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On the Linear Relaxation of the *p*-Median Problem II: Directed Graphs

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ON THE LINEAR RELAXATION OF THE p-MEDIAN PROBLEM II: DIRECTED GRAPHS

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ABSTRACT. We study a well-known linear programming relaxation of the p-median problem. We give a characterization of the directed graphs for which this system of inequalities defines an integral polytope. Our proof uses a similar result on oriented graphs that we gave in [2].

1. INTRODUCTION

This is the second of two papers dealing with a linear programming relaxation of the p-median problem. Our goal is to characterize the graphs for which this system of inequalities defines an integral polytope. In [2] we gave such a characterization for oriented graphs; these are graphs such that if (u, v) is in the arc-set then (v, u) is not in the arc-set. Here we give such a characterization for general directed graphs, we use the result on oriented graphs as a starting point.

Let $G = (V, A)$ be a *directed* graph, not necessarily connected, where each arc $(u, v) \in A$ has an associated cost $c(u, v)$. The *p-median problem* (pMP) consist of selecting p nodes, usually called *centers*, and then assign each non-selected node to a selected node. The goal is to select p nodes that minimize the sum of the costs yield by the assignment of the non-selected nodes. For more references on the pMP see [3, 2]. The graphs we consider do not contain multiple arcs, that is if (u, v) and (u', v') are two distinct arcs then we cannot have $u = u'$ and $v = v'$. The following is a natural linear programming relaxation for the pMP:

$$\text{minimize} \sum_{(u,v)\in A} c(u, v)x(u, v), \tag{1}$$

$$\sum_{v\in V} y(v) = p, \tag{2}$$

$$\sum_{v:(u,v)\in A} x(u, v) = 1 - y(u) \qquad \forall u \in V, \tag{3}$$

$$x(u, v) \leq y(v) \quad \forall (u, v) \in A, \tag{4}$$

$$y(v) \leq 1 \qquad \forall v \in V, \tag{5}$$

$$x(u, v) \geq 0 \quad \forall (u, v) \in A. \tag{6}$$

Denote by $P_p(G)$ the polytope defined by constraints (2)-(6), and let $pMP(G)$ be the convex hull of $P_p(G) \cap \{0, 1\}^{|A|+|V|}$. In this paper we characterize all directed graphs

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such that $P_p(G) = pMP(G)$. To state our main result we need some definitions and notation.

In Figure 1, we show four directed graphs and for each of them a fractional extreme point of $P_p(G)$. The numbers near the nodes correspond to the variables y, all the arcs variables are equal to $\frac{1}{2}$.

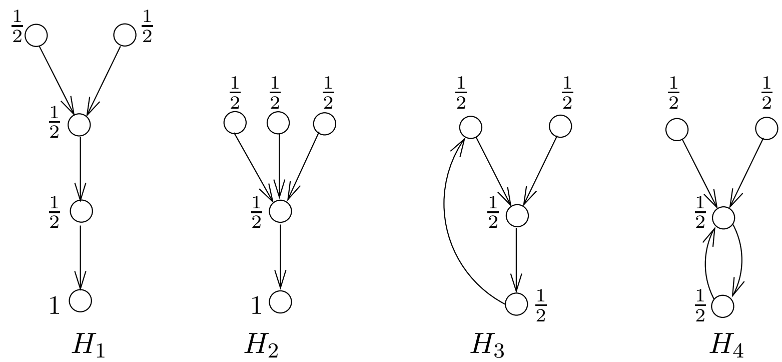

FIGURE 1. Fractional extreme points of $P_p(G)$.

A simple cycle C is an ordered sequence

$$v_0, a_0, v_1, a_1, \ldots, a_{p-1}, v_p,$$

where

- v_i, $0 \leq i \leq p-1$, are distinct nodes,
- a_i, $0 \leq i \leq p-1$, are distinct arcs,
- either v_i is the tail of a_i and v_{i+1} is the head of a_i, or v_i is the head of a_i and v_{i+1} is the tail of a_i, for $0 \leq i \leq p-1$, and
- $v_0 = v_p$.

We denote by $V(C)$ the nodes of C and by $A(C)$ the arcs of C. The *size* of C is p. By setting $a_p = a_0$, we associate with C three more sets as below.

- We denote by $\hat{C}$ the set of nodes v_i, such that v_i is the head of a_{i-1} and also the head of a_i, $1 \leq i \leq p$.
- We denote by $\dot{C}$ the set of nodes v_i, such that v_i is the tail of a_{i-1} and also the tail of a_i, $1 \leq i \leq p$.
- We denote by $\tilde{C}$ the set of nodes v_i, such that either v_i is the head of a_{i-1} and also the tail of a_i, or v_i is the tail of a_{i-1} and also the head of a_i, $1 \leq i \leq p$.

A cycle C is said to be *odd* if $|\tilde{C}| + |\hat{C}|$ is odd, otherwise it is said to be *even*. When $\hat{C} = \emptyset$ the cycle is a *directed* cycle. If we do not require $v_0 = v_p$, we have a *path*. In this case, the nodes $v_1, \ldots, v_{p-1}$ are called *internal* nodes.

The following definition extends to directed graphs, the definition of a Y-cycle given in [2] for oriented graphs.

Definition 1. *A simple cycle C is called a Y-cycle if for every $v \in \hat{C}$ at least one of the following hold:*

(i) *there exists an arc $(v, \bar{v}) \notin A(C)$, $\bar{v} \notin V(C)$, or*
(ii) *there exists an arc $(v, \bar{v}) \notin A(C)$, $\bar{v} \in \tilde{C}$ and $\bar{v}$ is one of the two neighbors of v in C.*

For a simple cycle C, denote by $\hat{C}_{(i)}$ the set of nodes in $\hat{C}$ that satisfy condition (i) of the above definition. Notice that we may have nodes in $\hat{C}$ that satisfy both (i) and (ii).

For a directed graph $G = (V, A)$ and a set $W \subset V$, we denote by $\delta^+(W)$ the set of arcs $(u, v) \in A$, with $u \in W$ and $v \in V \setminus W$. Also we denote by $\delta^-(W)$ the set of arcs (u, v), with $v \in W$ and $u \in V \setminus W$. We write $\delta^+(v)$ and $\delta^-(v)$ instead of $\delta^+(\{v\})$ and $\delta^-(\{v\})$, respectively. If there is a risk of confusion we use δ^+_G and δ^-_G. A node u with $\delta^+(u) = \emptyset$ is called a *pendent* node.

In Figure 2 we show a fractional extreme point of $P_p(G)$ different from those given in Figure 1. It consists of an odd Y-cycle with an arc having both of its endnodes outside the cycle. The values reported near each node represent the node variables, the arc variables are all equal to $\frac{1}{2}$. These values form a fractional extreme point of $P_p(G)$, with $p = 4$.

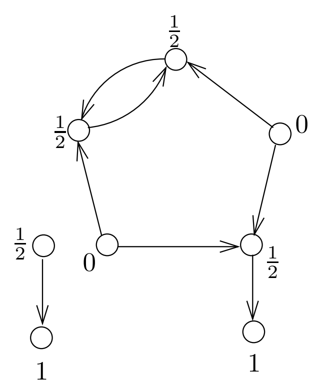

FIGURE 2. An odd Y-cycle with an arc outside the cycle .

The theorem below is the main result of this paper. It shows that the configurations in figures 1 and 2 are the only configurations that should be forbidden in order to have an integral polytope.

Theorem 2. *Let $G = (V, A)$ be a directed graph, then $P_p(G)$ is integral if and only if*

- *(i) it does not contain as a subgraph one of the graphs H_1, H_2, H_3 or H_4 of Figure 1, and*
- *(ii) it does not contain an odd Y-cycle C and an arc (u, v) with neither u nor v in $V(C)$.*

The proof of this theorem is given in Section 5. This proof uses the following main theorem of [2].

Theorem 3. *Let $G = (V, A)$ be an oriented graph, then $P_p(G)$ is integral if and only if*

- *(i) it does not contain as a subgraph one of the graphs H_1, H_2, H_3 of Figure 1, and*
- *(ii) it does not contain an odd Y-cycle C and an arc (u, v) with neither u nor v in $V(C)$.*

The paper is organized as follows. Section 2 contains preliminary definitions and notation. The graphs that satisfy conditions (i) and (ii) of Theorem 2 with no odd Y-cycle are considered in Section 3 and those containing an odd Y-cycle are studied in Section 4. Section 5 gives the proof of Theorem 2. In Section 6 we show how to test in polynomial time conditions (i) and (ii) of Theorem 2. Finally Section 7 concludes this paper with some remarks and a corollary in undirected graphs.

2. Preliminaries

Let $G = (V, A)$ be a directed graph. Let $l : V \cup A \to \{0, -1, 1\}$ be a labeling function that associates to each node and arc of G a label 0, -1 or 1.

A vector $(x, y) \in P_p(G)$ will be denoted by z, i. e. $z(u) = y(u)$ for all $u \in V$ and $z(u, v) = x(u, v)$ for all $(u, v) \in A$. Given a vector z and a labeling function l, we define a new vector z_l from z as follows:

$$z_l(u) = z(u) + l(u)\epsilon, \text{ for all } u \in V, \text{ and}$$

$$z_l(u, v) = z(u, v) + l(u, v)\epsilon, \text{ for all } (u, v) \in A,$$

where ϵ is a sufficiently small positive scalar. We say that an arc (u, v) is *tight* for $z \in P_p(G)$ if $z(u, v) = z(v)$.

The labeling procedure for even cycles [2]**.** We will recall the labeling procedure for even cycles introduced in [2] and some of its properties without proofs.

Let $C = v_0, a_0, v_1, a_1, \ldots, a_{p-1}, v_p$ be an even cycle, not necessarily a Y-cycle.

- If C is a directed cycle, assume that v_0 is the tail of a_0, then set $l(v_0) \leftarrow 1$; $l(a_0) \leftarrow -1$. Otherwise, assume $v_0 \in \dot{C}$ and set $l(v_0) \leftarrow 0$; $l(a_0) \leftarrow 1$.
- For $i = 1$ to $p - 1$ do the following:
  - If v_i is the head of a_{i-1} and is the tail of a_i, then $l(v_i) \leftarrow l(a_{i-1})$, $l(a_i) \leftarrow -l(v_i)$.
  - If v_i is the head of a_{i-1} and is the head of a_i, then $l(v_i) \leftarrow l(a_{i-1})$, $l(a_i) \leftarrow l(v_i)$.
  - If v_i is the tail of a_{i-1} and is the head of a_i, then $l(v_i) \leftarrow -l(a_{i-1})$, $l(a_i) \leftarrow l(v_i)$.
  - If v_i is the tail of a_{i-1} and is the tail of a_i, then $l(v_i) \leftarrow 0$, $l(a_i) \leftarrow -l(a_{i-1})$.

Remark 4. *If C is a directed even cycle, then $l(a_{p-1}) = l(v_0)$ and $\sum l(v_i) = 0$.*

This remark is easy to see. The second property is given in the following lemma and it concerns non-directed cycles.

Lemma 5. [2] *If C is a non-directed even cycle, then $l(a_{p-1}) = -l(a_0)$ and $\sum l(v_i) = 0$.*

Definition 6. *Let C be a Y-cycle in a directed graph $G = (V, A)$. A node $v \in V(C)$ is called a blocking node, (see Figure 3), if one of the following hold:*

(i) *$v \in \tilde{C}$, $(v, u) \in A(C)$, $(u, v) \in A \setminus A(C)$ and $u \in \tilde{C}$, or*
(ii) *$v \in \hat{C}$, $(u, v) \in A(C)$, $(w, v) \in A(C)$, $(v, u) \in A \setminus A(C)$, $(v, w) \in A \setminus A(C)$ and both u and w are in $\tilde{C}$.*

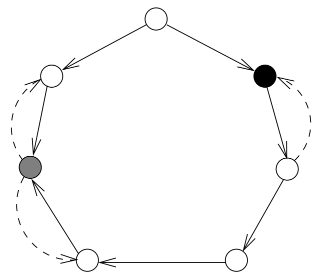

FIGURE 3. Solid lines represent an even Y-cycle. The black and the gray node are blocking nodes satisfying Definition 6 (i) and (ii), respectively.

Lemma 7. *Let $G = (V, A)$ be a directed graph with no multiple arcs and that satisfies condition (i) of Theorem 2. If the following assumptions hold:*

(a1) *G admits an even Y-cycle C of size greater or equal to three with no blocking node, and*

(a2) *$P_p(G)$ contains a vector $\bar{z}$ with:*
$0 < \bar{z}(v) < 1$ *for each node* $v \in \tilde{C} \cup \hat{C}$*;*
$0 < \bar{z}(u,v) < 1$ *for each arc* $(u,v) \in A(C)$*;*
and $0 < \bar{z}(u,v) < 1$ *for each arc* (u,v) *with* $u \in \hat{C}$*,*

then $\bar{z}$ is not an extreme point of $P_p(G)$.

Proof. Assume that the assumptions of the lemma are true. Let

$$C = v_0, a_0, v_1, a_1, \ldots, a_{p-1}, v_p$$

be an even Y-cycle with no blocking node.

Assign labels to the arcs and nodes of C following the labeling procedure above. Extend this labeling as follows: for each node $v_i \in \hat{C}$ if there is an arc $(v_i, u) \in A \setminus A(C)$ with $u \in \tilde{C}$, then $l(v_i, u) \leftarrow -l(v_i)$. Notice that $u = v_{i-1}$ or $u = v_{i+1}$ and since v_i is not a blocking node, such an arc is unique if it exists. If there is not such an arc, by the definition of a Y-cycle we must have an arc $(v_i, u) \in A \setminus A(C)$ with $u \notin V(C)$, in this case also set $l(v_i, u) \leftarrow -l(v_i)$. Now assign the label 0 for each node and arc with no label. Call this labeling function l.

Claim. $\bar{z}_l$ satisfies with equality each constraint among (2)-(6) that is satisfied with equality by $\bar{z}$.

Proof. Assumption (a2) shows that for the nodes and arcs that received a nonzero label, their corresponding variables take a fractional value. This implies that each inequality among (5) and (6) that is satisfied with equality by $\bar{z}$, is also satisfied with equality by $\bar{z}_l$.

Remark 4 and Lemma 5 imply $\sum l(v_i) = 0$, in both cases, whether C is directed or not. Hence equality (2) is satisfied by $\bar{z}_l$. When C is directed, equalities (3) are satisfied by $\bar{z}_l$ by definition. When it is not directed, by definition these equalities are satisfied for every node $v \neq v_0$. By Lemma 5 we have $l(a_{p-1}) = -l(a_0)$. This shows that equality (3) with respect to v_0 is also satisfied by $\bar{z}_l$.

Now we will show that every arc that is tight for $\bar{z}$ is also tight for $\bar{z}_l$. Let $(u,v) \in A(C)$, the labeling procedure gives $l(v) = l(u,v)$, hence $\bar{z}_l(u,v) = \bar{z}_l(v)$. Also, for every arc

$(u,v) \in A \setminus A(C)$ with $u, v \notin V(C)$, we have $l(u,v) = 0$ and $l(u) = l(v) = 0$. Let us examine the three other cases:

(i) $(u,v) \in A \setminus A(C)$, with u and v in $V(C)$. We have three sub-cases:
  - If $v \in \dot{C}$, then $l(v) = 0$ and $l(u,v) = 0$.
  - Suppose $v \in \tilde{C}$, since G does not contain any of the graphs H_1, H_3 and H_4 as a subgraph, the nodes u and v must be consecutive in C. So $(v,u) \in A(C)$. By assumption (a1), v is not a blocking node, so u must be in $\hat{C}$. Let u' be the other node of the cycle adjacent to u. The node u is not a blocking node. Thus if $(u,u') \in A$, then $u' \in \dot{C}$. Hence when extending the labeling of C, we get $l(u,v) = -l(u)$ which is equal to $l(v)$ by the labeling procedure of C.
  - The case $v \in \hat{C}$ cannot exist since G does not contain either H_2 or H_4 as a subgraph and it does not contain multiple arcs.

(ii) $(u,v) \in A \setminus A(C)$, with $u \in V(C)$ and $v \in V \setminus V(C)$. By definition $l(v) = 0$. If $u \in (\tilde{C} \cup \dot{C})$, then $l(u,v) = 0$. And if $u \in \hat{C}$, since G does not contain H_1, H_3 or H_4 as a subgraph, v must be a pendent node, so $\bar{z}(u,v) < \bar{z}(v) = 1$.

(iii) $(u,v) \in A \setminus A(C)$, with $u \in V \setminus V(C)$ and $v \in V(C)$. The node v must be in $\dot{C}$, otherwise one of the graphs H_1, H_2, H_3 or H_4 exists in G. Thus by the labeling procedure, $l(v) = 0$; and when extending this labeling (u,v) takes the label 0 since $u \notin V(C)$.

□

Since $\bar{z} \neq \bar{z}_l$, the claim above implies that $\bar{z}$ is not an extreme point of $P_p(G)$. □

3. Graphs with no odd Y-cycle

In this section we assume that $G = (V, A)$ is a directed graph satisfying condition (i) of Theorem 2, that is, it does not contain any of the graphs H_1, H_2, H_3 or H_4 of Figure 1 as a subgraph. Also we assume that G does not contain an odd Y-cycle.

This section is divided into two sub-sections. In Sub-section 3.1, we will proof the following lemma:

Lemma 8. *$P_p(G)$ does not contain a fractional extreme point $\bar{z}$ where $\bar{z}(u,v) = \bar{z}(v)$, for all (u,v) with v not a pendent node.*

This lemma is used to prove the following theorem in Sub-section 3.2:

Theorem 9. *If $G = (V, A)$ is a directed graph with no multiple arcs, no odd Y-cycle and satisfying condition (i) of Theorem 2, then $P_p(G)$ is integral.*

But first, let us remark some useful implicit properties of the graph $G = (V, A)$ defined above and of the polytope $P_p(G)$.

Remark 10. *Let $v \in V$, with $\delta^-(v) = \{(u_1, v), (u_2, v)\}$. If $(v,t) \in A$, then t is a pendent node or it coincides with u_1 or u_2.*

A bidirected path P of $G = (V, A)$, is an ordered sequence of nodes $P = v_1, \ldots, v_p$, where (v_i, v_{i+1}) and (v_{i+1}, v_i) belong to A, for $i = 1, \ldots, p-1$. The size of P is p. A node v_i of P is called *internal* if $i \notin \{1, p\}$.

Remark 11. *If $P = v_1, \ldots, v_p$ is a bidirected path of G, then for each internal node v_i we have $\delta^-(v_i) = \{(v_{i-1}, v_i), (v_{i+1}, v_i)\}$.*

Let us assume that $\bar{z}$ is a fractional extreme point of $P_p(G)$.

Lemma 12. *We can assume that $\bar{z}(u,v) > 0$ for all $(u,v) \in A$.*

Proof. Let G' be the graph obtained after removing all arcs (u,v) with $\bar{z}(u,v) = 0$. The graph G' has the same properties as G. Let z' be the restriction of $\bar{z}$ on G'. Then z' is a fractional extreme point of $P_p(G')$. □

Lemma 13. *We can assume that $\bar{z}(v) > 0$ for all $v \in V$ with $|\delta^-(v)| \geq 1$.*

Proof. It is straightforward from Lemma 12 and constraints (4). □

Lemma 14. *Let (u,v) and (v,w) be two arcs in G. Then $\bar{z}(v)$, $\bar{z}(u,v)$ and $\bar{z}(v,w)$ are fractional.*

Proof. Lemma 13 implies $\bar{z}(v) > 0$, and Lemma 12 implies $\bar{z}(v,w) > 0$ and $\bar{z}(u,v) > 0$. Using equation (3) with respect to v we get $\bar{z}(v) < 1$ and $\bar{z}(v,w) < 1$. And using inequalities (4) we obtain $\bar{z}(u,v) < 1$. □

Lemma 15. *We may assume that $|\delta^-(v)| \leq 1$ for every pendent node v in G.*

Proof. If v is a pendent node in G and $\delta^-(v) = \{(u_1,v), \ldots, (u_k,v)\}$, we can split v into k pendent nodes $\{v_1, \ldots, v_k\}$ and replace every arc (u_i,v) with (u_i,v_i). Then we define z' such that $z'(u_i,v_i) = z(u_i,v)$, $z'(v_i) = 1$, for all i, and $z'(u) = z(u)$, $z'(u,w) = z(u,w)$ for every other node and arc. Let G' be this new graph. This graph transformation does not create cycles nor any of the graphs $H_1, \ldots, H_4$. So G' has the same properties as G. Moreover, it is easy to check that z' is a fractional extreme point of $P_{p+k-1}(G')$. □

Lemma 16. *We can assume that G does not contain a bidirected path $P = v_1, v_2, v_3$, where $\delta^-(v_1) = \{(v_2,v_1)\}$, $\delta^-(v_3) = \{(v_2,v_3)\}$, the inner node v_2 is only adjacent to v_1 and v_3 and where all the arcs of P are tight for $\bar{z}$ except for (v_2,v_3) that may or may not be tight.*

Proof. Let P be the path defined in the lemma. Define G' as the graph obtained from G by identifying the nodes v_1 and v_3, call v^* the resulting node, and by removing the node v_2 with its incident arcs. Add a new node t and the arc (v^*,t), (see Figure 4).

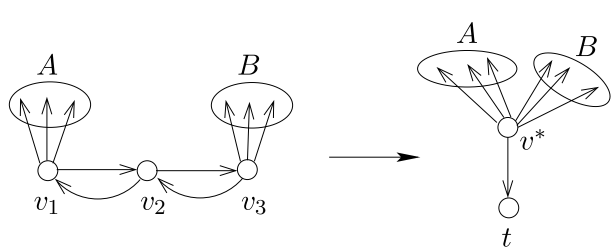

FIGURE 4. On the left the bidirected path P. On the right the graph G'.

Let $\delta = \bar{z}(v_3) - \bar{z}(v_2,v_3)$. Define z' from $\bar{z}$ as follows:

$$z'(v) = \begin{cases} \delta & \text{if } v = v^*, \\ 1 & \text{if } v = t, \\ \bar{z}(v) & \text{otherwise,} \end{cases} \quad ; \; z'(u,v) = \begin{cases} \bar{z}(v_1,v) & \text{if } u = v^* \text{ and } (v_1,v) \in A, \\ \bar{z}(v_3,v) & \text{if } u = v^* \text{ and } (v_3,v) \in A, \\ \bar{z}(v_2) & \text{if } u = v^* \text{ and } v = t, \\ \bar{z}(u,v) & \text{otherwise.} \end{cases}$$

Claim 1. G' has no multiple arcs, satisfies condition (i) of Theorem 2 and does not contain an odd Y-cycle.

Proof. (a) The graph G' does not contain multiple arcs. In fact, let a_1 and a_2 be two multiple arcs in G'. The node v^* must be their tail and let u be their head. Since $|\delta^-(u)| \geq 2$, by Lemma 15 u is not a pendent node. Let $(u,t') \in A$, by the definition of P, t' is different from v_1, v_2 and v_3. The cycle $C' = v_1,(v_1,v_2),v_2,(v_2,v_3),v_3,(v_3,u),u,(v_1,u),v_1$ is an odd Y-cycle ($u \in \hat{C}'$), which is not possible.

(b) If G' contains an odd Y-cycle C', we should assume that $v^* \in \dot{C}'$. Assume also that (v^*,u) and (v^*,v) are the two arcs in C' incident to v^*, where (v^*,u) was obtained from (v_1,u) and (v^*,v) was obtained from (v_3,v). Then by removing $(v^*,u),v^*,(v^*,v)$ from C' and adding $(v_1,u),v_1,(v_2,v_1),v_2,(v_2,v_3),v_3,(v_3,v)$, we obtain an odd Y-cycle in G, which is impossible.

(c) From (b) it follows that G' does not contain H_3. If G' contains one of the graphs H_1, H_2 or H_4 as a subgraph, then v^* belong to these graphs. Otherwise this subgraph exists in G too. By definition $\delta^-_{G'}(v^*) = \emptyset$. Suppose that G' contains H as a subgraph, where H is one of the graphs H_1, H_2 or H_4. Then $\delta^-_H(v^*) = \emptyset$. In this case, by replacing in H v^* by v_1 or v_3 with its corresponding arc in G, one obtain one of the graphs H_1, H_2 or H_4 as a subgraph in G, which is not possible.

□

Claim 2. z' is a fractional extreme point of $P_p(G')$.

Proof. Lemma 14 imply that $\bar{z}(v_2)$ is fractional. So at least $z'(v^*,t)$ is fractional.

Let us examine the validity of z'. By the definition of z', we only need to show that $\sum z'(v) = p$ and that equation (3) with respect to v^* is satisfied.

Notice that the validity of $\bar{z}$ implies that

$$\bar{z}(v_2) + \bar{z}(v_2,v_1) + \bar{z}(v_2,v_3) = 1. \tag{7}$$

Since $\bar{z}(v_2,v_1) = \bar{z}(v_1)$ and that $\bar{z}(v_2,v_3) = \bar{z}(v_3) - \delta$ when replacing in (7) we obtain

$$\bar{z}(v_2) + \bar{z}(v_1) + \bar{z}(v_3) = 1 + \delta, \tag{8}$$

$$\begin{aligned} \sum z'(v) &= \textstyle\sum_{v \in V} \bar{z}(v) - \bar{z}(v_1) - \bar{z}(v_2) - \bar{z}(v_3) + z'(v^*) + z'(t) \\ &= p - \bar{z}(v_1) - \bar{z}(v_2) - \bar{z}(v_3) + \delta + 1 \\ &= p. \end{aligned}$$

Now let us show that equation (3) with respect to v^* is satisfied as well.

The validity of $\bar{z}$ implies that

$$\bar{z}(\delta^+(v_1) \setminus \{(v_1, v_2)\}) + \bar{z}(v_1, v_2) + \bar{z}(v_1) = 1 \tag{9}$$
$$\bar{z}(\delta^+(v_3) \setminus \{(v_3, v_2)\}) + \bar{z}(v_3, v_2) + \bar{z}(v_3) = 1. \tag{10}$$

Adding equations (9) and (10) and replacing $\bar{z}(v_1, v_2)$ and $\bar{z}(v_3, v_2)$ by $\bar{z}(v_2)$, we obtain

$\bar{z}(\delta^+(v_1) \setminus \{(v_1, v_2)\}) + \bar{z}(\delta^+(v_3) \setminus \{(v_3, v_2)\}) + 2\bar{z}(v_2) + \bar{z}(v_1) + \bar{z}(v_3) = 2.$

By combining this last equation with (8), we obtain

$\bar{z}(\delta^+(v_1) \setminus \{(v_1, v_2)\}) + \bar{z}(\delta^+(v_3) \setminus \{(v_3, v_2)\}) + \bar{z}(v_2) + \delta = 1.$

By definition this last equation corresponds to equation (3) with respect to v^*.

Now, let us show that z' is an extreme point of $P_p(G')$. Suppose the contrary, then there must exist $z'' \in P_p(G')$ where every constraint tight for z' is also tight for z''. Let

$$\alpha = \sum_{u:(v_1,u)\in A} z''(v^*, u),$$

$$\beta = \sum_{u:(v_3,u)\in A} z''(v^*, u).$$

Notice that $z''(v^*) + z''(v^*, t) + \alpha + \beta = 1$. Let z^* be the extension of z'' to $P_p(G)$ defined as follows:

$$z^*(v) = \begin{cases} \beta + z''(v^*) & \text{if } v = v_1, \\ z''(v^*, t) & \text{if } v = v_2, \\ \alpha + z''(v^*) & \text{if } v = v_3, \\ z''(v) & \text{otherwise,} \end{cases}$$

$$z^*(u, v) = \begin{cases} z''(v^*, v) & \text{if } u = v_1 \text{ and } v \neq v_2, \\ z''(v^*, v) & \text{if } u = v_3 \text{ and } v \neq v_2, \\ z''(v^*, t) & \text{if } v = v_2 \text{ and } u = v_1 \text{ or } v_3, \\ \alpha & \text{if } u = v_2 \text{ and } v = v_3, \\ \beta + z''(v^*) & \text{if } u = v_2 \text{ and } v = v_1, \\ z''(u, v) & \text{otherwise.} \end{cases}$$

It is easy to check that $z^* \in P_p(G)$ and that every constraint tight for $\bar{z}$ is also tight for z^*, which contradicts the fact that $\bar{z}$ is an extreme point of $P_p(G)$. □

3.1. **Proof of Lemma 8.** In this sub-section we assume that $\bar{z}$ is a fractional extreme point of $P_p(G)$, such that

(11) $\bar{z}(u, v) = \bar{z}(v)$ for every arc $(u, v) \in A$, when v is not a pendent node.

The proof of Lemma 8 will be given at the end of this sub-section. Next, we give several lemmas useful for that proof.

Lemma 17. *Let (v, w), (w, v) and (w, t) be three arcs in A. Then $|\delta^+(v)| \geq 2$.*

Proof. Suppose the contrary, that is $\delta^+(v) = \{(v, w)\}$. Since v and w are not pendent nodes, assumption (11) implies $\bar{z}(w, v) = \bar{z}(v)$ and $\bar{z}(v, w) = \bar{z}(w)$. Constraint (3) with respect to v implies $\bar{z}(v, w) = 1 - \bar{z}(v)$. Thus $\bar{z}(w) = 1 - \bar{z}(v) = 1 - \bar{z}(w, v)$. Hence constraint (3) with respect to w implies that $\bar{z}(w, t) = 0$, which contradicts Lemma 12. □

Lemma 18. *We can assume that G does not contain a bidirected path P of size four, where its internal nodes are adjacent to only their neighbors in P.*

Proof. Assume the contrary. Let $P = v_1, v_2, v_3, v_4$ a bidirected path of size four, where $\delta^+(v_2) = \{(v_2, v_1), (v_2, v_3\}$, $\delta^-(v_2) = \{(v_1, v_2), (v_3, v_2)\}$, $\delta^+(v_3) = \{(v_3, v_2), (v_3, v_4)\}$ and $\delta^-(v_3) = \{(v_2, v_3), (v_4, v_3)\}$.

Consider the graph $G' = (V', A')$ obtained from G by identifying the nodes v_1 and v_4 and removing the nodes v_2 and v_3 (with their incident arcs). Call v^* the node that results from identifying v_1 and v_4. See Figure 5.

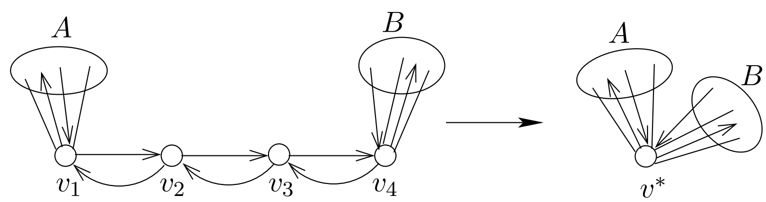

FIGURE 5. On the left the bidirected path of Lemma 18. On the right the graph G'.

Define z' from $\bar{z}$ as follows:

$$z'(v) = \begin{cases} \bar{z}(v_2, v_1) & \text{if } v = v^* \\ \bar{z}(v) & \text{otherwise,} \end{cases} \quad ; \; z'(u, v) = \begin{cases} \bar{z}(v_1, v) & \text{if } u = v^* \text{ and } (v_1, v) \in A, \\ \bar{z}(u, v_1) & \text{if } v = v^* \text{ and } (u, v_1) \in A, \\ \bar{z}(v_4, v) & \text{if } u = v^* \text{ and } (v_4, v) \in A, \\ \bar{z}(u, v_4) & \text{if } v = v^* \text{ and } (u, v_4) \in A, \\ \bar{z}(u, v) & \text{if } u \neq v^* \text{ and } v \neq v^*. \end{cases}$$

We will prove that G' has the same properties as G and that z' is a fractional extreme point of $P_{p'}(G')$, for some positive integer p'.

Claim 1. v_1 and v_4 have no neighbor in common.

Proof. Let u be a common neighbor of v_1 and v_4. We have four cases to consider:

(a) (v_1, u) and (u, v_4) are in A. Then the ordered sequence $v_1, u, v_4, v_3, v_2, v_1$ defines and odd directed cycle, which is not possible.

(b) (u, v_1) and (v_4, u) are in A. By symmetry we get the same contradiction as in (a).

(c) (u, v_1) and (u, v_4) are in A. By Lemma 17, $|\delta^+(v_1)| \geq 2$. Thus there must exist an arc (v_1, v'), with $v' \notin \{v_1, v_2, v_3, v_4\}$. Suppose $v' = u$. Then the ordered sequence u, v_4, v_3, v_2, v_1, u defines a directed odd cycle in G, which is impossible. And if $v' \neq u$, then the cycle $C' = u, (u, v_1), v_1, (v_2, v_1), v_2, (v_2, v_3), v_3, (v_3, v_4), v_4, (u, v_4), u$ is an odd Y-cycle, (v_1 and v_4 are in $\hat{C}'$ and $v_3 \in \tilde{C}'$). This contradicts the fact that G does not contain an odd Y-cycle.

(d) (v_1, u) and (v_4, u) are in A. Lemma 15 implies that u is not a pendent node. Thus we must have an arc $(u, v) \in A$. The node v is different from v_2 and v_3. Suppose that v is different from v_1 and v_4. Then $C' = u, (v_1, u), v_1, (v_1, v_2), v_2, (v_3, v_2), v_3, (v_4, v_3), v_4, (v_4, u), u$ is an odd Y-cycle (u and v_2 are in $\hat{C}'$ and v_3 in $\tilde{C}'$). If $v = v_4$, then the ordered sequence u, v_4, v_3, v_2, v_1, u define an odd directed cycle. Also if $v = v_1$ one can construct by symmetry and odd directed cycle. In all cases, G contain an odd Y-cycle, which is not possible.

□

Claim 2. G' does not contain an odd Y-cycle.

Proof. Assume the contrary and let C' be an odd Y-cycle in G'. The cycle C' must contain the node v^*, otherwise C' is an odd Y-cycle in G too, which is impossible. We distinguish four cases as shown in Figure 6.

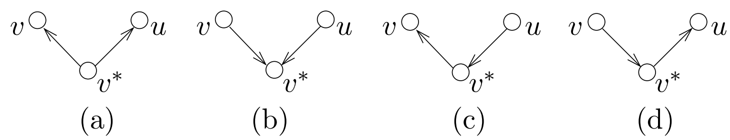

FIGURE 6. v^* with its incident arcs in C'.

(a) $v^* \in \dot{C}'$. Let $(v_1, v) \in A$ and $(v_4, u) \in A$. Let C be the Y-cycle in G obtained from C' by removing the node v^* and the arcs (v^*, u) and (v^*, v), and by adding the nodes v_1, v_2, v_3, v_4 and the arcs (v_1, v), (v_1, v_2), (v_2, v_3), (v_4, v_3) and (v_4, u). We have $|V(C)| = |V(C')|+3$ and $|\hat{C}| = |\hat{C}'|+1$. These imply that $|V(C)|+|\hat{C}| = |V(C')| + |\hat{C}'| + 4$. Thus C is odd, which is impossible.

(b) $v^* \in \hat{C}'$. Let $(v, v_1) \in A$ and $(u, v_4) \in A$. We have two sub-cases:
  - Suppose that there is an arc $(v^*, t) \in A'$, $t \notin V(C')$. Suppose that (v^*, t) was obtained from $(v_1, t) \in A$. Let C be the Y-cycle in G obtained from C' by removing the node v^* and the arcs (u, v^*) and (v, v^*), and by adding the nodes v_1, v_2, v_3, v_4 and the arcs (v, v_1), (v_2, v_1), (v_2, v_3), (v_4, v_3) and (u, v_4). We have that $|V(C)| + |\hat{C}| = |V(C')| + |\hat{C}'| + 4$. So C is an odd Y-cycle of G.
  - If the arc $(v^*, t) \in A'$, $t \notin V(C')$, does not exist, we have that $u \in \tilde{C}'$, say. Also $(v^*, u) \in A'$. Let C be the Y-cycle in G obtained from C' by removing the node v^* and the arcs (u, v^*) and (v, v^*), and by adding the nodes v_1, v_2, v_3, v_4 and the arcs (v, v_1), (v_1, v_2), (v_3, v_2), (v_3, v_4) and (u, v_4). We have that $|V(C)| + |\hat{C}| = |V(C')| + |\hat{C}'| + 4$. Thus C is odd, which is impossible.

(c) $v^* \in \tilde{C}'$. Let $(v_1, v) \in A$ and $(u, v_4) \in A$. Let C be the Y-cycle in G obtained from C' by removing the node v^* and the arcs (u, v^*) and (v^*, v), and by adding the nodes v_1, v_2, v_3, v_4 and the arcs (v_1, v), (v_2, v_1), (v_2, v_3), (v_3, v_4) and (u, v_4). We have that C is an odd Y-cycle, a contradiction.

(d) This case is similar to the case (c).

□

Claim 3. G' does not contain any of the graphs H_i, $1 \le i \le 4$, as a subgraph.

Proof. By Claim 2 G' cannot contain H_3. Remark that $|\delta^-(v^*)| \le 2$, otherwise G contains H_4 as a subgraph. When $|\delta^-(v^*)| \le 1$, the claim is straightforward. Hence we assume that $|\delta^-(v^*)| = 2$. Let $\delta^-(v^*) = \{(u, v^*), (v, v^*)\}$, then in G we must have $\delta^-(v_1) = \{(u, v_1), (v_2, v_1)\}$ and $\delta^-(v_4) = \{(v, v_4), (v_3, v_4)\}$, otherwise G contains H_4. If G' contains one of the graphs H_1, H_2 or H_4, then v^* must belong to these graphs otherwise these graphs exist in G too. We also suppose that v^* is the head of at least two arcs in these graphs, the other cases are straightforward. Since $|\delta^-(v^*)| = 2$, then G' cannot contain H_2 nor H_4.

Assume that G' contains H_1. Let (u, v^*), (v, v^*), (v^*, w) and (w, t) the four arcs that compose H_1. Assume that (u, v_1) and (v, v_4) are in G. We must have (v_1, w) or (v_4, w) in G. Say (v_1, w) is an arc of G. Then the four arcs (u, v_1), (v_2, v_1), (v_1, w) and (w, t) are in G. Thus G contains H_1 as a subgraph, which is impossible. □

Claim 4. $z' \in P_{p-1}(G')$.

Proof. The definition of P, assumption (11) and equalities (3) with respect to v_1, v_2, v_3 and v_4 imply the following:

$$\bar{z}(v_2) + \bar{z}(v_2, v_1) + \bar{z}(v_2, v_3) = 1, \tag{12}$$
$$\bar{z}(v_1) = \bar{z}(v_2, v_1), \tag{13}$$
$$\bar{z}(\delta^+(v_1) \setminus \{(v_1, v_2)\}) = \bar{z}(v_2, v_3), \tag{14}$$
$$\bar{z}(v_3) = \bar{z}(v_2, v_3), \tag{15}$$
$$\bar{z}(v_4) = \bar{z}(v_2, v_1), \tag{16}$$
$$\bar{z}(\delta^+(v_4) \setminus \{(v_4, v_3)\}) = \bar{z}(v_2). \tag{17}$$

Any constraint that does not contain $z'(v^*)$ is satisfied by definition. Let us examine those constraints that contain $z'(v^*)$.

- Let us show that z' satisfies equality (2).

$$\begin{aligned}\sum_{v \in V'} z'(v) &= \sum_{v \in V \setminus \{v_1, v_2, v_3, v_4\}} \bar{z}(v) + z'(v^*) \\ &= p - \bar{z}(v_1) - \bar{z}(v_2) - \bar{z}(v_3) - \bar{z}(v_4) + z'(v^*).\end{aligned}$$

  By (13) $\bar{z}(v_1) = \bar{z}(v_2, v_1)$ and by (15) $\bar{z}(v_3) = \bar{z}(v_2, v_3)$. Replacing this in (12), we obtain $\bar{z}(v_1) + \bar{z}(v_2) + \bar{z}(v_3) = 1$. Also from (16) and the definition of $z'(v^*)$ we have that $\bar{z}(v_4) = z'(v^*)$. Thus $\sum_{v \in V'} z'(v) = p - 1$.
- Let us show that z' satisfies equality (3) with respect to v^*. We have

$$z'(\delta^+(v^*)) + z'(v^*) = \bar{z}(\delta^+(v_1) \setminus \{(v_1, v_2)\}) + \bar{z}(\delta^+(v_4) \setminus \{(v_4, v_3)\}) + z'(v^*).$$

  If we combine the above equality with (14) and (17), we obtain

$$z'(\delta^+(v^*)) + z'(v^*) = \bar{z}(v_2, v_3) + \bar{z}(v_2) + z'(v^*).$$

  Now replace $z'(v^*)$ of the right hand side of the above equality by its value and evaluate this side using (12), we get

$$z'(\delta^+(v^*)) + z'(v^*) = 1.$$

- Finally, let us show that z' verifies (4) with respect to v^*. Let (u, v^*) be an arc in G' and let us show that $z'(u, v^*) \le z'(v^*)$.

  By definition $z'(u, v^*) = \bar{z}(u, v_1)$ or $z'(u, v^*) = \bar{z}(u, v_4)$. The definition of $z'(v^*)$, (13) and (16) imply $z'(v^*) = \bar{z}(v_1) = \bar{z}(v_4)$. Hence the fact that $\bar{z}(u, v_1) \le \bar{z}(v_1)$ or $\bar{z}(u, v_4) \le \bar{z}(v_4)$ imply immediately $z'(u, v^*) \le z'(v^*)$. Also remark that $z'(v^*, u) \le z'(u)$ for all $(v^*, u) \in A'$.

□

Claim 5. z' is a fractional extreme point of $P_{p-1}(G')$.

Proof. By Claim 4, we have $z' \in P_{p-1}(G')$. Lemma 14 and the definition of z' imply that z' is fractional. Suppose that z' is not an extreme point of $P_{p-1}(G')$. Thus there must exist $z'' \in P_{p-1}(G')$, $z'' \neq z'$, where each constraint that is tight for z' is also tight for z''. Let

$$\alpha = \sum_{u:(v_1,u)\in A} z''(v^*,u),$$
$$\beta = \sum_{u:(v_4,u)\in A} z''(v^*,u).$$

Notice that $z''(v^*)+\alpha+\beta = 1$. Let z^* be the extension of z'' to $P_p(G)$ defined as follows:

$$z^*(v) = \begin{cases} z''(v^*) & \text{if } v = v_1 \text{ or } v = v_4, \\ \beta & \text{if } v = v_2, \\ \alpha & \text{if } v = v_3, \\ z''(v) & \text{otherwise,} \end{cases}$$

$$z^*(u,v) = \begin{cases} z''(v^*,v) & \text{if } u = v_1 \text{ and } v \neq v_2, \\ z''(u,v^*) & \text{if } u \neq v_2 \text{ and } v = v_1, \\ z''(v^*,v) & \text{if } u = v_4 \text{ and } v \neq v_3, \\ z''(u,v^*) & \text{if } u \neq v_3 \text{ and } v = v_4, \\ \beta & \text{if } v = v_2 \text{ and } u = v_1 \text{ or } v_3, \\ \alpha & \text{if } v = v_3 \text{ and } u = v_2 \text{ or } v_4, \\ z''(v^*) & \text{if } (u,v) = (v_2,v_1) \text{ or } (v_3,v_4), \\ z''(u,v) & \text{otherwise.} \end{cases}$$

It is easy to check that $z^* \in P_p(G)$ and that every constraint tight for $\bar{z}$ is also tight for z^*, which contradicts the fact that $\bar{z}$ is an extreme point of $P_p(G)$. □

Claim 1 implies that G' has no multiple arcs. Hence Claims 1, 2 and 3 show that G' has the same properties as G. Claim 5 shows that z' is a fractional extreme point of $P_{p-1}(G')$. This completes the proof of this lemma. □

Lemma 19. *G does not contain a bidirected path $P = v_1, v_2, v_3$, satisfying the following:*

(i) $(v_3,t) \in A$ *with t a pendent node, and*
(ii) $\delta^-(v_1) = \{(v_2,v_1)\}$.

Proof. Suppose the contrary and let $P = v_1, v_2, v_3$ be a bidirected path satisfying (i) and (ii). Let l be a labeling function, where the node v_2 with the arcs (v_1,v_2) and (v_3,v_2) receive the label 1; the node v_1 with the arcs (v_2,v_1) and (v_3,t) receive the label -1; and all other nodes and arcs receive the label 0.

The vector $\bar{z}_l$ satisfies with equality each constraint among (2)-(6) that was satisfied with equality by $\bar{z}$. In fact, Lemma 14 implies that the value of $\bar{z}$, corresponding to the nodes and arcs that received a label different from 0, is fractional. This implies that any inequalities (5) or (6) that are satisfied with equality by $\bar{z}$ remain satisfied with equality by $\bar{z}_l$. Let us see that equations (3) are satisfied. The arcs that receive a non-zero label are incident to the nodes v_1, v_2 and v_3. Equation (3) with respect to v_1 is satisfied since v_1 and (v_1,v_2) receive opposite labels, the same holds for v_2. Also, the unique arcs incident to v_3 that receive a non-zero label are (v_3,v_2) and (v_3,t), and they receive opposite labels. Since v_3 receives a zero label, then equation (3) with respect to v_3 is satisfied. Equality (2) is satisfied since v_1 and v_2 received opposite labels and the other nodes received the label 0.

The unique nodes with labels different from 0 are v_1 and v_2. Notice that (v_2, v_1) received the same label as v_1 and by hypothesis (ii) is the unique arc directed into v_1. Also by Remark 11 the only arcs directed into v_2 are (v_1, v_2) and (v_3, v_2) and they received the same label as v_2. Hence any inequality (4) that is satisfied with equality by $\bar{z}$ remains satisfied with equality by $\bar{z}_l$. This is in contradiction with the fact that $\bar{z}$ is an extreme point of $P_p(G)$. □

Lemma 20. *G does not contain a bidirected path $P = v_1, v_2, v_3, v_4$, such that v_1 and v_4 are adjacent to a pendent node.*

Proof. Suppose the contrary and let $P = v_1, v_2, v_3, v_4$ be a bidirected path such that (v_1, t) and (v_4, t') are in A, where t and t' are pendent nodes.

Assign the label 1 to the node v_3 and the arcs (v_1, t), (v_2, v_3) and (v_4, v_3); assign the label -1 to the node v_2 and the arcs (v_1, v_2), (v_3, v_2) and (v_4, t'); assign to the other nodes and arcs the label 0. Call this labeling l.

As in the proof of Lemma 19, one can easily check that $\bar{z}_l$ satisfies with equality any constraint among (2)-(6) that is satisfied with equality by $\bar{z}$. This contradicts the fact that $\bar{z}$ is an extreme point of $P_p(G)$. □

Lemma 21. *If G contains a cycle of size at least three, then it contains a Y-cycle of the same size.*

Proof. Let $C' = v_0, a_0, v_1, a_1, \ldots, a_{p-1}, v_p$, be a simple cycle with $p \geq 3$. Suppose that C' is a not a Y-cycle. There must exist a node $v_i \in \hat{C}'$ where conditions (i) and (ii) of Definition 1 are not satisfied. Let (v_{i-1}, v_i) and (v_{i+1}, v_i) be the two arcs of C' directed into v_i. By Lemma 13, $\bar{z}(v_i) > 0$. Since v_i is not a pendent node, there must exist an arc (v_i, u) in G. The fact that (i) is not satisfied implies that $u \in V(C')$. If u is different from v_{i-1} and v_{i+1}, then C' is of size at least four. In this case, G must contain one of the graphs H_1 or H_3 as a subgraph, which is impossible. Thus $\delta^+(v_i)$ consists of one of the arcs (v_i, v_{i-1}) or (v_i, v_{i+1}), or both. Assume $(v_i, v_{i-1}) \in A$, since Definition 1 (ii) is not satisfied v_{i-1} must be in $\dot{C}'$, so $(v_{i-1}, v_{i-2}) \in A(C')$ with $v_{i-2} \in V(C')$. Then Lemma 17 implies that $(v_i, v_{i+1}) \in A$. Also $(v_i, v_{i+1}) \in A$ implies $v_{i+1} \in \dot{C}'$, so $(v_{i+1}, v_{i+2}) \in A(C')$, (see Figure 7).

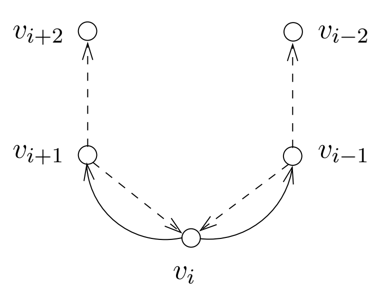

FIGURE 7. Dashed lines represent arcs in C'.

Thus we may suppose that for any node $v_i \in \hat{C}'$ that does not satisfy Definition 1 (i) and (ii), $\delta^+(v_i) = \{(v_i, v_{i-1}), (v_i, v_{i+1})\}$ and both nodes v_{i-1} and v_{i+1} are in $\dot{C}'$. Define C from C', recursively, following the procedure below:

Step 0. $A(C) \leftarrow A(C')$, $V(C) \leftarrow V(C')$, $C \leftarrow C'$.
Step 2. If there exist $v_i \in \hat{C}$, a node not satisfying Definition 1 (i) and (ii), go to Step 3. Otherwise stop, C is a Y-cycle.
Step 3. $A(C) \leftarrow (A(C) \setminus \{(v_{i-1}, v_i), (v_{i+1}, v_i)\}) \cup \{(v_i, v_{i-1}), (v_i, v_{i+1})\}$. C is the new cycle defined by $A(C)$. Go to Step 2.

Each Step 3 decreases by one the number of nodes in $\hat{C}$. Thus the procedure must end. □

Lemma 22. *Let $C = v_0, a_0, v_1, a_1, \ldots, a_{p-1}, v_p$, $p \geq 3$, be an even Y-cycle with $|\hat{C}_{(i)}|$ maximum. Then C does not contain a blocking node.*

Proof. Suppose that C contains a blocking node v_i.

Case 1. v_i is a blocking node satisfying Definition 6 (i). Thus $v_i \in \tilde{C}$, (v_{i-1}, v_i), (v_i, v_{i+1}) in $A(C)$, (v_{i+1}, v_i) in $A \setminus A(C)$ and $v_{i+1} \in \tilde{C}$. Thus $(v_{i+1}, v_{i+2}) \in A(C)$ (see Figure 8). Notice that $v_{i+2} \neq v_{i-1}$, otherwise C is a directed odd cycle.

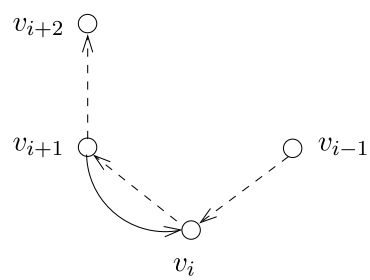

FIGURE 8. Dashed lines represent arcs in C.

Claim 1. If $(v_i, u) \in A$, then $u \in V(C)$.

Proof. Suppose the contrary, let $(v_i, u) \in A$ with $u \notin V(C)$. The node v_{i+2} is not in $\tilde{C}$, otherwise the cycle C', where $V(C') = V(C)$ and $A(C') = (A(C) \setminus \{(v_i, v_{i+1})\}) \cup \{(v_{i+1}, v_i)\}$, is an odd Y-cycle. Thus v_{i+2} must be in $\hat{C}$. If the cycle C' as defined previously is a Y-cycle, then it is odd. Thus C' is not a Y-cycle, which implies that $(v_{i+2}, v_{i+1}) \in A \setminus A(C)$ and $v_{i+2} \notin \hat{C}_{(i)}$. Replace the arcs (v_i, v_{i+1}) and (v_{i+1}, v_{i+2}) by (v_{i+1}, v_i) and (v_{i+2}, v_{i+1}). Call C'' the resulting cycle. It is easy to check that C'' is a Y-cycle with $|\hat{C}''_{(i)}| = |\hat{C}_{(i)}| + 1$, this contradicts the fact that C is a Y-cycle with $|\hat{C}_{(i)}|$ maximum. □

Claim 2. $\delta^+(v_i) = \{(v_i, v_{i-1}), (v_i, v_{i+1})\}$ and $\delta^-(v_i) = \{(v_{i-1}, v_i), (v_{i+1}, v_i)\}$.

Proof. Lemma 17 implies that $|\delta^+(v_i)| \geq 2$. Let $(v_i, u) \in \delta^+(v_i)$, where $u \neq v_{i+1}$. Claim 1 implies that $u \in V(C)$. If $u \neq v_{i-1}$, then G contains one of the graphs of Figure 1 as a subgraph. Thus $u = v_{i-1}$ and $\delta^+(v_i) = \{(v_i, v_{i-1}), (v_i, v_{i+1})\}$. To finish the proof of this claim, remark that since G does not contain H_4 as a subgraph the only arcs in A directed into v_i are (v_{i-1}, v_i) and (v_{i+1}, v_i). □

Claim 3. $v_{i-1} \in \dot{C}$.

Proof. Suppose that $v_{i-1} \notin \dot{C}$. It follows that $v_{i-1} \in \tilde{C}$, thus $(v_{i-2}, v_{i-1}) \in A(C)$. Remark that v_{i-1} is a blocking node satisfying Definition 6 (i). Thus Claim 2 may be applied to v_{i-1}, so $\delta^+(v_{i-1}) = \{(v_{i-1}, v_{i-2}), (v_{i-1}, v_i)\}$ and $\delta^-(v_{i-1}) = \{(v_{i-2}, v_{i-1}), (v_i, v_{i-1})\}$. Thus the sequence $P = v_{i-2}, v_{i-1}, v_i, v_{i+1}$ is a bidirected path of size four, where its internal nodes v_i and v_{i-1} are adjacent to only their neighbors in P. This contradicts Lemma 18. □

Thus v_{i-1} must be in $\dot{C}$ and $(v_{i-1}, v_{i-2}) \in A(C)$, as shown by Figure 9. Notice that $v_{i-2} \neq v_{i+2}$, otherwise the Y-cycle C will be odd.

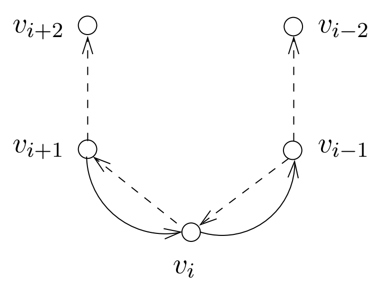

FIGURE 9. Dashed lines represent arcs in C.

$P = v_{i-1}, v_i, v_{i+1}$ is a bidirected path of size three. Lemma 16 implies that at least one of the arcs (u, v_{i-1}) or (u, v_{i+1}) exists, with $u \neq v_i$.

Suppose $(u, v_{i-1}) \in A$. The case when $(u, v_{i+1}) \in A$ is symmetric. Since v_{i-2} is not a pendent node, Remark 10 implies that $u = v_{i-2}$, so $\delta^-(v_{i-1}) = \{(v_i, v_{i-1}), (v_{i-2}, v_{i-1})\}$. If $\delta^+(v_{i-1}) = \{(v_{i-1}, v_i), (v_{i-1}, v_{i-2})\}$, then $P = v_{i-2}, v_{i-1}, v_i, v_{i+1}$ is a bidirected path that contradicts Lemma 18. Hence we may assume that $(v_{i-1}, t) \in A$ and t is a pendent node.

If $\delta^-(v_{i+1}) = \{(v_i, v_{i+1})\}$, then $P = v_{i-1}, v_i, v_{i+1}$ is a bidirected path satisfying the conditions (i) and (ii) of Lemma 19, which is impossible. Thus we must have an arc $(u, v_{i+1}) \in A$ with $u \neq v_i$. Since v_{i+2} is not a pendent node, Remark 10 implies that $u = v_{i+2}$. There must exist also an arc $(v_{i+1}, t') \in A$, where t' is a pendent node, otherwise the bidirected path $P = v_{i-1}, v_i, v_{i+1}, v_{i+2}$ contradicts Lemma 18. The situation is summarized in Figure 10.

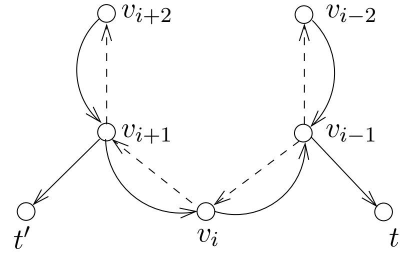

FIGURE 10. Dashed lines represent arcs in C.

If $v_{i+2} \in \tilde{C}$, then v_{i+1} is a blocking node satisfying Definition 6 (i). But since $(v_{i+1}, t') \in A$ and $t' \notin V(C)$, this contradicts Claim 1. Thus $v_{i+2} \in \hat{C}$. We claim that $v_{i+2} \notin \hat{C}_{(i)}$. Indeed, suppose the contrary let $(v_{i+2}, u) \in A$ and $u \notin V(C)$. The node u must be a pendent node, otherwise G contains one of the graphs H_1, H_3 or H_4 as a subgraph. Thus, the sequence $P = v_{i-1}, v_i, v_{i+1}, v_{i+2}$ is a bidirected path of size four, where v_{i-1} and v_{i+2} are adjacent to pendent nodes, which is impossible by Lemma 20.

Now it is easy to check that the cycle C' obtained from C by removing (v_{i+1}, v_{i+2}) and adding (v_{i+2}, v_{i+1}) is a Y-cycle with $|\hat{C'}_{(i)}| = |\hat{C}_{(i)}| + 1$. This contradicts the fact that C is chosen so that $|\hat{C}_{(i)}|$ is maximum.

Case 2. v_i is a blocking node satisfying Definition 6 (ii). Thus $v_i \in \hat{C}$; (v_{i-1}, v_i), (v_{i+1}, v_i) belong to $A(C)$; (v_i, v_{i+1}), (v_i, v_{i-1}) belong to $A \setminus A(C)$; and $v_{i-1}, v_{i+1} \in \tilde{C}$. It follows that (v_{i+2}, v_{i+1}) and (v_{i-2}, v_{i-1}) are in $A(C)$ (see Figure 11). Notice that $v_{i+2} \neq v_{i-2}$, otherwise C is an odd Y-cycle.

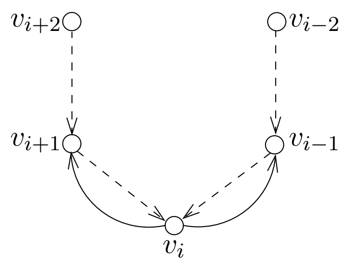

FIGURE 11. Dashed lines represent arcs in C.

Lemma 17 implies that $(v_{i-1}, u) \in A$ and $(v_{i+1}, u') \in A$, with $u \neq v_i$, $u' \neq v_i$. By Remark 10, u is a pendent node or $u = v_{i-2}$, and also u' is a pendent node or $u' = v_{i+2}$. Also both nodes v_{i-1} and v_{i+1} cannot be adjacent to a pendent node. Otherwise, the cycle obtained from C by removing (v_{i-1}, v_i) and (v_{i+1}, v_i), and by adding (v_i, v_{i-1}) and (v_i, v_{i+1}) is an odd Y-cycle, which is not possible. Thus we have two sub-cases; at least

(a) $u = v_{i-2}$ and v_{i-1} is not adjacent to a pendent node, or
(b) $u' = v_{i+2}$ and v_{i+1} is not adjacent to a pendent node.

Below we treat sub-case (a), the sub-case (b) is symmetric. Let $u = v_{i-2}$, v_{i-1} is not adjacent to a pendent node and $(v_{i-1}, v_{i-2}) \in A \setminus A(C)$. The node v_i must be adjacent to a pendent node t, otherwise the bidirected path $P = v_{i-2}, v_{i-1}, v_i, v_{i+1}$ contradicts Lemma 18. The situation is described in Figure 12

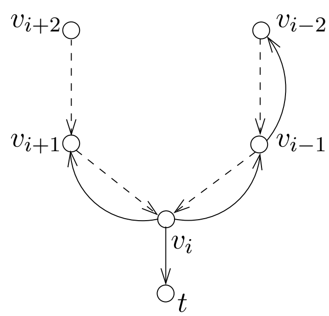

FIGURE 12. Dashed lines represent arcs in C.

The node v_{i-2} must be in $\dot{C}$. Otherwise, v_{i-2} is a blocking node by Definition 6 (i), which is impossible as shown in Case 1. Thus, $(v_{i-2}, v_{i-3}) \in A(C)$. By Lemma 19, we must have an arc (u', v_{i-2}), $u' \neq v_{i-1}$. Since v_{i-3} is not a pendent node, Remark 10 implies $u' = v_{i-3}$. Also, Lemma 18 implies that v_{i-2} is adjacent to a pendent node t'.

If $v_{i-3} = v_{i+2}$ then the cycle C' obtained from C, by replacing the arc (v_{i-2}, v_{i-3}) by (v_{i-3}, v_{i-2}) and replacing the arc (v_{i-2}, v_{i-1}) by (v_{i-1}, v_{i-2}), is an odd Y-cycle. Thus $v_{i-3} \neq v_{i+2}$.

If $(v_{i-3}, v_{i-4}) \in A(C)$ and $v_{i-4} \in \tilde{C}$, then the cycle C' as defined above is an odd Y-cycle. A contradiction.

Suppose $(v_{i-3}, v_{i-4}) \in A(C)$ and $v_{i-4} \in \hat{C}$. Since $v_{i+2} \notin \hat{C}$, we have $v_{i-4} \neq v_{i+2}$. If the cycle C' as defined above is a Y-cycle, then it is odd. Thus C' is not a Y-cycle, which implies that $(v_{i-4}, v_{i-3}) \in A \backslash A(C)$ and $v_{i-4} \notin \hat{C}_{(i)}$. Thus the cycle C'' defined from C by replacing (v_{i-3}, v_{i-4}) by (v_{i-4}, v_{i-3}), replacing (v_{i-2}, v_{i-3}) by (v_{i-3}, v_{i-2}) and replacing (v_{i-2}, v_{i-1}) by (v_{i-1}, v_{i-2}) is a Y-cycle with $|\hat{C}''_{(i)}| = |\hat{C}_{(i)}| + 1$, which contradicts the fact that $|\hat{C}_{(i)}|$ is maximum. Figure 13 illustrates this case.

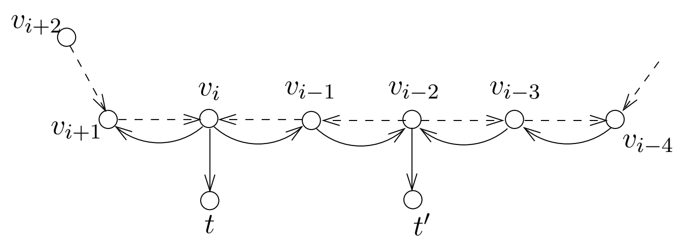

FIGURE 13. Dashed lines represent arcs in C. The node v_{i-4} is not in $\hat{C}_{(i)}$.

Now suppose $(v_{i-4}, v_{i-3}) \in A(C)$, so $v_{i-3} \in \hat{C}$. In this case, the cycle obtained from C by replacing (v_{i-2}, v_{i-3}) by (v_{i-3}, v_{i-2}) is an odd Y-cycle, this is again a contradiction. □

Lemma 23. *G does not contain a cycle of size at least three.*

Proof. Assume the contrary. Suppose that G admits such a cycle. From Lemma 21, we may assume that G contains an even Y-cycle. Among all these Y-cycles, let $C = v_0, a_0, v_1, a_1, \ldots, a_{p-1}, v_p$ be an even Y-cycle such that $|\hat{C}_{(i)}|$ is maximum. Lemma 22 implies that C does not contain a blocking node. Hence assumption (a1) of Lemma 7 is satisfied. Also $\bar{z} \in P_p(G)$ and Lemma 14 implies that assumption (a2) of Lemma 7 is satisfied. Also the graph G is a directed graph with no multiple arcs and satisfies (i) of Theorem 2. It follows from Lemma 7 that $\bar{z}$ is not an extreme point of $P_p(G)$, a contradiction. □

Now we can prove the main result of this sub-section.

Proof of Lemma 8:

Denote by $Pair(G)$ the set of pair of nodes $\{u, v\}$ such that both arcs (u, v) and (v, u) belong to A.

The proof is by induction on $|Pair(G)|$. If $|Pair(G)| = 0$ then G is an oriented graph that satisfies conditions (i) and (ii) of Theorem 3. Thus by Theorem 3, $P_p(G)$ has no fractional extreme point so the lemma is true. Suppose that the lemma is true for every directed graph H with no multiple arcs, no odd Y-cycle and satisfying condition (i) of Theorem 2 and $|Pair(H)| \leq m$, $m \geq 0$. Let $G = (V, A)$ be a directed graph with no multiple arcs, no odd Y-cycle, satisfying condition (i) of Theorem 2 and such that $|Pair(G)| = m + 1$.

Let $\bar{z}$ be a fractional extreme point of $P_p(G)$ where $\bar{z}(u, v) = \bar{z}(v)$ for each arc (u, v) with v not a pendent node. Notice that Lemma 23 applies, so G does not contain a cycle.

Let (u, v) and (v, u) be two arcs in G. Denote by $G(u, v)$ the graph obtained from G by removing the arc (u, v) and adding a new arc (u, t), where t is a new pendent node. Define $\tilde{z} \in P_{\tilde{p}}(G(u, v))$, $\tilde{p} = p + 1$, to be $\tilde{z}(u, t) = \bar{z}(u, v)$, $\tilde{z}(t) = 1$ and $\tilde{z}(r) = \bar{z}(r)$, $\tilde{z}(r, s) = \bar{z}(r, s)$ for every other node and arc.

The graph $G(u, v)$ is directed with no multiple arcs and satisfies condition (i) of Theorem 2. Since G does not contain a cycle, we have that $G(u, v)$ has no odd Y-cycle. Moreover $|Pair(G(u, v))| \leq m$, hence the induction hypothesis applies for $G(u, v)$. We have that $\tilde{z}$ is a fractional vector in $P_{\tilde{p}}(G(u, v))$ with $\tilde{z}(u, v) = \tilde{z}(v)$ for each arc (u, v), with v not pendent. By the induction hypothesis $\tilde{z}$ is not an extreme point. Thus, there must exist a set of extreme points of $P_{\tilde{p}}(G(u, v))$, $z^1, \ldots, z^k$, where each constraint that is tight for $\tilde{z}$ is also tight for each of $z^1, \ldots, z^k$, and $\tilde{z}$ is a convex combination of $z^1, \ldots, z^k$. Let us see that all this extreme points are in 0-1. In fact, suppose that z^1 is a fractional extreme point of $P_{\tilde{p}}(G(u, v))$. By the induction hypothesis, we must have an arc (u', v') in $G(u, v)$ with v' is not a pendent node and $z^1(u', v') < z^1(v')$. Since v' is not a pendent node, then by construction the arc (u', v') is in G too. Thus we must have $\tilde{z}(u', v') < \tilde{z}(v')$. But this implies that v' must be a pendent node, a contradiction.

Since all the extreme points $z^1, \ldots, z^k$ are in 0-1 and $\tilde{z}(v, u) > 0$, there must exist one vector among $z^1, \ldots, z^k$, say z^1, with $z^1(v, u) = 1$. From z^1 define $z'' \in P_p(G)$ as follows: $z''(u, v) = z^1(u, t)$ and $z''(r, s) = z^1(r, s)$, $z''(r) = z^1(r)$, for all other nodes and arcs. All constraints that are tight for $\bar{z}$ are also tight for z''. To see this, it suffices to remark that $z''(v) = z^1(v) = 0$ and $z''(u, v) = z^1(u, t) = 0$. This contradicts the fact that $\bar{z}$ is an extreme point of $P_p(G)$. Thus the proof of Lemma 8 is complete. □

3.2. **The proof of Theorem 9.** Assume that $\bar{z}$ is a fractional extreme point of $P_p(G)$. In this subsection, we will not further suppose that $\bar{z}(u, v) = \bar{z}(v)$ when v is not a pendent node.

Lemma 24. *Let (u, v) be an arc such that v is not pendent. Let G' be the graph obtained from G by removing (u, v) and adding a new pendent node v' and the arc (u, v'). If G' does not contain an odd Y-cycle, then we can assume that $\bar{z}(u, v) = \bar{z}(v)$.*

Proof. Suppose that $\bar{z}(u, v) < \bar{z}(v)$. Define $z'(u, v') = \bar{z}(u, v)$, $z'(v') = 1$, and $z'(s, t) = \bar{z}(s, t)$, $z'(r) = \bar{z}(r)$ for all other arcs and nodes. It is easy to check that G' share the same properties as G and that z' is a fractional extreme point of $P_{p+1}(G')$. □

Lemma 25. *Let $G = (V, A)$ be a directed graph with no odd Y-cycle. If (u, v) and (v, u) are two arcs in A and $\delta^-(u) = \{(v, u)\}$, then the graph G' obtained from G by removing (u, v) and adding a new pendent node v' and the arc (u, v') does not contain an odd Y-cycle.*

Proof. It is easy to see that any odd Y-cycle in G' is also an odd Y-cycle in G. This is because the node u cannot belong to $\hat{C}$ for any cycle C in G'. □

Let v a node in G. We call v a *knot* if $\delta^-(v) = \{(u, v), (w, v)\}$, $u \neq w$ and both (v, u) and (v, w) belong to $\delta^+(v)$. Let $\bar{z}$ be a fractional extreme point of $P_p(G)$. Recall that $\bar{z}(u, v) > 0$ for all (u, v). Let v be a knot in G as defined above, recall that from

Lemma 14, $\bar{z}(v)$, $\bar{z}(u,v)$ and $\bar{z}(w,v)$ are fractional. If $\bar{z}(u,v) < \bar{z}(v)$ or $\bar{z}(w,v) < \bar{z}(v)$, then v is called a *fragile knot* and we say that the pair $(G, \bar{z})$ contains a fragile knot.

Let $(G, \bar{z})$ be a pair containing a fragile knot v. The arcs incident to v are (u,v), (v,u), (w,v), (v,w) and there is possibly other arcs (v,t) with t different from u and w. Assume that $\bar{z}(u,v) < \bar{z}(v)$. Define the graph $G(v)$ form G as follows. Remove v and its incident arcs. Add four nodes v', v'', s' and a pendent node t'. Add the arcs (u,v'), (v',u), (v',s'), (s',t'), (v'',w), (w,v'') and the set of arcs (v'',t) whenever (v,t) belongs to G, $t \neq u,w$.

Lemma 26. *If G has no multiple arcs, no odd Y-cycle and satisfies condition (i) of Theorem 2, then $G(v)$, as defined above, has the same properties.*

Proof. Remark first that if there exists an arc (v,t) with t different from u and w, then t is a pendent node. Otherwise G contains H_1 or H_3 as a subgraph. Using this remark, one can see that the nodes v' and v'' cannot belong to any cycle of size at least three in $G(v)$. Thus if there is an odd Y-cycle in $G(v)$, then this is also an odd Y-cycle in G, which is not possible.

By definition $G(v)$ does not contain multiple arcs.

Now suppose that $G(v)$ contains one of the graphs H_1, H_2 or H_4 as a subgraph, call it H. Remark that H cannot contain (s',t'). If it contains (v',s'), then by replacing it by (v,w) one obtains the same subgraph in G. If H does not contain (v',s') and contains the node v', then the set of nodes in H where v' is replaced by v induces the same subgraph in G, which is not possible. Similar arguments can be used with v''. Finally, If H does not contain v' nor v'', then H is also a subgraph in G. □

Lemma 27. *Let $G = (V,A)$ be a directed graph. If $P_p(G)$ admits a fractional extreme point $\bar{z}$, where $(G,\bar{z})$ contains a fragile knot v, $(\bar{z}(u,v) < \bar{z}(v))$, then $P_{\tilde{p}}(G(v)) \neq \tilde{p}MP(G(v))$, with $\tilde{p} = p+2$.*

Proof. Suppose that $P_{\tilde{p}}(G(v)) = \tilde{p}MP(G(v))$. Define $\tilde{z} \in P_{\tilde{p}}(G(v))$ to be

$$\tilde{z}(l) = \begin{cases} \bar{z}(v) & \text{if } l = v' \text{or } l = v'', \\ 1-\bar{z}(v) & \text{if } l = s', \\ 1 & \text{if } l = t', \\ \bar{z}(l) & \text{otherwise} \end{cases} \quad ; \quad \tilde{z}(l,k) = \begin{cases} \bar{z}(u,v) & \text{if } (l,k) = (u,v'), \\ \bar{z}(v,u) & \text{if } (l,k) = (v',u), \\ 1-\bar{z}(v) & \\ \quad -\bar{z}(v,u) & \text{if } (l,k) = (v',s'), \\ \bar{z}(v,w) & \text{if } (l,k) = (v'',w), \\ \bar{z}(w,v) & \text{if} (l,k) = (w,v''), \\ \bar{z}(v,t) & \text{if } (v,t) \in A,\ t \neq u,w, \\ & \text{ and } (l,k) = (v'',t), \\ \bar{z}(l,k) & \text{otherwise.} \end{cases}$$

The vector $\tilde{z}$ is fractional, so $\tilde{z}$ is not an extreme point of $P_{\tilde{p}}(G(v))$. Since $P_{\tilde{p}}(G(v))$ is integral, there is a 0-1 vector $z^* \in P_{\tilde{p}}(G(v))$ with $z^*(v',s') = 1$ so that the same constraints that are tight for $\tilde{z}$ are also tight for z^*. From z^* define $z'' \in P_p(G)$ as follows:

$$z''(l) = \begin{cases} z^*(v'') & \text{if } l = v, \\ z^*(l) & \text{otherwise.} \end{cases} \quad ; \quad z''(l,k) = \begin{cases} z^*(u,v') & \text{if } (l,k) = (u,v), \\ z^*(v',u) & \text{if } (l,k) = (v,u), \\ z^*(v'',w) & \text{if } (l,k) = (v,w), \\ z^*(w,v'') & \text{if} (l,k) = (w,v), \\ z^*(v'',t) & \text{if } (v,t) \in A,\ t \neq u,w, \\ & \quad \text{and } (l,k) = (v,t), \\ z^*(l,k) & \text{otherwise.} \end{cases}$$

All constraints that are tight for $\bar{z}$ are also tight for z''. To see this, it suffices to notice that $\sum_{v \in V} z''(v) = p$ since $z^*(s') = z^*(t') = 1$. Also remark that $z^*(u,v') = z^*(v') = 0$ and that $z^*(v'')$ may be equal to 0 or 1, so we may have $z''(u,v) = 0 < z''(v) = 1$ but this inequality was not tight for $\bar{z}$. □

Lemma 28. *Let (u,v) and (v,u) be two arcs in G. If $\delta^+(u) = \{(u,v)\}$ and $\bar{z}(v,u) = \bar{z}(u)$, then $\bar{z}(u,v) = \bar{z}(v)$ for $\bar{z} \in P_p(G)$.*

Proof. Immediate from the validity of $\bar{z}$. □

All the material defined above permits us to characterize $pMP(G)$ in a special class of graphs defined in the following theorem. This theorem will be used to prove Theorem 9.

Theorem 29. *Let $G = (V,A)$ be a directed graph with no multiple arcs, no odd Y-cycle and satisfying condition (i) of Theorem 2. If G does not contain a knot, then $P_p(G)$ is integral*

Proof. Suppose that the theorem is false. Let $\bar{z}$ be a fractional extreme point of $P_p(G)$. By Lemma 8, there must exist an arc (u,v) with $\bar{z}(u,v) < \bar{z}(v)$ and v is not a pendent node. Lemma 24 implies that the graph G' obtained from G by removing (u,v) and adding a pendent node v' with the arc (u,v') contains an odd Y-cycle C. Also, since G contains no knot, this implies in G that $\delta^+(u) = \{(u,v)\}$ and $\delta^-(u) = \{(s,u),(v,u)\}$, where s and v are the nodes that are adjacent to u in C. Remark that v must be in $\dot{C}$, otherwise C is also an odd Y-cycle in G, which is not possible.

We have that $\delta^-(v) = \{(u,v)\}$. In fact, since $v \in \dot{C}$ we must have an arc (v,w) in C. Because G has no knot this implies that the arc (w,v) cannot exist. So suppose (w',v) is an arc of G with $w' \neq w$, $w' \neq u$. Since w is in C, it is not a pendent node and hence G does not satisfies condition (i) of Theorem 2.

Now, if we remove (v,u) and we add a new pendent node u' and the arc (v,u') the resulting graph does not contain an odd Y-cycle, so Lemma 24 implies that $\bar{z}(v,u) = \bar{z}(u)$. But in this case, Lemma 28 implies that $\bar{z}(u,v) = \bar{z}(v)$, a contradiction. □

Now we prove the main result of this sub-section.

Proof of Theorem 9:

Denote by $knot(G)$ the set of knots in G. The proof is by induction on $|knot(G)|$. If $|knot(G)| = 0$, then by Theorem 29 $P_p(G)$ is integral.

Suppose that the theorem is true for every directed graph with no multiple arcs, with no odd Y-cycle, satisfying condition (i) of Theorem 2 and having at most m knots, with $m \geq 0$. Let $G = (V,A)$ be a directed graph, with no multiple arcs, no odd Y-cycle, satisfying condition (i) of Theorem 2 and $|knot(G)| = m+1$. Assume that $\bar{z}$ is a fractional extreme point of $P_p(G)$.

Claim 1. $(G,\bar{z})$ does not contain a fragile knot.

Proof. Suppose the contrary and let v be a fragile knot. We have that $|knot(G(v))| \leq m$ and by Lemma 26 the graph $G(v)$ has no multiple arcs, no odd Y-cycle and satisfies condition (i) of Theorem 2. Thus the induction hypothesis applies, so $P_{p+2}(G(v))$ is integral. This contradicts Lemma 27. □

By Lemma 8, G must contain an arc (v_2, v_3) with $\bar{z}(v_2, v_3) < \bar{z}(v_3)$ and v_3 is not a pendent node. Lemma 24 implies that the graph G' obtained from G by removing (v_2, v_3) and adding a new pendent node v_3' and the arc (v_2, v_3') contains an odd Y-cycle C. The fact that G does not contain an odd Y-cycle implies that C is an odd cycle in G where $v_2 \in \hat{C}$ and v_2 does not satisfy either Definition 1 (i) or (ii). Hence $\delta^-(v_2) = \{(v_3, v_2), (v_1, v_2)\}$, otherwise the graph H_4 is present. Also $v_3 \in \dot{C}$. Let v_4 be the other node in C adjacent to v_3, so (v_3, v_4) is an arc of C.

Suppose that $(u, v_3) \in A$, with $u \neq v_2$. We must have $u = v_4$, otherwise G does not satisfies condition (i) of Theorem 2. Thus v_3 is a fragile knot, which is impossible by Claim 1. It follows that we may assume that $\delta^-(v_3) = \{(v_2, v_3)\}$.

Lemma 24 together with Lemma 25 imply that $\bar{z}(v_3, v_2) = \bar{z}(v_2)$. Now Lemma 28 implies that we must have an arc (v_2, u) different from (v_2, v_3). Since v_2 is in $\hat{C}$ and it does not satisfy either Definition 1 (i) or (ii) and G satisfies condition (i) of Theorem 2, we must have $u = v_1$ and $v_1 \in \dot{C}$. If $\bar{z}(v_1, v_2) < \bar{z}(v_2)$, then v_2 is a fragile knot, which is not possible by Claim 1. And if $\bar{z}(v_2, v_1) < \bar{z}(v_1)$, then the labeling function l that assign 1 to (v_2, v_3), -1 to (v_2, v_1) and 0 to each other node and arc, implies that $\bar{z}_l$ satisfies with equality the same constraints that are satisfied with equality for $\bar{z}$. This contradicts the fact that $\bar{z}$ is an extreme point.

Let us summarize the above discussion. We have

- $\delta^-(v_2) = \{(v_1, v_2), (v_3, v_2)\}$; $\delta^+(v_2) = \{(v_2, v_1), (v_2, v_3)\}$; $\delta^-(v_3) = \{(v_2, v_3)\}$,
- $\bar{z}(v_1, v_2) = \bar{z}(v_3, v_2) = \bar{z}(v_2)$; $\bar{z}(v_2, v_1) = \bar{z}(v_1)$ and $\bar{z}(v_2, v_3) < \bar{z}(v_3)$.

Since v_2 does not satisfy either Definition 1 (i) or (ii), the node v_1 must be in $\dot{C}$, so we must have (v_1, v_0) in $A(C)$. Lemma 16 implies that we must have an arc (u, v_1) with $u \neq v_2$. Condition (i) of Theorem 2, implies that $u = v_0$. We must have $\bar{z}(v_0, v_1) = \bar{z}(v_1)$, otherwise v_1 is a fragile knot which is impossible by Claim 1. Suppose that $\bar{z}(v_1, v_0) < \bar{z}(v_0)$ (resp. There exist an arc (v_1, t) and t a pendent node). Define the following labeling function l. Assign the label 1 to the arcs (v_1, v_0) (resp. (v_1, t)) and (v_2, v_3) and to the node v_3; assign the label -1 to the arcs (v_1, v_2) and (v_3, v_2) and to the node v_2; for all other arcs and nodes assign the label 0. Then any constraint that is tight for $\bar{z}$ is also tight for $\bar{z}_l$, which contradicts the fact that $\bar{z}$ is an extreme point. Hence we must have $\bar{z}(v_1, v_0) = \bar{z}(v_0)$ and $\delta^+(v_1) = \{(v_1, v_0), (v_1, v_2)\}$.

Finally we have a bidirected path $P = v_0, v_1, v_2, v_3$, where the inner nodes v_1 and v_2 are incident to only their neighbors in P and all the arcs of P are tight for $\bar{z}$ except the arc (v_2, v_3).

Define G' the graph obtained from G by identifying the nodes v_0 and v_3, call v^* the resulting node, and by removing the nodes v_1 and v_2 with their incident arcs.

Define z' from $\bar{z}$ as follows:

$$z'(v) = \begin{cases} \bar{z}(v_3) & \text{if } v = v^* \\ \bar{z}(v) & \text{otherwise,} \end{cases} \quad ; \; z'(u,v) = \begin{cases} \bar{z}(v_0,v) & \text{if } u = v^* \text{ and } (v_0,v) \in A, \\ \bar{z}(u,v_0) & \text{if } v = v^* \text{ and } (u,v_0) \in A, \\ \bar{z}(v_3,v) & \text{if } u = v^* \text{ and } (v_3,v) \in A, \\ \bar{z}(u,v) & \text{if } u \neq v^* \text{ and } v \neq v^*. \end{cases}$$

Claim 2. G' has no multiple arcs, satisfies condition (i) of Theorem 2 and does not contain an odd Y-cycle.

Proof. The proof is given by claims 1, 2 and 3 in the proof of Lemma 18. □

Claim 3. z' is a fractional vector in $P_{p-1}(G')$.

Proof. Lemma 14 imply that $\bar{z}(v_3)$ is fractional. So at least $z'(v^*)$ is fractional.

Let us examine the validity of z'. By definition any constraint where $z(v^*)$ does not appear is satisfied. Let us show that $\sum z'(v) = p-1$ and that equation (3) with respect to v^* is satisfied.

We have that $\sum z'(v) = \sum_{v \in V} \bar{z}(v) - \bar{z}(v_0) - \bar{z}(v_1) - \bar{z}(v_2)$. Notice that the validity of $\bar{z}$ implies that

$$\bar{z}(v_1) + \bar{z}(v_1,v_0) + \bar{z}(v_1,v_2) = 1 \tag{18}$$

Since all the arcs of P are tight for $\bar{z}$ except (v_2,v_3), the equation (18) is equivalent to

$$\bar{z}(v_1) + \bar{z}(v_0) + \bar{z}(v_2) = 1 \tag{19}$$

Then we have that $\sum z'(v) = \sum_{v \in V} \bar{z}(v) - \bar{z}(v_0) - \bar{z}(v_1) - \bar{z}(v_2) = p-1$. Now let us see that equation (3) with respect to v^* is satisfied, that is $z'(v^*) + z'(\delta^+(v^*)) = 1$.

By definition we have

$$z'(v^*) + z'(\delta^+(v^*)) = \bar{z}(v_3) + \bar{z}(\delta^+(v_3) \setminus \{(v_3,v_2)\}) + \bar{z}(\delta^+(v_0) \setminus \{(v_0,v_1)\}).$$

Equations (3) with respect to v_0 and v_3 imply

$$\bar{z}(v_0) + \bar{z}(\delta^+(v_0) \setminus \{(v_0,v_1)\}) + \bar{z}(v_0,v_1) = 1, \tag{20}$$

$$\bar{z}(v_3) + \bar{z}(\delta^+(v_3) \setminus \{(v_3,v_2)\}) + \bar{z}(v_3,v_2) = 1. \tag{21}$$

Since $\bar{z}(v_0,v_1) = \bar{z}(v_1)$ and $\bar{z}(v_3,v_2) = \bar{z}(v_2)$, when we replace (19) in the sum of (20) and (21), we obtain $\bar{z}(v_3) + \bar{z}(\delta^+(v_3) \setminus \{(v_3,v_2)\}) + \bar{z}(\delta^+(v_0) \setminus \{(v_0,v_1)\}) = 1$. Hence $z'(v^*) + z'(\delta^+(v^*)) = 1$.

To finish the proof of this claim, we need also to show that $z'(u,v^*) \leq z'(v^*)$ for any arc (u,v^*) in G'.

The validity of $\bar{z}$ implies that

$$\bar{z}(v_2) + \bar{z}(v_2,v_1) + \bar{z}(v_2,v_3) = 1. \tag{22}$$

Since $\bar{z}(v_2,v_1) = \bar{z}(v_1)$, then equation (22) is equivalent to

$$\bar{z}(v_2) + \bar{z}(v_1) + \bar{z}(v_2,v_3) = 1 \tag{23}$$

Combining (19) with (23) we obtain

$$\bar{z}(v_2, v_3) = \bar{z}(v_0). \tag{24}$$

If (u, v^*) is an arc in G', then (u, v_0) is an arc in G. The validity of $\bar{z}$ and (24) imply that $\bar{z}(u, v_0) \le \bar{z}(v_0) = \bar{z}(v_2, v_3) \le \bar{z}(v_3)$ and with the definition of z' we have $z'(u, v^*) = \bar{z}(u, v_0) \le \bar{z}(v_3) = z'(v^*)$. □

Notice that $|knot(G')| \le m$. It follows from Claim 2 that the induction hypothesis applies. Thus Claim 3 implies that z' is not an extreme point of $P_p(G')$. So z' can be written as a convex combination of 0-1 vectors that satisfy with equation each constraint that is satisfied with equation by z'. If there is an arc (u, v^*) in G', then by the definition of z' and Lemma 12 we have $z'(u, v^*) > 0$. Hence one can choose among the 0-1 solutions above a solution z^* with $z^*(u, v^*) = 1$. This also implies that $z^*(v^*) = 1$. Otherwise, since $z'(v^*) > 0$ one can also choose a solution z^* with $z^*(v^*) = 1$. From z^* define $z'' \in P_p(G)$ to be as follows:

$$z''(v) = \begin{cases} 0 & \text{if } v \in \{v_1, v_2\}, \\ 1 & \text{if } v \in \{v_0, v_3\}, \\ z^*(v) & \text{otherwise.} \end{cases} \quad ; \quad z''(u,v) = \begin{cases} 1 & \text{if } (u,v) \in \{(v_1, v_0), (v_2, v_3)\}, \\ 0 & \text{if } (u,v) \in \{(v_0, v_1), (v_2, v_1), \\ & \qquad (v_1, v_2), (v_3, v_2)\}, \\ z^*(u,v) & \text{otherwise.} \end{cases}$$

It is easy to check that $z'' \in P_p(G)$ and any constraint that is satisfied as equality for $\bar{z}$ is also satisfied as equality for $\bar{z}$. It suffices to see that if there is an arc (u, v_0) with $u \neq v_1$, then this arc is unique and by definition $z''(u, v_0) = z''(v_0) = 1$. Thus we have a contradiction with the fact that $\bar{z}$ is an extreme point.

4. Graphs with odd Y-cycles

In this section we assume that $G = (V, A)$ is a directed graph with no multiple arcs, satisfying conditions (i) and (ii) of Theorem 2. Also we assume that G contains an odd Y-cycle $C = v_0, a_0, v_1, a_1, \ldots, a_{p-1}, v_p$. We plan to prove that conditions (i) and (ii) of Theorem 2 are sufficient when G contains an odd Y-cycle. Let $\bar{z}$ be a fractional extreme point of $P_p(G)$. First we need several lemmas.

Lemma 30. *We can assume that*

- $\bar{z}(u, v) > 0$ *for all* $(u, v) \in A$,
- $\bar{z}(v) > 0$ *for all* $v \in V$ *with* $|\delta^-(v)| \ge 1$, *and*
- $|\delta^-(v)| \le 1$ *for every pendent node* $v \in V$.

Proof. Similar to the proofs of Lemmas 12, 13 and 15. □

Let v_k and v_l be two nodes in $V(C)$. Call P_1 and P_2 the two paths in C from v_k to v_l. We are going to prove that if there is another path between v_k and v_l whose internal nodes are not in $V(C)$, then this path consists of just one arc and v_k and v_l should be consecutive in C. Assume the contrary, and let $P = v_k, b_1, u_1, \ldots, u_{r-1}, b_r, v_l$ be another path between v_k and v_l. Assume that all internal nodes of P are not in $V(C)$. Notice that because of (ii) P cannot have more than two arcs. We call C_1 (resp. C_2) the cycle defined by P_1 and P (resp. P_2 and P).

Lemma 31. *Assume that v_k and v_l are not consecutive in C or P contains two arcs, then if an arc of P is directed into (resp. away from) v_k (or v_l) then this node must be in $\dot{C}$ (resp. $\dot{C} \cup \tilde{C}$).*

Proof.

- Suppose first that b_1 is directed into v_k, thus $b_1 = (u_1, v_k)$. Assume that v_k and v_l are not consecutive or that P consists of two arcs.

  Let $v_k \in \hat{C}$. If $v_k \in \hat{C}_{(i)}$ (resp. $v_k \notin \hat{C}_{(i)}$) then G contains H_2 (resp. H_4) as a subgraph.

  Now assume that $v_k \in \tilde{C}$. Let (v_{k-1}, v_k) and (v_k, v_{k+1}) be the two arcs of C incident to v_k. The node v_{k+1} is not a pendent node, so there is an arc (v_{k+1}, u). If $u \in \{v_k, v_{k-1}, u_1\}$ (resp. $u \notin \{v_k, v_{k-1}, u_1\}$) then the graph defined by (v_{k-1}, v_k), (u_1, v_k), (v_k, v_{k+1}) and (v_{k+1}, u) corresponds to H_3 or H_4 (resp. H_1). Therefore $v_k \in \dot{C}$.
- Suppose now that b_1 is directed away from v_k, thus $b_1 = (v_k, u_1)$. Suppose that $v_k \in \hat{C}$, and (v_{k-1}, v_k) and (v_{k+1}, v_k) are the two arcs of C incident to v_k.

  Assume first that P consists of two arcs.
  - Assume that (u_1, v_l) is the second arc of P. If v_l coincides with v_{k+1} or v_{k-1}, then we have H_3 as a subgraph, otherwise we have H_1 as a subgraph.
  - Assume now that (v_l, u_1) is the second arc of P. Since $|\delta^-(u_1)| \geq 2$, by Lemma 30 u_1 is not a pendent node, so there is an arc (u_1, u). If $u = v_k$ we have H_4 as a subgraph; if u coincides with v_{k-1} or v_{k+1} we have H_3 as a subgraph; otherwise we have H_1 as a subgraph.

  Assume now that P consists of one arc and that v_k and v_l are not consecutive. So $u_1 = v_l$. Since b_1 is directed into v_l, we have seen above that v_l must be in $\dot{C}$. In this case we must have H_1 or H_3 as a subgraph.

□

Lemma 32. *If v_k and v_l are not consecutive in C, then P cannot consist of just one arc.*

Proof. Let $P = v_k, (v_k, v_l), v_l$. By Lemma 31, $v_l \in \dot{C}$ and $v_k \in \dot{C} \cup \tilde{C}$. We then consider two cases: (a) $v_k \in \dot{C}$ and (b) $v_k \in \tilde{C}$, as shown in Figure 14.

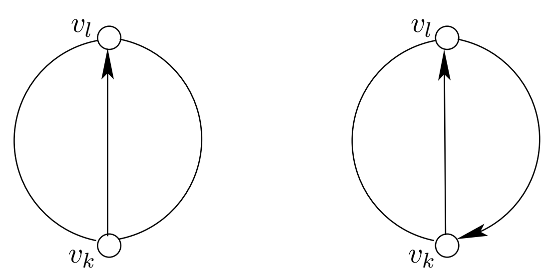

FIGURE 14. Cases (a) and (b).

(a) C_1 and C_2 are both Y-cycles and exactly one of them is odd. The fact that G satisfies (ii) implies that the even cycle contains three arcs. Let C_1 be the even cycle. Thus $C_1 = v_k, (v_k, v_l), v_l, (v_l, v), v, (v_k, v), v_k$, where $v \in \hat{C}$. Since both nodes v_k and v_l are in $\dot{C}$, there is an arc $(v, \bar{v})$, where $\bar{v}$ is a pendent node, $\bar{v} \notin V(C)$. Therefore condition (ii) is violated by C_2 and $(v, \bar{v})$.

(b) Let (u, v_k) and (v_k, v) be the two arcs in $A(C)$ incident to v_k. Notice that there is no arc from v to v_k, otherwise G contains H_1 or H_3 as a subgraph. Thus C_1 and C_2 are both Y-cycles. The parity of C implies that exactly one of these cycles is odd. If one is odd the fact that G satisfies (ii) implies that the other cycle must contain three arcs. So the odd cycle must be the one containing the arc (u, v_k) call it C_2. Let $C_1 = v_k, (v_k, v_l), v_l, (v_l, v), v, (v_k, v), v_k$. Since C and C_1 are both Y-cycles, there is an arc $(v, \bar{v})$, where $\bar{v}$ is a pendent node, $\bar{v} \notin V(C)$. Thus condition (ii) is violated by C_2 and $(v, \bar{v})$.

□

Lemma 33. *The path P cannot consist of two arcs.*

Proof. Let $P = v_k, b_1, u_1, b_2, v_l$. We have to study three cases:

(1) $b_1 = (u_1, v_k)$ and $b_2 = (u_1, v_l)$. By Lemma 31, both v_k and v_l are in $\dot{C}$. Both C_1 and C_2 are Y-cycles and exactly one of them must be odd, otherwise C is an even Y-cycle. Suppose that C_1 is odd. Then C_2 is even and must contain four arcs, otherwise G does not satisfies (ii). Now it is easy to see that $|\hat{C}_2| + |\tilde{C}_2| = 3$, a contradiction.

(2) $b_1 = (v_k, u_1)$ and $b_2 = (u_1, v_l)$. The case where $b_1 = (u_1, v_k)$ and $b_2 = (v_l, u_1)$ may be treated by symmetry. By Lemma 31, $v_l \in \dot{C}$ and $v_k \in \dot{C} \cup \tilde{C}$. So we have to distinguish two sub-cases: (a) $v_k \in \dot{C}$ and (b) $v_k \in \tilde{C}$. They are shown below in Figure 15.

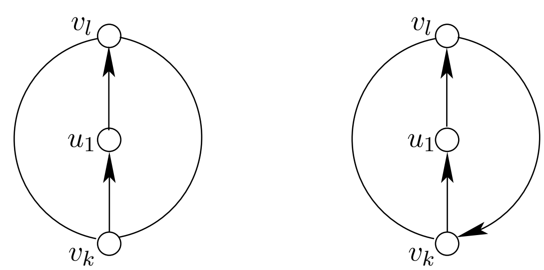

FIGURE 15. The sub-cases (a) and (b).

(a) C_1 and C_2 are both Y-cycles. The parity of C implies that exactly one of C_1 or C_2 is odd. Suppose C_1 is odd. As in the previous case we have that $|\hat{C}_2| + |\tilde{C}_2| = 3$, a contradiction.

(b) Let (u, v_k) and (v_k, v) be the two arcs of C incident to v_k. First we need several claims.

  ∗ Claim: u and v are different from v_l.
  Proof. Since $v_l \in \dot{C}$, we have $v \neq v_l$. Suppose $u = v_l$. Then the cycle $v_l, (v_l, v_k), v_k, (v_k, u_1), u_1, (u_1, v_l), v_l$ is an odd Y-cycle. Since G satisfies (ii), the arcs (v_l, v) and (v_k, v) must be in $A(C)$. But this implies that C is an even Y-cycle, a contradiction. It follows that both u and v are different from v_l.

  ∗ Claim: C_1 and C_2 are Y-cycles.
  Proof. Let C_1 be the cycle containing (v_k, v) and let C_2 be the cycle containing (u, v_k). It is easy to see that C_2 is a Y-cycle. Let us see that C_1 is also a Y-cycle. If $v \in \tilde{C}$ then clearly C_1 is a Y-cycle. Suppose $v \in \hat{C}$. Thus $v \in \hat{C}_1$. We need to show that v verifies (i) or (ii) of Definition 1 with respect to C_1. Suppose the contrary, then $(v, v_k) \in A$. It follows that the graph defined by the arcs (v, v_k), (u, v_k), (v_k, u_1) and (u_1, v_l) corresponds to H_1, which is not possible. Hence $v \in \hat{C}_{(i)}$

or there is an arc $(v, \bar{v})$, where $\bar{v}$ is the other node adjacent to v in C, and $\bar{v} \in \tilde{C}$. In either case we have that C_1 is a Y-cycle.

∗ Claim: C_2 is a directed Y-cycle of size four.
Proof. The parity of C implies that exactly one of the cycles C_1 or C_2 is odd. If C_2 is odd then, as in the previous cases $|\hat{C_1}|+|\tilde{C_1}| = 3$, which is impossible. So suppose that C_1 is odd. Then C_2 is a directed Y-cycle of size four, $C_2 = v_k, (v_k, u_1), u_1, (u_1, v_l), v_l, (v_l, u), u, (u, v_k), v_k$, see Figure 16.

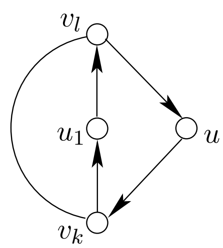

FIGURE 16. The Case (2b)

Now suppose there is an arc not in $A(C_2)$ directed into a node in C_2. Call this arc (w, t). If $w \notin V(C_2)$, then G contains H_1; and if w and t are not consecutive in C_2, then G contains H_3. So assume $(w, t) \in A \setminus A(C_2)$ and t and w are two consecutive nodes in $V(C_2)$.
Let C_2' be the cycle obtained from C_2 by adding (w, t) and removing (t, w). We have two sub-cases:

∗ Assume that C_2' is an odd Y-cycle. This implies that C_1 must be of size four, otherwise G does not satisfy (ii). Thus the arcs (v_l, v) and (v_k, v) are in $A(C_1)$ and if C_1 is of size four, it was proved above that $v \in \hat{C}_{(i)}$. Let $(v, \bar{v}) \in A$ with $\bar{v} \notin V(C)$. If $\bar{v} \neq u_1$, then the pair C_2' and $(v, \bar{v})$ violates condition (ii) of Theorem 2. And if $\bar{v} = u_1$, then the graph defined by (v, u_1), (u_1, v_l), (v_l, v) and (v_k, u_1) corresponds to H_3, which is not possible.
∗ The case when C_2' is not a Y-cycle is obtained when $(w, t) = (v_l, u_1)$ or $(w, t) = (v_k, u)$; and in both cases $\delta^+(t) = \{(t, w)\}$. Suppose that $\bar{z}(w, t) = \bar{z}(t)$. Thus constraint (3) with respect to t implies that

$$\bar{z}(t) + \bar{z}(t, w) = 1 = \bar{z}(t, w) + \bar{z}(w, t). \tag{25}$$

Since w is one of the nodes v_k or v_l, then there is an arc (w, t') where t' is another node in C different from t. Lemma 30 implies that $\bar{z}(w, t') > 0$. Hence from constraint (3) with respect to w

$$\bar{z}(w) + \bar{z}(w, t) < 1. \tag{26}$$

Combining (25) with (26) we obtain

$$\bar{z}(t, w) > \bar{z}(w). \tag{27}$$

But this contradicts the validity of $\bar{z}$.
Hence we may suppose that if there is an arc (w, t) not in C_2 directed into a node in C_2, then $\bar{z}(w, t) < \bar{z}(t)$. Assign labels to the nodes and

arcs in C_2 following the labeling procedure of an even cycle. Extend this labeling by assigning the label 0 to each node and arc with no label. Call this labeling l. The constraints that are satisfied with equality by $\bar{z}$ are also satisfied with equality by $\bar{z}_l$. This contradicts the fact that $\bar{z}$ is an extreme point of $P_p(G)$. Notice that we do not need $\bar{z}_l \in P_p(G)$.

(3) $b_1 = (v_k, u_1)$ and $b_2 = (v_l, u_1)$. Notice that by Lemma 30, u_1 is not a pendent node. Since G satisfies condition (ii), there is an arc (u_1, t) with $t \in V(C)$. If t is different from v_k and v_l then one can easily create a subgraph in G that is one of the subgraphs of Figure 1. So t must coincide with v_k or v_l, say $t = v_l$. If we take the path $P' = v_k, (v_k, u_1), u_1, (u_1, v_l), v_l$, instead of P, this reduces to the case (2) above.

□

Lemma 34. *The node set of any cycle of size at least three in G coincides with $V(C)$.*

Proof. The proof is straightforward from Lemmas 32 and 33 and condition (ii) of Theorem 2. □

The following lemma permits the reduction to oriented graphs.

Let (u, v) and (v, u) be two arcs in A. Denote by $G(u, v)$ the graph obtained from G by removing the arc (u, v) and adding a new arc (u, t), where t is a new pendent node.

Lemma 35. *Let $G = (V, A)$ be a directed graph and (u, v) and (v, u) two arcs in A. If $P_p(G)$ admits a fractional extreme point $\bar{z}$ with $\bar{z}(v, u) > 0$, then $P_{\tilde{p}}(G(u, v)) \neq \tilde{p}MP(G(u, v))$, where $\tilde{p} = p + 1$.*

Proof. Let $\bar{z}$ be a fractional extreme point of $P_p(G)$ with $\bar{z}(v, u) > 0$. Suppose that $P_{\tilde{p}}(G(u, v)) = \tilde{p}MP(G(u, v))$. Define $\tilde{z} \in P_{\tilde{p}}(G(u, v))$ to be $\tilde{z}(u, t) = \bar{z}(u, v)$, $\tilde{z}(t) = 1$ and $\tilde{z}(r) = \bar{z}(r)$, $\tilde{z}(r, s) = \bar{z}(r, s)$ for all other nodes and arcs. The solution $\tilde{z}$ is fractional, so $\tilde{z}$ is not an extreme point of $P_{\tilde{p}}(G(u, v))$. Since $P_{\tilde{p}}(G(u, v))$ is integral, there must exist a 0-1 vector $z^* \in P_{\tilde{p}}(G(u, v))$ with $z^*(v, u) = 1$, so that the same constraints that are tight for $\tilde{z}$ are also tight for z^*. From z^* define $z'' \in P_p(G)$ as follows: $z''(u, v) = z^*(u, t)$ and $z''(r) = z^*(r)$, $z''(r, s) = z^*(r, s)$, for all other nodes and arcs. All constraints that are tight for $\bar{z}$ are also tight for z''. To see this, it suffices to remark that $z''(v) = z^*(v) = 0$ and $z''(u, v) = z^*(u, t) = 0$. This contradicts the fact that $\bar{z}$ is an extreme point of $P_p(G)$. □

Now we can prove the main result of this section.

Theorem 36. *If $G = (V, A)$ is a directed graph with no multiple arcs, satisfying condition (i) and (ii) of Theorem 2 and containing an odd Y-cycle, then $P_p(G)$ is integral.*

Proof. Denote by $Pair(G)$ the set of pair of nodes $\{u, v\}$ such that both arcs (u, v) and (v, u) belong to A. The proof is by induction on $|Pair(G)|$. If $|Pair(G)| = 0$ then G is an oriented graph that satisfies conditions (i) and (ii) of Theorem 3. Hence the result follows from Theorem 3.

Suppose that Theorem 36 is true for every directed graph H with no multiple arcs, satisfying conditions (i) and (ii) of Theorem 2, containing an odd Y-cycle and $|Pair(H)| \leq$

m, $m \geq 0$. Let $G = (V, A)$ be a directed graph with no multiple arcs, satisfying conditions (i) and (ii) of Theorem 2, containing an odd Y-cycle and $|Pair(G)| = m + 1$. Assume that $\bar{z}$ is a fractional extreme point of $P_p(G)$.

Let (u, v) and (v, u) be two arcs in A. Lemma 30 implies $\bar{z}(v, u) > 0$, so Lemma 35 applies and implies that

$$P_{\tilde{p}}(G(u, v)) \neq \tilde{p}MP(G(u, v)). \tag{28}$$

Claim. The graph $G(u, v)$ satisfies conditions (i) and (ii) of Theorem 2.

Proof. To see that $G(u, v)$ satisfies condition (i) is easy, it follows from the definition of $G(u, v)$ and the fact that G satisfies (i) too. Let us see that $G(u, v)$ satisfies (ii). The graph $G = (V, A)$ satisfies conditions (i) and (ii) and contains an odd Y-cycle, call it C. Lemma 34 implies that $V = U \cup V(C)$, where $U = \{u_1, \cdots, u_k\}$, and $|\delta^+(u_i)| \leq 1$, $|\delta^-(u_i)| \leq 1$, for $i = 1, \cdots, k$. Moreover, if $(t, u_i) \in \delta^-(u_i)$ then $t \in V(C)$, if $(u_i, t) \in \delta^+(u_i)$ then $t \in V(C)$, and if $(u_i, t) \in \delta^+(u_i)$ and $(t', u_i) \in \delta^-(u_i)$ then $t = t'$; for $i = 1, \cdots, k$.

Thus we can assume that $u \in V(C)$. Suppose that (ii) is violated with respect to $G(u, v)$. Then in $G(u, v)$ we must have an odd Y-cycle C' with (s, w) an arc in $G(u, v)$ with both s and w not in $V(C')$. The new arc (u, t) and the new node t of $G(u, v)$ cannot be in $V(C')$ since t is a pendent node. So C' is a cycle in G, too. Lemma 34 implies that $V(C) = V(C')$. But then the pair C and (s, w) violate condition (ii) with respect to G, which is not possible. □

By the claim above and Theorem 9, $G(u, v)$ must contain an odd Y-cycle. Since $|Pair(G(u, v))| = m$, we can apply the induction hypothesis and so $P_{\tilde{p}}(G(u, v)) = \tilde{p}MP(G(u, v))$. This contradicts (28). □

5. Proof of Theorem 2

In this section we put all pieces together and prove Theorem 2, the main result of this paper.

Necessity. Let $G = (V, A)$ be a directed graph. Let H be a subgraph of G that corresponds to one of the graphs H_1, H_2, H_3 or H_4 of Figure 1. Define $\bar{z}$ to be the solution obtained by extending the fractional extreme point associated with H, defined in Figure 1, as follows: $\bar{z}(u) = 1$ for each node u not in H; $\bar{z}(u, v) = 0$ for each arc (u, v) not in H. Then it is easy to check in all cases that $\bar{z}$ is a fractional extreme point of $P_{|V|-2}(G)$.

Now suppose that G contains an odd Y-cycle C with an arc $(t, w) \in A \setminus A(C)$, with t and w not in $V(C)$. Define $\bar{z}$ as follows: $\bar{z}(t) = \frac{1}{2}$, $\bar{z}(t, w) = \frac{1}{2}$ and $\bar{z}(w) = 1$; $\bar{z}(v) = \frac{1}{2}$ for each node $v \in \hat{C} \cup \tilde{C}$ and $\bar{z}(v) = 0$ for each node $v \in \dot{C}$; $\bar{z}(u, v) = \frac{1}{2}$ for each arc $(u, v) \in A(C)$; for each node $v \in \hat{C}_{(i)}$ by the definition of a Y-cycle it must exist and arc $(v, \bar{v}) \notin A(C)$ with $\bar{v}$ a pendent node, so let $\bar{z}(v, \bar{v}) = \frac{1}{2}$ and $\bar{z}(\bar{v}) = 1$; for each node $v \in \hat{C} \setminus \hat{C}_{(i)}$ by the definition of a Y-cycle there must exist an arc $(v, \bar{v})$ with $\bar{v} \in \tilde{C}$, so let $\bar{z}(v, \bar{v}) = \frac{1}{2}$. For all other node v and arc (u, v), let $\bar{z}(v) = \bar{z}(u, v) = 0$.

It is straightforward and is left to the reader to see that $\bar{z}$ is a fractional extreme point of $P_p(G)$, where $p = |V| - |\dot{C}| - \frac{(|\hat{C}|+|\tilde{C}|+1)}{2}$.

Sufficiency. It is straightforward from theorems 9 and 36.

6. Recognizing the graphs defined in Theorem 2

In this section we show how to decide if a graph satisfies conditions (i) and (ii) of Theorem 2. Clearly Condition (i) can be tested in polynomial time. Thus we assume that we have a graph satisfying Condition (i), then we pick an arc (u, v), we remove u and v, and look for an odd Y-cycle in the new graph. We repeat this for every arc. It remains to show how to find an odd Y-cycle.

In [1] we gave a procedure that finds an odd cycle if there is any. We remind the reader that a cycle C is odd if $|V(C)| + |\hat{C}|$ is odd. Since an odd cycle is not necessarily a Y-cycle, we are going to modify the graph so that an odd cycle in the new graph gives an odd Y-cycle in the original graph. The main difficulty resides in how to deal with nodes that satisfy condition (ii) of Definition 1. Such a node should appear in a pair $\{(u, v), (v, u)\}$. Instead of working with such a pair we are going to work with a maximal bidirected path $P = v_1, \ldots, v_q$. Notice that if the graph contains a bidirected cycle, then it is easy to derive an odd Y-cycle. So in what follows we assume that there is no bidirected cycle. The transformation is based on the following two remarks.

Remark 37. *There is at most one arc (u, v_1), $u \notin P$, and at most one arc (v, v_q), $v \notin P$. Otherwise the graph H_4 is present.*

Remark 38. *If the arc (u, v_1) is in A, $u \notin P$, and there is an arc (v_1, w) also in A, $w \notin P$, then w is a pendent node. Otherwise we obtain one of the graphs in Figure 1.*

Let C be a Y-cycle that goes through P. We have three cases to study.

Case 1. $\delta^-(P) = \{(u, v_1), (v, v_q)\}$. In this case C contains all nodes in P and also the arcs (u, v_1) and (v, v_q). Since C contains all nodes from P, the only variable that can change the parity of C is the parity of $|\hat{C} \cap P|$.

Notice that if $q \geq 5$ and if there is a Y-cycle going through P then we can always change the parity of it if needed. In fact, we can always join the nodes v_1 and v_q using arcs of P in such a way that $|\hat{C} \cap P| = 1$ as shown in Figure 17 (a), or $|\hat{C} \cap P| = 2$ as shown in Figure 17 (b). It follows that if there is a cycle C' going through P then there is a cycle C of the same parity, whose nodes in $|\hat{C} \cap P|$ satisfy Definition 1 (ii).

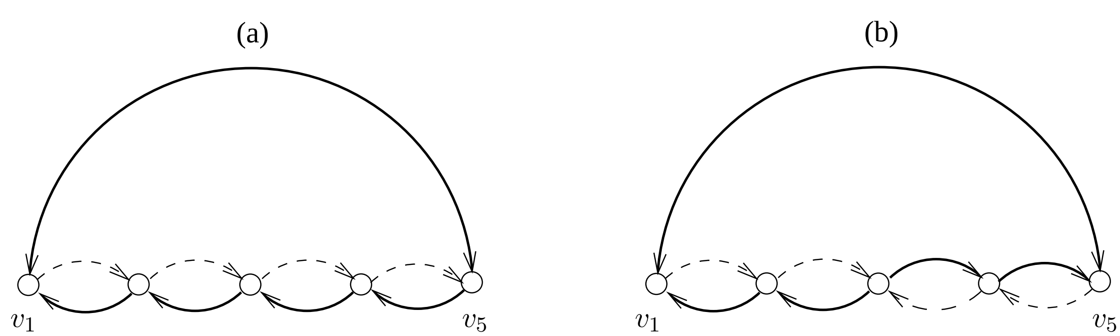

Figure 17. Case 1, $q \geq 5$. In bold the Y-cycle C. In dashed line the other arcs of P.

It remains to analyze the cases when $q \leq 4$. The only cases when a transformation is required, are the following two:

- $q = 4$ and neither v_1 nor v_4 is adjacent to a pendent node. In this case we should have $|\hat{C} \cap P| = 1$. To impose that when looking for an odd cycle, we replace P by a bidirected path with two nodes. See Figure 18.

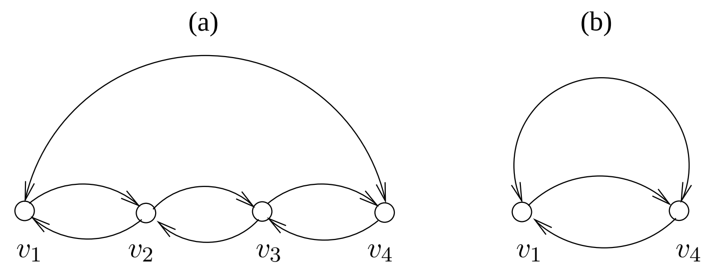

FIGURE 18. Case 1, $q = 4$. (a): before transformation. (b): after transformation.

Let P' the new bidirected path. Any cycle C' with $|\hat{C}' \cap P'| = 1$ can be extended to a cycle C with $|\hat{C} \cap P| = 1$ and where the node in $\hat{C} \cap P$ satisfies Definition 1 (ii).

- $q = 3$ and at most one of v_1 or v_3 is adjacent to a pendent node. Also here we have $|\hat{C} \cap P| = 1$. To impose that when looking for an odd cycle, we remove the arc (v_2, v_3).

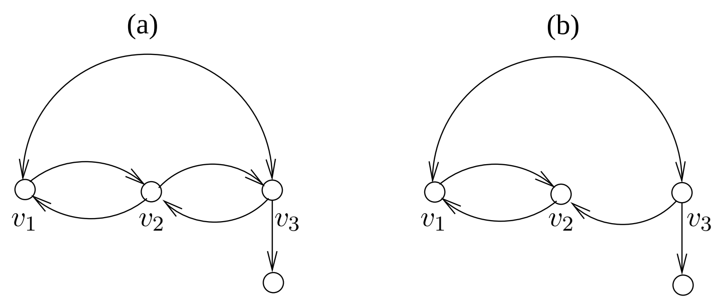

FIGURE 19. Case 1, $q = 3$. (a): before transformation. (b): after transformation.

In Figure 19, we supposed that v_3 is adjacent to a pendent node and v_1 is not.

The two remaining cases below follow the same philosophy as above.

Case 2. $\delta^-(P) = \{(u, v_1)\}$. In this case C contains (u, v_1), all the nodes in P and one arc (v_q, v), $v \notin P$. Here we have two cases to analyze.

- $q \geq 3$ or $q = 2$ and v_1 is adjacent to a pendent node. If $|\hat{C} \cap P|$ is even, we can assume that $|\hat{C} \cap P| = 0$. If $|\hat{C} \cap P|$ is odd, we can assume that $|\hat{C} \cap P| = 1$. Here no transformation is needed.
- $q = 2$ and v_1 is not adjacent to a pendent node. Here we should have $|\hat{C} \cap P| = 0$. To impose that when looking for an odd cycle, we remove (v_2, v_1).

Case 3. $\delta^-(P) = \emptyset$. In this case C contains an arc (v_1, u), $u \notin P$, all nodes in P, and an arc (v_q, v), $v \notin P$. Again we have two cases to analyze.

- $q \neq 3$ or $q = 3$ and v_2 is adjacent to a pendent node. If $|\hat{C} \cap P|$ is even, we can assume that $|\hat{C} \cap P| = 0$. If $|\hat{C} \cap P|$ is odd, we can assume that $|\hat{C} \cap P| = 1$. Here no transformation is needed.

- $q = 3$ and v_2 is not adjacent to a pendent node. Here we should have $|\hat{C} \cap P| = 0$. To impose that when looking for an odd cycle, we remove (v_1, v_2) and (v_3, v_2).

After preprocessing the graph as in cases 1, 2, and 3, we have to split all pendent nodes as in Lemma 15. This is to avoid having a pendent node in $\hat{C}$. Then we look for an odd cycle; if there is one, it gives an odd Y-cycle in the original graph.

7. Concluding remarks

We have characterized the graphs for which the system (2)-(6) defines an integral polytope. The proof of Theorem 2 consists of three major steps as follows. In [3] we proved a similar theorem for Y-free graphs, this is used in [2] as the starting point for proving a similar theorem for oriented graphs. The theorem on oriented graphs has been used here as the starting point for proving our main result.

We conclude with a simple corollary. For a undirected graph $G = (V, E)$ we denote by $\overleftrightarrow{G} = (V, A)$ the directed graph obtained from G by replacing each edge $uv \in E$ by two arcs (u, v) and (v, u).

Corollary 39. *Let G be a connected undirected graph. Then $P_p(\overleftrightarrow{G})$ is integral for all p if and only if G is a path or a cycle.*

Proof. If G is a path or a cycle, then $\overleftrightarrow{G}$ satisfies conditions (i) and (ii) of Theorem 2 and so $P_p(\overleftrightarrow{G})$ is integral.

Suppose G is not a path nor a cycle. Then G contains a node of degree at least 3. Thus $\overleftrightarrow{G}$ contains H_4 as a subgraph. Again Theorem 2 implies that $P_p(\overleftrightarrow{G})$ is not integral for all p. □

References

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