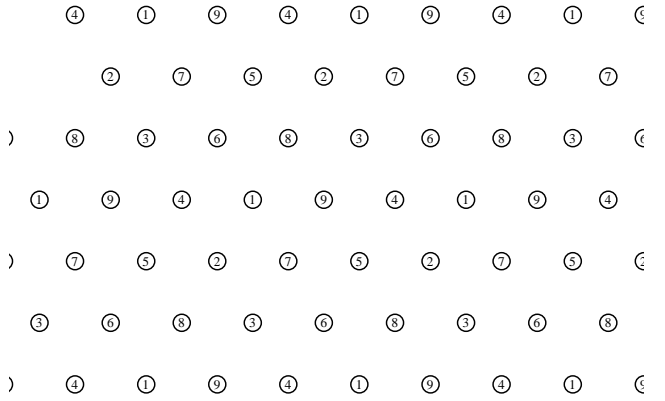


Fig. 8. A 9-covering of the plane optimized using  $s = 2$ .  $d_{\min} = \frac{1}{3}$ .

be used, with the additional constraint that the sensors in the original  $k$ -covering remain fixed. In Fig. 9 we show how the 3-covering in Fig. 3 is extended to a 4-covering. Note that it is suboptimal to the 4-covering in Fig. 4.

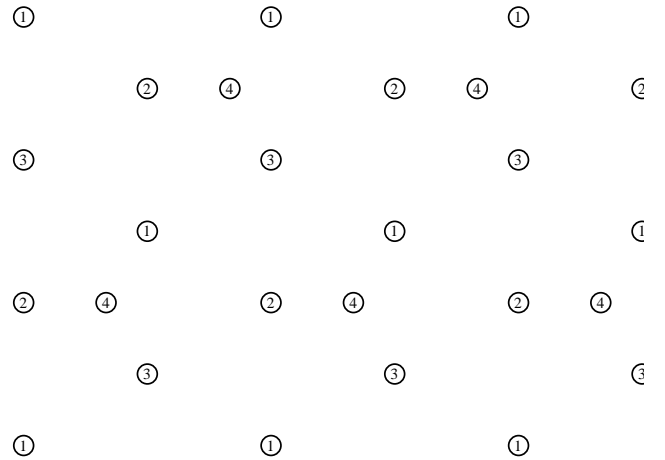


Fig. 9. A 4-covering of the plane with  $d_{\min} = \frac{1}{3}$ . This covering is obtained by adding an additional 1-covering to the 3-covering in Fig. 3.

So far we have applied the algorithm to the case where

the underlying 1-covering is the hexagonal lattice (we'll call this the *base* covering). This approach can also be applied to other base 1-coverings. For instance, Fig. 10 shows a 4-covering obtained by translating 4 square grid 1-coverings and optimizing the arrangement by minimizing  $E_{10}$ . We see that the totality of sensors form a hexagonal lattice. A more irregular looking 6-covering based on 6 squared grid lattices is shown in Fig. 11.

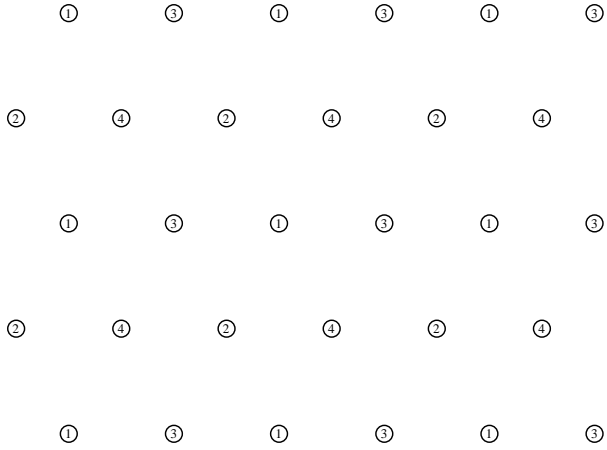


Fig. 10. A 4-covering of the plane. The base 1-covering is the square grid lattice and  $d_{\min} = \frac{1}{2\sqrt{2}}$ .

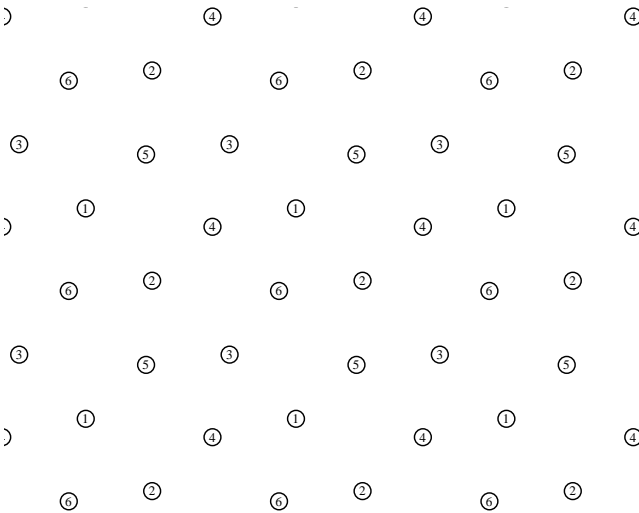


Fig. 11. A 6-covering of the plane. The base 1-covering is the square grid lattice and  $d_{\min} = 0.2708$ .

## VII. RELATIONSHIP WITH DITHER ARRAY HALFTONING

There is a close relationship between these  $k$ -coverings and dispersed dither digital halftoning [8]. In dispersed dither, a pattern is tiled on the plane such that it forms a visually pleasing pattern. The dispersed ordered dither patterns are regular patterns that are repeated and are similar to the  $k$ -coverings where all the sensors lie on a regular grid (e.g. Fig.

10) and blue noise dispersed dither [9] looks similar to  $k$ -coverings in other cases (e.g. Fig. 11). The stacking constraint in these dither patterns dictates that a pattern with more dots is a superset of a pattern with less dots and this requirement corresponds to the case of incremental  $k$ -coverings. Another indication of the close relationship between these two areas is that many algorithms for generating blue noise dither patterns are based on energy minimization [10]. The main difference between  $k$ -coverings and dither halftoning is that in a  $k$ -covering, the sensors can be positioned at any point on the plane resulting in a nonlinear programming problem, whereas in a dispersed dither pattern, the dots are constrained to lie on the printer addressability grid (e.g. a 600 dpi or a 1200 dpi grid), resulting in an integer programming problem. We could envision a scenario where the sensors must be placed on a specific grid in which case the algorithms in [10] can be useful.

## VIII. CONCLUDING REMARKS

We presented a method to generate  $k$ -covering of sensors on the plane using Riesz energy functions. In general the union of the sensors do not form a regular lattice. For  $k \in \mathbb{Q}'$ , they do form a hexagonal lattice which is optimal and such configurations can be used to recursively generate  $k$ -coverings for large  $k$ . We also examine the case where a  $k$ -covering is incrementally constructed by adding 1-coverings and illustrate the relationship with dispersed dither digital halftoning.

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