

IBM Research Report

On the Linear Relaxation of the p -Median Problem I: Oriented Graphs

Mourad Baiou

CNRS

LIMOS Complexe scientifique des Czeaux
63177 Aubiere cedex,
France

Francisco Barahona

IBM Research Division

Thomas J. Watson Research Center

P.O. Box 218

Yorktown Heights, NY 10598



Research Division

Almaden - Austin - Beijing - Haifa - India - T. J. Watson - Tokyo - Zurich

ON THE LINEAR RELAXATION OF THE p -MEDIAN PROBLEM I: ORIENTED GRAPHS

MOURAD BAÏOU AND FRANCISCO BARAHONA

ABSTRACT. We study a well known linear programming relaxation of the p -median problem. We characterize the oriented graphs for which this linear relaxation defines an integral polytope.

1. INTRODUCTION

This is the first of two papers that deal with a linear relaxation of the p -median problem. Our goal is to characterize the graphs for which this linear relaxation defines an integral polytope. This first paper deals with oriented graphs, i.e., if (u, v) belongs to the arc-set then (v, u) does not belong to it. The second paper deals with general directed graphs.

Let $G = (V, A)$ be a directed graph, not necessarily connected. We assume that G is simple, i.e., between any two nodes u and v there is at most one arc directed from u to v . Also for each arc $(u, v) \in A$ there is an associated cost $c(u, v)$. The p -median problem (p MP) consists of selecting p nodes, usually called *centers*, and then assign each non-selected node to a selected node. The goal is to select p nodes that minimize the sum of the costs yield by the assignment of the non-selected nodes. This problem has several applications such as location of bank accounts [7], placement of web proxies in a computer network [13], semistructured data bases [12, 11]. When the number of centers is not specified and each opened center induces a given cost, this is called the *uncapacitated facility location problem* (UFLP).

The facets of the p -median polytope have been studied in [1] and [9]. The facets of the uncapacitated facility location polytope have been studied in [10], [8], [5], [6], [4].

Consider the following linear system:

$$\begin{aligned} (1) \quad & \sum_{v \in V} y(v) = p, \\ (2) \quad & \sum_{v: (u,v) \in A} x(u, v) + y(u) = 1 \quad \forall u \in V, \\ (3) \quad & x(u, v) \leq y(v) \quad \forall (u, v) \in A, \\ (4) \quad & 0 \leq y(v) \leq 1 \quad \forall v \in V, \\ (5) \quad & x(u, v) \geq 0 \quad \forall (u, v) \in A. \end{aligned}$$

Inequalities (1)-(5) give a linear programming relaxation of the p MP. Analogously (2)-(5) give a linear programming relaxation of the UFLP.

Date: November 1, 2007.

Key words and phrases. p -median problem, uncapacitated facility location problem.

Denote by $P_p(G)$ the polytope defined by (1)-(5), also let $P(G)$ be the polytope defined by (2)-(5). A polyhedron is called integral if all its extreme points are integral. In this paper we characterize the class of oriented graphs G for which $P_p(G)$ is integral.

Now we need to state several definitions. For a directed graph $G = (V, A)$ and a set $W \subset V$, we denote by $\delta^+(W)$ the set of arcs $(u, v) \in A$, with $u \in W$ and $v \in V \setminus W$. Also we denote by $\delta^-(W)$ the set of arcs (u, v) , with $v \in W$ and $u \in V \setminus W$. We write $\delta^+(v)$ and $\delta^-(v)$ instead of $\delta^+(\{v\})$ and $\delta^-(\{v\})$, respectively. If there is a risk of confusion we use δ_G^+ and δ_G^- . A node u with $\delta^+(u) = \emptyset$ is called a *pendent* node.

A simple cycle C is an ordered sequence $v_0, a_0, v_1, a_1, \dots, a_{p-1}, v_p$, where

- v_i , $0 \leq i \leq p-1$, are distinct nodes,
- a_i , $0 \leq i \leq p-1$, are distinct arcs,
- either v_i is the tail of a_i and v_{i+1} is the head of a_i , or v_i is the head of a_i and v_{i+1} is the tail of a_i , for $0 \leq i \leq p-1$, and
- $v_0 = v_p$.

Let $V(C)$ and $A(C)$ denote the nodes and the arcs of C , respectively. By setting $a_p = a_0$, we associate with C three more sets as below.

- We denote by \hat{C} the set of nodes v_i , such that v_i is the head of a_{i-1} and also the head of a_i , $1 \leq i \leq p$.
- We denote by \check{C} the set of nodes v_i , such that v_i is the tail of a_{i-1} and also the tail of a_i , $1 \leq i \leq p$.
- We denote by \tilde{C} the set of nodes v_i , such that either v_i is the head of a_{i-1} and also the tail of a_i , or v_i is the tail of a_{i-1} and also the head of a_i , $1 \leq i \leq p$.

Notice that $|\hat{C}| = |\check{C}|$. A cycle will be called *odd* if $|\tilde{C}| + |\hat{C}|$ is odd, otherwise it will be called *even*. A cycle C with $V(C) = \tilde{C}$ is a *directed cycle*. A cycle C is called a *Y-cycle* if for any node v in \hat{C} , there exists an arc $(v, \bar{v}) \in A$ with $\bar{v} \notin V(C)$. Remark that when $\hat{C} = \emptyset$ then C is a directed cycle and also a Y-cycle.

A path is defined in a similar way, but without asking the condition $v_0 = v_p$. For a path from v_0 to v_p , the nodes v_1, \dots, v_{p-1} are called *internal*.

In Figure 1 we show three graphs, for each of them we show a fractional extreme point of $P_p(G)$. The numbers close to the nodes correspond to the y variables and the numbers close to the arcs correspond to the x variables. In Figure 2 we show an odd Y-cycle and one arc, we also show a fractional extreme point of $P_p(G)$.

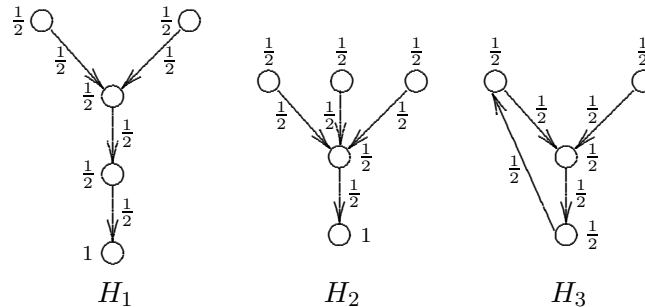


FIGURE 1. Three fractional extreme points.

The main result of this paper is the following.

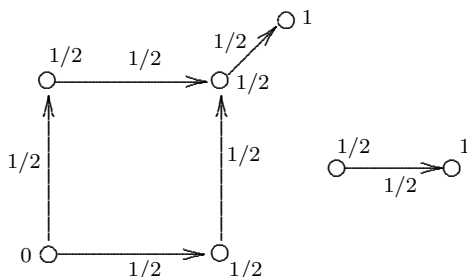


FIGURE 2. An odd Y -cycle and one arc. A fractional extreme point is shown.

Theorem 1. Let $G = (V, A)$ be an oriented graph, then $P_p(G)$ is integral if and only if

- (i) it does not contain as a subgraph any of the graphs H_1 , H_2 or H_3 of Figure 1, and
- (ii) it does not contain an odd Y -cycle C and an arc (u, v) with neither u nor v in $V(C)$.

A variation of the p -MP that is common in the literature is when V is partitioned into V_1 and V_2 . The set V_1 corresponds to the customers, and the set V_2 corresponds to the potential facilities. Each customer in V_1 should be assigned to an opened facility in V_2 . This is obtained by considering $A \subseteq V_1 \times V_2$, and using the following linear programming relaxation

$$(6) \quad \sum_{v \in V_2} y(v) = p,$$

$$(7) \quad \sum_{(u,v) \in A} x(u, v) = 1 \quad \forall u \in V_1,$$

$$(8) \quad x(u, v) \leq y(v) \quad \forall (u, v) \in A,$$

$$(9) \quad y(v) \geq 0 \quad \forall v \in V_2,$$

$$(10) \quad y(v) \leq 1 \quad \forall v \in V_2,$$

$$(11) \quad x(u, v) \geq 0 \quad \forall (u, v) \in A.$$

We call this the *bipartite case*. Here we also characterize the bipartite graphs for which (6)-(11) defines an integral polytope.

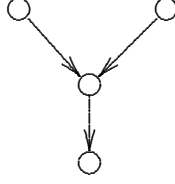
This paper is organized as follows. In Section 2 we give some preliminary results. In Section 3 we give the necessity proof of Theorem 1. Sections 4 and 5 are devoted to the sufficiency proof. Section 6 deals with the bipartite case. In Section 7 we discuss the recognition of this class of graphs.

2. PRELIMINARIES

This section contains some preliminary results and definitions.

2.1. Y -free graphs. An oriented graph $G = (V, A)$, not necessarily connected, is called Y -free if it does not contain as subgraph the graph of Figure 3. Notice that if C is a cycle in a Y -free graph, then all nodes in \hat{C} are pendent.

The consequences of Theorems 14 and 20 in [3] are summarized in the following theorem. This will be used in the proof of Theorem 1.

FIGURE 3. The graph Y .

Theorem 2. *If G is a Y -free graph with no odd directed cycle, then $P_p(G)$ and $P(G)$ are integral.*

2.2. The labeling procedure. Given a vector in $P_p(G)$ we plan to add and subtract a value ϵ from some of its components in such a way that any inequality (1)-(3) that was satisfied with equality remains satisfied with equality. This will be done according to a labeling procedure that we describe below. Let C be an even cycle, not necessarily a Y -cycle. The following labeling procedure will assign labels to the nodes and arcs in C .

- Given an even cycle $C = v_0, a_0, v_1, a_1, \dots, a_{p-1}, v_p$. If C is a directed cycle then, set $l(v_0) \leftarrow 1$; $l(a_0) \leftarrow -1$. Otherwise, assume $v_0 \in \dot{C}$ and set $l(v_0) \leftarrow 0$; $l(a_0) \leftarrow 1$.
- For $i = 1$ to $p - 1$ do the following:
 - If v_i is the head of a_{i-1} and is the tail of a_i , then $l(v_i) \leftarrow l(a_{i-1})$, $l(a_i) \leftarrow -l(v_i)$.
 - If v_i is the head of a_{i-1} and is the head of a_i , then $l(v_i) \leftarrow l(a_{i-1})$, $l(a_i) \leftarrow l(v_i)$.
 - If v_i is the tail of a_{i-1} and is the head of a_i , then $l(v_i) \leftarrow -l(a_{i-1})$, $l(a_i) \leftarrow l(v_i)$.
 - If v_i is the tail of a_{i-1} and is the tail of a_i , then $l(v_i) \leftarrow 0$, $l(a_i) \leftarrow -l(a_{i-1})$.

The remark below is easy to see.

Remark 3. *If C is a directed even cycle then $l(a_{p-1}) = l(v_0)$, and $\sum l(v_i) = 0$.*

It remains the case when C is not directed.

Lemma 4. *If C is a non-directed even cycle then $l(a_{p-1}) = -l(a_0)$ and $\sum l(v_i) = 0$.*

Proof. Let $v_{j(0)}, v_{j(1)}, \dots, v_{j(k)}$ be the ordered sequence of nodes in \dot{C} , with $v_{j(0)} = v_{j(k)}$. A path in C

$$v_{j(i)}, a_{j(i)}, \dots, a_{j(i+1)-1}, v_{j(i+1)}$$

from $v_{j(i)}$ to $v_{j(i+1)}$ will be called a *segment* and denoted by S_i . A segment is *odd* (resp. *even*) if it contains an *odd* (resp. *even*) number of arcs. Let

$$l(S_i) = \sum_{v \in S_i \cap V} l(v).$$

Let r be the number of even segments and t the number of odd segments. We have that $r + t = |\dot{C}|$, and since the parity of $|V(C)|$ is equal to the parity of t , we have that $t + |\dot{C}|$ is even. Therefore $r = |\dot{C}| - t$ is also even. The labeling has the following properties:

- a) If the segment is odd then $l(a_{j(i)}) = -l(a_{j(i+1)-1})$.
- b) If the segment is even then $l(a_{j(i)}) = l(a_{j(i+1)-1})$.
- c) If S_i is odd then $l(S_i) = 0$.

- d) If S_i is even then $l(S_i) = l(a_{j(i)})$.
e) Let S_1, \dots, S_r be the ordered sequence of even segments in C . Then $l(S_i) = -l(S_{i+1})$, for $i = 1, \dots, r - 1$.

Since there is an even number of even segments, properties a) and b) imply $l(a_0) = l(a_{p-1})$. Properties c) and e) imply $\sum l(v_i) = 0$. \square

2.3. Some basic polyhedral facts. Consider a polyhedron P defined by

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}.$$

Denote by $A^\equiv x \leq b^\equiv$ a maximal subsystem of $Ax \leq b$ such that $A^\equiv x = b^\equiv$ for all $x \in P$. Then the dimension of P is

$$n - \text{rank}(A^\equiv).$$

A *face* F of P is obtained by setting into equation some of the inequalities defining P . Clearly F is a polyhedron. An extreme point of P is a face of dimension 0. A *polytope* is a bounded polyhedron.

Lemma 5. *Let P be a polytope defined by*

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\},$$

whose extreme points are all 0–1 vectors. Let P' be defined by

$$P' = \{x \in P \mid cx = d\}.$$

If \hat{x} is an extreme point of P' then all its components are in

$$\{0, 1, \alpha, 1 - \alpha\},$$

for some number $\alpha \in [0, 1]$.

Proof. Let $A^\equiv x \leq b^\equiv$ be a maximal subsystem of $Ax \leq b$ such that $A^\equiv \hat{x} = b^\equiv$. If $\text{rank}(A^\equiv) = n$ then \hat{x} is an extreme point of P and it is a 0–1 vector. If $\text{rank}(A^\equiv) = n - 1$ then

$$F = \{x \in P \mid A^\equiv x = b^\equiv\}$$

is a face of P of dimension 1. Therefore \hat{x} is a convex combination of two extreme points of P . \square

3. NECESSITY PROOF OF THEOREM 1

To prove that conditions (i) and (ii) of Theorem 1 are necessary, it suffices to construct a fractional extreme point of $P_p(G)$ when G contains as a subgraph any of the graphs H_1, H_2, H_3 or if G contains an odd Y -cycle C with an arc (u, v) where neither u nor v is in $V(C)$. In the following we give a fractional vector (x^*, y^*) of $P_p(G)$ in each case. The proof that (x^*, y^*) is in fact an extreme point is easy and it is left to the reader.

- Suppose that $G = (V, A)$ contains H_1, H_2 or H_3 as a subgraph. Extend the fractional vector defined in each case of Figure 1 by setting $y^*(u) = 1$ for all other nodes, and $x^*(u, v) = 0$ for all other arcs. Then (x^*, y^*) is a fractional extreme point of $P_{|V|-2}(G)$.

- Let C be an odd Y -cycle in $G = (V, A)$ and let $(t, w) \in A$ such that neither t nor w is in $V(C)$. Define $x^*(u, v) = 1/2$ for each arc in $A(C)$. Define $y^*(v) = 1/2$ if $v \in \hat{C} \cup \tilde{C}$ and $y^*(v) = 0$ if $v \in \dot{C}$. By the definition of a Y -cycle, if $v \in \hat{C}$ then there must exist an arc $(v, \bar{v}) \in A$ with $\bar{v} \notin V(C)$. Thus for each $v \in \hat{C}$, set $x^*(v, \bar{v}) = 1/2$. Set $y^*(t) = 1/2$ and $x^*(t, w) = 1/2$. Set to zero all remaining components of x^* and to one the remaining components of y^* . Thus (x^*, y^*) is a fractional extreme point of $P_p(G)$ with $p = |V| - |\dot{C}| - (|\hat{C}| + |\tilde{C}| + 1)/2$.

Now it remains to prove that conditions (i) and (ii) are sufficient. For simplicity, in what follows we use z to denote the vector (x, y) , i.e., $z(u) = y(u)$ and $z(u, v) = x(u, v)$. The next four sections are devoted to the sufficiency proof.

4. SUFFICIENCY PROOF WHEN G DOES NOT CONTAIN AN ODD Y -CYCLE.

In this section we prove the following simplified version of our main result.

Theorem 6. *If G is an oriented graph that satisfies (i) and (ii) and has no odd Y -cycle, then $P_p(G)$ is integral.*

The proof is done by induction on the number of Y -nodes. If $|Y_G| = 0$ then, the graph is Y -free with no odd directed cycle, it follows from Theorem 2 that $P_p(G)$ is integral. Assume that $P_p(G')$ is integral for any positive integer p and for any oriented graph G' , with $|Y_{G'}| < |Y_G|$, that satisfies condition (i) and does not contain an odd Y -cycle. Now we suppose that z is a fractional extreme point of $P_p(G)$ and we plan to obtain a contradiction.

Lemma 7. *We may assume that $z(u, v) > 0$ for all $(u, v) \in A$.*

Proof. Let G' be the graph obtained after removing all arcs (u, v) with $z(u, v) = 0$. The graph G' has the same properties as G , that is, G' does not contain one of the graphs H_1 , H_2 or H_3 as a subgraph and it does not contain an odd Y -cycle. Let z' be the restriction of z on G' . Then z' is a fractional extreme point of $P_p(G')$. One can then consider the pair G' and z' instead of G and z . \square

Lemma 8. *We may assume that $|\delta^-(v)| \leq 1$ for every pendent node v in G .*

Proof. Let v be a pendent node in G and $\delta^-(v) = \{(u_1, v), \dots, (u_k, v)\}$. We can split v into k pendent nodes $\{v_1, \dots, v_k\}$ and replace every arc (u_i, v) with (u_i, v_i) . Then we define z' such that $z'(u_i, v_i) = z(u_i, v)$, $z'(v_i) = 1$, for all i , and $z'(u) = z(u)$, $z'(u, w) = z(u, w)$ for all other nodes and arcs. Let G' be this new graph. By definition, if G satisfies (i) and does not contain an odd Y -cycle then, these properties also hold for G' . It is easy to see that z' is a fractional extreme point of $P_{p+k-1}(G')$. Then one can consider G' , z' and $P_{p+k-1}(G')$ instead of G , z and $P_p(G)$. \square

Lemma 9. *We can assume that $z(u, v) = z(v)$, when v is not a pendent node.*

Proof. Suppose that $z(u, v) < z(v)$ and v is not a pendent node. We can remove (u, v) and add a new node v' and the arc (u, v') . Define $z'(u, v') = z(u, v)$, $z'(v') = 1$, and $z'(s, t) = z(s, t)$, $z'(r) = z(r)$ for all other arcs and nodes. Let G' be this new graph. It is easy to check that G' shares the same properties as G and that z' is a fractional extreme point of $P_{p+1}(G')$. \square

Lemma 10. *G does not contain a cycle.*

Proof. Let $C = v_0, a_0, v_1, a_1, \dots, a_{p-1}, v_p$ be a cycle in G . If $u \in \hat{C}$ then by Lemma 8 u is not a pendent node in G , since $|\delta^-(u)| \geq 2$. Hence there must exist an arc $(u, v) \in A$. If $v \in V(C)$ then G must contain one of the graphs H_1, H_2 or H_3 as a subgraph, which is not possible since G satisfies condition (i). Thus for each $u \in \hat{C}$ we have $(u, v) \in A$ and $v \notin V(C)$. Therefore C is a Y -cycle, and it must be even, that is $|\hat{C}| + |\tilde{C}|$ is even.

To obtain a contradiction we are going to define a new vector that satisfies with equality the same constraints that are tight for z . We assign labels to the nodes and arcs of C following the labeling procedure of Section 2. Define z^* as follows. For every arc a_i of C , $i = 0, \dots, p-1$, let $z^*(a_i) = z(a_i) + l(a_i)\epsilon$. For every node v_i , $i = 0, \dots, p-1$, let $z^*(v_i) = z(v_i) + l(v_i)\epsilon$. Also for every node $u \in \hat{C}$, pick an arc $(u, v) \in \delta^+(u)$ and let $z^*(u, v) = z(u, v) - l(u)\epsilon$. Finally let $z^*(u) = z(u)$, $z^*(u, v) = z(u, v)$ for all other nodes and arcs of G .

We have to see that any constraint that is satisfied with equality by z is also satisfied with equality by z^* . We treat the case when C is not directed. This proof hold when C is directed, by considering that \hat{C} and \tilde{C} as empty sets.

Let us see first that the inequalities (4) and (5) that are satisfied with equality by z remain satisfied with equality by z^* . For this, two remarks suffice. First, remark that if $(u, v) \in A(C)$ then $0 < z(u, v) < 1$. Lemma 7 implies that $z(u, v) > 0$. If $z(u, v) = 1$ then $z(u) = 0$ and thus for the second arc of C that is incident to u its variable must take the value zero which impossible by Lemma 7. The second remark is that $l(v_i) = 0$ for $v_i \in \tilde{C}$. Thus to prove that inequalities (4) that are tight for z are also tight for z^* , it suffices to see that $0 < z(v) < 1$ for all $v \in \hat{C} \cup \tilde{C}$. Let $v \in \hat{C} \cup \tilde{C}$ and $(u, v) \in A(C)$. Lemma 7 implies that $z(u, v) > 0$. But we have that $z(u, v) \leq z(v)$, so $z(v) > 0$. If $z(v) = 1$, Lemma 7 implies that $\delta^+(v) = \emptyset$ which implies that $v \notin V(C)$.

Now let us consider inequalities (3). Remark that for every node $v \in \tilde{C}$, $|\delta^-(v)| = 1$, otherwise H_1 or H_3 is present. Thus the unique arc directed into v belongs to $A(C)$. Also, if $v \in \hat{C}$ then the only arcs directed into v are in $A(C)$, otherwise H_2 is present. This shows that every constraint (3) that is tight for z is also tight for z^* .

Now we deal with constraints (2). By definition z^* satisfies constraints (2) for every node $v \neq v_0$. Lemma 4 states that $l(a_{p-1}) = -l(a_0)$, so equality (2) with respect to v_0 is satisfied by z^* .

Also the same lemma states that $\sum l(v_i) = 0$, this implies that equation (1) is satisfied by z^* .

We conclude that $z^* \in P_p(G)$ and that any constraint among (1)-(5) that was tight for z remains tight for z^* . This contradicts the fact that z is an extreme point of $P_p(G)$. \square

Let $t \in V$. The node t is called a Y -node in $G = (V, A)$ if there exists three different nodes u_1, u_2, w in V such that (u_1, t) , (u_2, t) and (t, w) belong to A . Denote by Y_G the set of Y -nodes in G .

Lemma 11. *The graph G must contain at least one Y -node.*

Proof. Suppose the contrary. Then G is a Y -free graph with no odd directed cycle. Theorem 2 implies that $P_p(G)$ is an integral polytope. This contradicts the fact that z is a fractional extreme point of $P_p(G)$. \square

Lemma 12. *There is a Y -node t in G , and arcs (u_1, t) , (u_2, t) , (t, w) , such that:*

- V can be partitioned into W_1 and W_2 so that $\{u_1, t, w\} \subseteq W_1$ and $u_2 \in W_2$.

- The only arc in G between W_1 and W_2 is (u_2, t) . See Figure 4.

Proof. Let t be a Y -node in G , Lemma 11 shows that such a node exists. Let $G_1 = (S_1, A_1)$ be the connected component of G that contains t . It follows from Lemma 10 that (u_2, t) does not belong to any cycle in G . Hence if we remove (u_2, t) from G then we disconnect G_1 into two connected components. Let S'_1 and S'_2 be the node sets of these two components, containing u_1 and u_2 , respectively. Define W_1 to be S'_1 and $W_2 = V \setminus S'_1$. \square

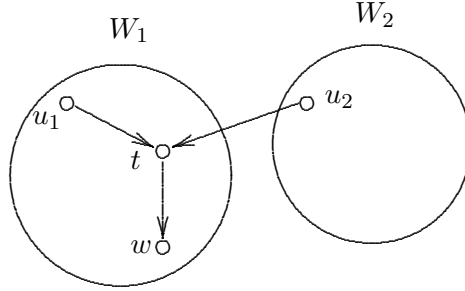


FIGURE 4

Lemma 13. $P(G)$ is an integral polytope.

Proof. $P(G)$ is a face of the polytope studied in [2]. Here G has no odd Y -cycle. It follows from Lemma 8 that any odd cycle is a Y -cycle, hence G has no odd cycle. It was shown in [2] that if G has no odd cycle then $P(G)$ is integral. \square

Lemma 14. The values of z are in $\{0, 1, \alpha, 1 - \alpha\}$, for some number $\alpha \in [0, 1]$.

Proof. Since Lemma 13 shows that $P(G)$ is an integral polytope and $P_p(G)$ is obtained from $P(G)$ by adding exactly one equation, the result follows from Lemma 5. \square

Lemma 15. $z(t) = \frac{1}{2}$.

Proof. Based on Lemma 12, we define the graphs G^1 and G^2 as follows. Let $A(W_1)$ and $A(W_2)$ be the set of arcs in G having both endnodes in W_1 and W_2 , respectively. Let $G^1 = (W_1, A(W_1))$ and $G^2 = (W_2 \cup \{t', v', w'\}, A(W_2) \cup \{(u_2, t'), (t', v'), (v', w')\})$, see Figure 5. Let $G' = G^1 \cup G^2$.

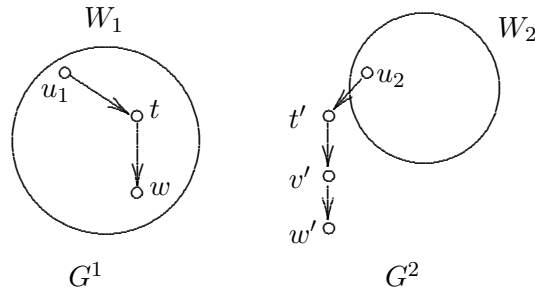


FIGURE 5

Notice that from Lemma 9 we have $z(u_1, t) = z(u_2, t) = z(t)$. Define z' to be $z'(u_2, t') = z(u_2, t)$, $z'(t') = z(u_2, t)$, $z'(t', v') = 1 - z(u_2, t)$, $z'(v') = 1 - z(u_2, t)$,

$z'(v', w') = z(u_2, t)$, $z'(w') = 1$ and $z'(u) = z(u)$, $z'(u, v) = z(u, v)$ for all other nodes and arcs. We have that $z' \in P_{p+2}(G')$ and G' is a graph satisfying (i) and does not contain an odd Y -cycle, with $|Y_{G'}| < |Y_G|$. The induction hypothesis implies that z' is not an extreme point of $P_{p+2}(G')$. Thus there must exist a 0-1 vectors z'_1, \dots, z'_r in $P_{p+2}(G')$ such that z' is a convex combination of z'_1, \dots, z'_r , and all constraints that are tight for z' are also tight for z'_1, \dots, z'_r . Thus

$$(12) \quad z' = \sum_{i=1}^r \lambda_i z'_i,$$

$$(13) \quad \sum_{i=1}^r \lambda_i = 1,$$

$$(14) \quad \lambda_i \geq 0, \quad i = 1, \dots, r.$$

If there exists a vector z'_k with $z'_k(t) = z'_k(t')$, then we can define from z'_k a 0-1 vector $z'' \in P_p(G)$ such that the same constraints tight for z are also tight for z'' . The vector z'' is obtained setting $z''(t) = z'_k(t) = z'_k(t')$, and $z''(u) = z'_k(u)$, $z''(u, v) = z'_k(u, v)$, for all other nodes and arcs in G .

Thus we may suppose that for all z'_i , $i = 1 \dots, r$, we have $z'_i(t) \neq z'_i(t')$. Let $z'_i(t) = 1$, $z'_i(t') = 0$, for $i = 1, \dots, r_1$, and $z'_i(t) = 0$, $z'_i(t') = 1$, for $i = r_1 + 1, \dots, r$. Then

$$(15) \quad z'(t) = \sum_{i=1}^{r_1} \lambda_i,$$

$$(16) \quad z'(t') = \sum_{i=r_1+1}^r \lambda_i,$$

and since by definition $z'(t) = z'(t')$ and $z'(t) + z'(t') = \sum_{i=1}^r \lambda_i = 1$, the result is obtained. \square

Lemma 16. *Each component of z is in $\{0, 1, \frac{1}{2}\}$.*

Proof. Immediate from Lemma 9, Lemma 14 and Lemma 15. \square

Define $p_1 = \sum_{v \in W_1} z(v)$ and $p_2 = \sum_{v \in W_2} z(v)$, so $p = p_1 + p_2$. We distinguish two cases: p_1 and p_2 are integer; and they are not.

Lemma 17. *If the numbers p_1 and p_2 are integer then z cannot be an extreme point.*

Proof. Consider the graphs G^1 and G^2 of Figure 5, as defined above. Let z_1 be the restriction of z to G^1 . Clearly $z_1 \in P_{p_1}(G^1)$. Define z_2 as follows, $z_2(u_2, t') = z(u_2, t) = \frac{1}{2}$, $z_2(t') = \frac{1}{2}$, $z_2(t', v') = \frac{1}{2}$, $z_2(v') = \frac{1}{2}$, $z_2(v', w') = \frac{1}{2}$, $z_2(w') = 1$ and $z_2(u) = z(u)$, $z_2(u, v) = z(u, v)$ for all other nodes and arcs of G^2 . We have that $z_2 \in P_{p_2+2}(G^2)$.

Both graphs G^1 and G^2 satisfy (i) and do not contain an odd Y -cycle. Moreover, $|Y_{G^1}| < |Y_G|$ and $|Y_{G^2}| < |Y_G|$. Since z_1 and z_2 are both fractional, the induction hypothesis implies that they are not extreme points of $P_{p_1}(G^1)$ and $P_{p_2+2}(G^2)$, respectively. Thus there must exist a 0-1 vector $z'_1 \in P_{p_1}(G^1)$ with $z'_1(t) = 0$ so that the same constraints that are tight for z_1 are also tight for z'_1 . Also there must exist a 0-1 vector $z'_2 \in P_{p_2+2}(G^2)$ with $z'_2(t') = 0$ such that the same constraints that are tight for z_2 are also tight for z'_2 . Combine z'_1 and z'_2 to define a solution $z' \in P_p(G)$ as follows.

$$\begin{aligned}
z'(u) &= z'_1(u), & \text{for every node } u \text{ of } G^1, \\
z'(u, v) &= z'_1(u, v), & \text{for every arc } (u, v) \text{ of } G^1, \\
z'(u_2, t) &= 0, \\
z'(v) &= z'_2(v), & \text{for every node } v \in W_2, \\
z'(u, v) &= z'_2(u, v), & \text{for every arc } (u, v) \in A(W_2).
\end{aligned}$$

Remark that $\sum_{v \in V} z'(v) = p$. Also any constraint among (2)-(5), that is tight for z is also tight for z' . Then the same constraints of $P_p(G)$ that are tight for z are also tight for z' . This contradicts the fact that z is an extreme point of $P_p(G)$. \square

Lemma 18. *If the numbers p_1 and p_2 are not integer then z cannot be an extreme point.*

Proof. Thus from Lemma 16, $\sum_{v \in W_1} z(v) = p_1 = \alpha + \frac{1}{2}$ and $\sum_{v \in W_2} z(v) = p_2 = \beta - \frac{1}{2}$, where α and β are integers and $\alpha + \beta = p$. Define G^1 and G^2 from G as follows. $G^1 = (W_1 \cup \{u'_1\}, (A(W_1) \setminus \{(u_1, t)\}) \cup \{(u_1, u'_1), (u'_1, t)\})$ and $G^2 = (W_2 \cup \{t', w'\}, A(W_2) \cup \{(u_2, t'), (t', w')\})$, see Figure 6.

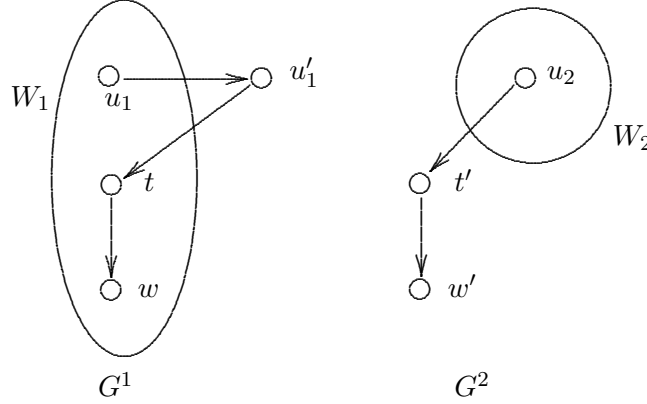


FIGURE 6

Define z^1 to be:

$$\begin{aligned}
z^1(u_1, u'_1) &= z^1(u'_1) = z^1(u'_1, t) = \frac{1}{2}, \\
z^1(u) &= z(u) \quad \text{for all other nodes of } G^1, \\
z^1(u, v) &= z(u, v) \quad \text{for all other arcs of } G^1.
\end{aligned}$$

Let z^2 be defined by:

$$\begin{aligned}
z^2(u_2, t') &= z^2(t') = z^2(t', w') = \frac{1}{2}, \\
z^2(w') &= 1, \\
z^2(u) &= z(u) \quad \text{for all other nodes of } G^2, \\
z^2(u, v) &= z(u, v) \quad \text{for all other arcs of } G^2.
\end{aligned}$$

Notice that $z^1 \in P_{\alpha+1}(G^1)$ and $z^2 \in P_{\beta+1}(G^2)$. Notice also that the graphs G^1 and G^2 satisfy (i) and do not contain an odd Y -cycle. The induction hypothesis may be applied to G^1 and G^2 since $|Y_{G^1}| < |Y_G|$ and $|Y_{G^2}| < |Y_G|$. Thus there must exist a 0-1 vector $\bar{z}^1 \in P_{\alpha+1}(G^1)$ such that the same constraints that are tight for z^1 are also tight for \bar{z}^1 , and such that $\bar{z}^1(u_1, u'_1) = 0$. Also there must exist a 0-1 vector $\bar{z}^2 \in P_{\beta+1}(G^2)$ such that

the same constraints that are tight for z^2 are also tight for \bar{z}^2 and such that $\bar{z}^2(t') = 0$. Now from \bar{z}^1 and \bar{z}^2 define $\bar{z} \in P_p(G)$ as follows.

$$\begin{aligned} \bar{z}(u_2, t) &= 0, \\ \bar{z}(u) &= \bar{z}^1(u), & \text{for all } u \in W_1 \setminus \{t\}, \\ \bar{z}(u, v) &= \bar{z}^1(u, v), & \text{for all } (u, v) \in A(W_1) \setminus \{(u_1, t), (t, w)\}, \\ \bar{z}(t) &= 0, \\ \bar{z}(u_1, t) &= 0, \\ \bar{z}(t, w) &= 1, \\ \bar{z}(u) &= \bar{z}^2(u), & \text{for all } u \in W_2, \\ \bar{z}(u, v) &= \bar{z}^2(u, v), & \text{for all } (u, v) \in A(W_2). \end{aligned}$$

It is easy to see that $\bar{z} \in P_p(G)$ and that the same constraints that are tight for z are also tight for \bar{z} . Thus z cannot be an extreme point. \square

These last two lemmas give the desired contradiction, so the proof of Theorem 6 is complete.

5. SUFFICIENCY PROOF WHEN G CONTAINS AN ODD Y -CYCLE

The graph we consider in this section is oriented, satisfies conditions (i) and (ii) of Theorem 1 and contains an odd Y -cycle. In this case we will prove that $P_p(G)$ is integral. This with Theorem 6 will complete the proof of Theorem 1. Assume that $z = (x, y)$ is a fractional extreme point of $P_p(G)$.

Lemma 19. *We can suppose that $z(u, v) > 0$ for all $(u, v) \in A$.*

Proof. Let G' be the graph obtained after removing all arcs (u, v) with $z(u, v) = 0$. Let z' be the restriction of z to G' . Since z is a fractional extreme point of $P_p(G)$ this implies that z' is a fractional extreme point of $P_p(G')$. Notice that G' satisfies conditions (i) and (ii) of Theorem 1. Also G' contains an odd Y -cycle, otherwise from Theorem 6 we have that $P_p(G')$ is integral, this would contradict the fact that z' is an extreme point of $P_p(G')$. \square

We also remark that from the lemma above and constraints (3) we should have $z(v) > 0$ for all $v \in V$ with $|\delta^-(v)| \geq 1$. This remark and Lemma 19 will be used implicitly when we define a new solution from z .

Lemma 20. *We can assume that for a pendent node v in G , $|\delta^-(v)| \leq 1$.*

Proof. If C is an odd Y -cycle then v cannot be in $V(C)$. So the graph G' as constructed in the proof of Lemma 8 has the same properties as G : it satisfies conditions (i) and (ii) and it contains an odd Y -cycle. Hence the pair G' and z' as defined in the proof of Lemma 8 may be considered instead of G and z . \square

Let $C = v_0, a_0, v_1, a_1, \dots, a_{p-1}, v_p$ be an odd Y -cycle in G . Let v_k and v_l be two nodes in $V(C)$. Call P_1 and P_2 the two paths in C from v_k to v_l . We plan to prove that there is no other path between v_k and v_l . Assume the contrary, and let $P = v_k, b_1, u_1, \dots, u_{r-1}, b_r, v_l$ be another path between v_k and v_l . Assume that all internal nodes of P are not in $V(C)$, (otherwise we change v_l). Notice that because of (ii) P cannot have more than two arcs. We call C_1 (resp. C_2) the cycle defined by P_1 and P (resp. P_2 and P). We need several lemmas.

Lemma 21. *If an arc of the path P is directed into (resp. away from) v_k (or v_l) then this node must be in \hat{C} (resp. $\hat{C} \cup \tilde{C}$).*

Proof. Let $b_1 = (u_1, v_k)$. If $v_k \in \hat{C}$, C being a Y -cycle implies that $(v_k, \bar{v}_k) \in A \setminus A(C)$ and \bar{v}_k is a pendent node. The graph induced by the two neighbors of v_k in C , the nodes v_k, u_1 and \bar{v}_k corresponds to H_2 , which is not possible since G satisfies (i). So assume $v_k \in \tilde{C}$. Let (v_{k-1}, v_k) and (v_k, v_{k+1}) be the two arcs of C incident to v_k . The node v_{k+1} is not a pendent node, so there is an arc (v_{k+1}, u) . If $u \in \{v_{k-1}, u_1\}$ (resp. $u \notin \{v_{k-1}, u_1\}$) then the graph induced by the nodes, v_{k-1}, v_k, v_{k+1}, u and u_1 corresponds to H_3 (resp. H_1).

Let $b_1 = (v_k, u_1)$. Suppose that $v_k \in \hat{C}$. Let (v_{k-1}, v_k) and (v_{k+1}, v_k) be the two arcs of C incident to v_k . If $u_1 = v_l$, then as seen above $v_l \in \hat{C}$ and the graph induced by $v_{k-1}, v_k, v_{k+1}, u_1$ and one of the neighbors of v_l in C corresponds to H_1 or H_3 . If P contains the arc (u_1, v_l) then as before one can construct a graph H_1 or H_3 . It remains the case when P contains the arc (v_l, u_1) . By Lemma 20, u_1 is not a pendent node and there is an arc (u_1, u) . If $u \in \{v_{k-1}, v_{k+1}\}$ (resp. $u \notin \{v_{k-1}, v_{k+1}\}$), then the graph induced by the nodes v_{k-1}, v_k, v_{k+1}, u and u_1 corresponds to H_3 (resp. H_1). \square

Lemma 22. *The path P cannot consist of exactly one arc.*

Proof. Let $P = v_k, (v_k, v_l), v_l$. By the lemma above, $v_l \in \hat{C}$ and $v_k \in \hat{C} \cup \tilde{C}$. We then consider two cases: (1) $v_k \in \hat{C}$ and (2) $v_k \in \tilde{C}$, as shown in Figure 7.

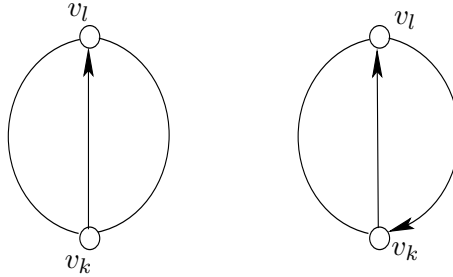


FIGURE 7. Cases (1) and (2).

- (1) C_1 and C_2 are both Y -cycles and exactly one of them is odd. The fact that G satisfies (ii) implies that the even cycle is of size three. Let C_1 be the even cycle. Thus $C_1 = v_k, (v_k, v_l), v_l, (v_l, v), v, (v_k, v), v_k$, where $v \in \hat{C}$. Since v is not pendent, there is an arc (v, t) with $t \notin V(C)$. Then the cycle C_2 and the arc (v, t) violate condition (ii) of Theorem 1.
- (2) Let (u, v_k) and (v_k, v) be the two arcs in $A(C)$ incident to v_k . The cycles C_1 and C_2 are both Y -cycles. The parity of C implies that exactly one of these cycles is odd. If one is odd the fact that G satisfies (ii) implies that the other cycle must be of size three. So the odd cycle must be the one containing the arc (u, v_k) , call it C_2 . Now as in case (1), $C_1 = v_k, (v_k, v_l), v_l, (v_l, v), v, (v_k, v), v_k$, and there is an arc (v, t) with $t \notin V(C)$. This violates condition (ii) of Theorem 1.

\square

Lemma 23. *The path P cannot consist of exactly two arcs.*

Proof. We have to study three cases:

- (1) $b_1 = (u_1, v_k)$ and $b_2 = (u_1, v_l)$. By Lemma 21, both v_k and v_l are in \dot{C} . The cycles C_1 and C_2 are Y -cycles and exactly one of them must be odd, otherwise C is an even Y -cycle. Suppose C_1 odd. Then C_2 must be of size four, otherwise G does not satisfy (ii). Now it is easy to see that $|\hat{C}_2| + |\tilde{C}_2| = 3$, which is impossible.
- (2) $b_1 = (v_k, u_1)$ and $b_2 = (u_1, v_l)$. The case where $b_1 = (u_1, v_k)$ and $b_2 = (v_l, u_1)$ may be treated by symmetry. Lemma 21 implies $v_l \in \dot{C}$ and $v_k \in \dot{C} \cup \tilde{C}$. So we have to distinguish two sub-cases: (a) $v_k \in \dot{C}$ and (b) $v_k \in \tilde{C}$. They are shown below in Figure 8.

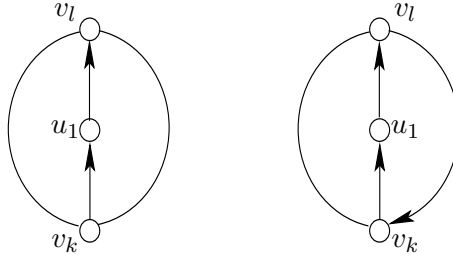


FIGURE 8. The sub-cases (a) and (b).

- (a) C_1 and C_2 are both Y -cycles. The parity of C implies that exactly C_1 or C_2 is odd. Suppose C_1 is odd. As in the previous case one can check that $|\hat{C}_2| + |\tilde{C}_2| = 3$, which is impossible.
- (b) Let (u, v_k) and (v_k, v) be the two arcs of C incident to v_k . Suppose $u = v_l$. Then the cycle $v_l, (v_l, v_k), v_k, (v_k, u_1), u_1, (u_1, v_l), v_l$ is an odd Y -cycle. Since G satisfies (ii), the arcs (v_l, v) and (v_k, v) must be in C . But this implies that C is an even Y -cycle which is impossible. It follows that both u and v are different from v_l .

Let C_1 be the cycle containing (v_k, v) and C_2 the cycle containing (u, v_k) . Both cycles are Y -cycles. The parity of C implies that exactly one of the cycles C_1 or C_2 is odd. If C_2 is odd then, as in the previous cases $|\hat{C}_1| + |\tilde{C}_1| = 3$, which is impossible. So suppose C_1 is odd. Then C_2 is a directed cycle of size four, $C_2 = v_k, (v_k, u_1), u_1, (u_1, v_l), v_l, (v_l, u), u, (u, v_k), v_k$, see Figure 9.

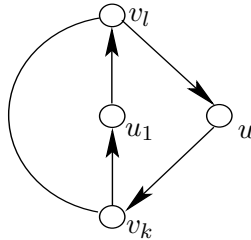


FIGURE 9. The sub-case (b).

If there is an arc not in C_2 directed into a node in C_2 , then the graph H_1 or H_3 is present. Thus we may suppose that the only arc that is directed into

a node in C_2 belongs to C_2 . Define z^* as follows:

$$z^*(u, v) = \begin{cases} z(u, v) + \epsilon & \text{if } (u, v) \in \{(v_k, u_1), (v_l, u)\}, \\ z(u, v) - \epsilon & \text{if } (u, v) \in \{(u_1, v_l), (u, v_k)\}, \\ z(u, v) & \text{otherwise;} \end{cases}$$

$$z^*(v) = \begin{cases} z(v) + \epsilon & \text{if } v \in \{u_1, u\}, \\ z(v) - \epsilon & \text{if } v \in \{v_k, v_l\}, \\ z(v) & \text{otherwise.} \end{cases}$$

The constraints that are satisfied with equality by z are also satisfied with equality by z^* . This contradicts the fact that z is an extreme point of $P_p(G)$.

- (3) $b_1 = (v_k, u_1)$ and $b_2 = (v_l, u_1)$. Notice that by Lemma 20, u_1 is not a pendent node. Thus there is an arc (u_1, t) where t is a pendent node. Therefore $t \notin V(C)$. But in this case C is an odd Y -cycle and (u_1, t) an arc in G where both u_1 and t are not in C , this contradicts the fact that G satisfies (ii). □

Lemma 24. *Let $C = v_0, a_0, v_1, a_1, \dots, a_{p-1}, v_p$ be an odd Y -cycle in G . Let v_k and v_l be two nodes in $V(C)$. Let P_1 and P_2 be the two paths in C from v_k to v_l , then there is no other path between v_k and v_l .*

Proof. This follows from condition (ii) and lemmas 22 and 23. □

Lemma 25. *The graph G contains exactly one odd Y -cycle.*

Proof. The proof is straightforward from Lemma 24 and the fact that G satisfies (ii). □

Now suppose that the graph G satisfies (i) and (ii) and contains exactly one odd Y -cycle. We can redefine $G = (V, A)$ as follows. Let C be the unique odd Y -cycle of G . Let A' be the set of arcs directed from a node not in $V(C)$ to a node in $V(C)$. Let A'' be the set of arcs directed from a node in $V(C)$ to a node not in $V(C)$. Let V' (resp. V'') be the node set defined by the tails (resp. heads) of the arcs A' (resp. A''). The graph could also contain a set V''' of isolated nodes. We have $V = V(C) \cup V' \cup V'' \cup V'''$ and $A = A(C) \cup A' \cup A''$. Notice the following properties of G .

- $V' \cap V'' = \emptyset$. Otherwise, Lemma 24 is contradicted.
- The nodes in V'' are pendent. This follows from condition (ii) and Lemma 24.
- The nodes in V' are adjacent to only nodes in \dot{C} . Otherwise G contains H_1 , H_2 or H_3 as a subgraph.
- Each node in V' is adjacent to exactly one node in \dot{C} . Otherwise we have a contradiction with Lemma 24.
- Each node in $V(C)$ is adjacent to at most one node in V'' . Otherwise, suppose $v \in V(C)$ and let u_1 and u_2 be two nodes in V'' adjacent to v . Define $\bar{z}(v, u_1) = z(v, u_1) + \epsilon$; $\bar{z}(v, u_2) = z(v, u_2) - \epsilon$ and $\bar{z}(u) = z(u)$, $\bar{z}(u, v) = z(u, v)$ for all other nodes and arcs. Then all inequalities that are satisfied with equality by z are also satisfied with equality by \bar{z} , this contradicts the fact that z is an extreme point of $P_p(G)$.
- A node $v \in \dot{C}$ can be adjacent to at most one node in $V' \cup V''$. Otherwise, suppose that there is an arc (v, w) with $w \in V''$ and an arc (t, v) with $t \in V'$. Define $\bar{z}(v) = z(v) + \epsilon$, $\bar{z}(v, w) = z(v, w) - \epsilon$, $\bar{z}(t, v) = z(t, v) + \epsilon$ and $\bar{z}(t) = z(t) - \epsilon$. And define $\bar{z}(u) = z(u)$, $\bar{z}(s, t) = z(s, t)$ for all other nodes and arcs. Then all

constraints that are tight for z are also tight for \bar{z} . If there are two nodes in V' adjacent to v , then we obtain the graph H_1 .

Clearly the nodes in V''' can be ignored. A graph with the above properties will be called an *extended odd Y -cycle*.

Lemma 26. *For each arc $(u, v) \in A(C)$ we have $z(u, v) = z(v)$.*

Proof. Suppose $z(u, v) < z(v)$. Consider the graph G' obtained from G by removing the arc (u, v) and adding the arc (u, w) , with w a new node. Let z' be defined as $z'(u, w) = z(u, v)$, $z'(w) = 1$ and $z'(u) = z(u)$, $z'(u, v) = z(u, v)$ for all other nodes and arcs. We have that $z' \in P_{p+1}(G')$. The graph G' satisfies (i) and does not contain an odd Y -cycle. Theorem 6 implies that z' is not an extreme point of $P_{p+1}(G')$. Hence, there exists a vector $z^* \in P_{p+1}(G')$, $z^* \neq z'$, such that all constraints that are tight for z' are also tight for z^* . Define $\bar{z}(u, v) = z^*(u, w)$ and $\bar{z}(u) = z^*(u)$, $\bar{z}(u, v) = z^*(u, v)$ for all other nodes and arcs of G . Then $\bar{z} \neq z$ and all constraints that are tight for z are also tight for \bar{z} . This is impossible since z is an extreme point of $P_p(G)$. Notice that we do not need that $\bar{z} \in P_p(G)$. \square

Now it remains the case when G is an extended odd Y -cycle. In the following two sub-sections we establish several properties of the extreme points of $P(G)$ and $P_p(G)$. The last sub-section contains the end of the proof. Let C be the Y -cycle of G . For a vector z and a labeling function l that associates integer values to the elements of $V \cup A$, we define a new vector z' as $z'(u) = z(u) + l(u)\epsilon$ and $z'(u, v) = z(u, v) + l(u, v)\epsilon$ for all nodes and arcs.

5.1. Fractional extreme points of $P(G)$. We suppose that z is a fractional extreme point of $P(G)$ such that $z(u, v) = z(v)$, for all $(u, v) \in A(C)$.

Lemma 27. *If $u \in \hat{C}$, then $z(u) = 0$.*

Proof. Suppose that $z(u) > 0$ and $u \in \hat{C}$. We give the label $l(u) = -2$ to the node u , then the label $+1$ to one of the arcs incident to u in C and extend the labels along C with the procedure of Section 2. If there is an arc a entering u , we give the label $l(a) = -2$ to a , and the label $+2$ to the tail of a . Also for each node $w \in \hat{C}$, we give the label $-l(w)$ to the arc whose tail is w . We set to zero the labels for all remaining nodes and arcs. With these labels we define a new vector z' that satisfies with equality each constraint that z satisfies with equality.

In order to see that the labels around C are correct, we proceed as follows. Let (u, v) and (u, t) be the two arcs incident to u in C . If we add extra node u' and replace (u, t) by (u, u') and (u', t) we have an even cycle. The labeling procedure gives $l(u, v) = -l(u, u') = l(u', t)$, therefore $l(u, v) = l(u, t)$ in the original graph. \square

Lemma 28. *If $u \in \tilde{C} \cup \hat{C}$ and (u, v) is an arc where v is pendent, then $z(u, v) = 0$.*

Proof. Suppose that $u \in \tilde{C}$ and $z(u, v) > 0$. Give the label -2 to (u, v) , give the label $+1$ to u , and the label $+1$ to the arc in C that leaves u . Then extend the labels around C . Also for each node $w \in \hat{C}$, we give the label $-l(w)$ to the arc whose tail is w . We set to zero the labels for all remaining nodes and arcs. As before, these labels define a new vector that leads to a contradiction.

To see that the labels around C are correct, we do the following. Let (u, w) and (t, u) be the arcs incident to u in C . We can add a new node u' and replace (t, u) by (t, u') and

(u', u) . The new cycle is even, then $l(u, w) = -l(u', u) = l(t, u')$. Thus $l(u, w) = l(t, u)$ in the original graph.

The case $u \in \dot{C}$ can be treated with a similar labeling, except that $l(u) = 0$. \square

Lemma 29. *If $(u, v) \in A(C)$ then $z(u, v) = 1/2$. Also $z(u) = 1/2$, if $u \in \tilde{C} \cup \dot{C}$.*

Proof. From lemmas 27 and 28, it follows that the values $z(u, v)$, for $(u, v) \in A(C)$, are the solution of a system of equations like

$$x(i) + x(i + 1) = 1, \text{ for } 0 \leq i \leq 2q, \quad x(2q + 1) = x(0),$$

where $2q + 1 = |\tilde{C}| + |\dot{C}|$. \square

5.2. Extreme points of $P_p(G)$. Here we show that several configurations cannot exist. We denote by A_1'' the arcs in A'' that are incident to a node in $\tilde{C} \cup \dot{C}$. Also let \dot{C}^+ be the set of nodes $v \in \dot{C}$ with $z(v) > 0$.

Lemma 30. $|A' \cup A_1''| \leq 1$.

Proof. Consider the case when u and v are two nodes in \tilde{C} , and $a_1 = (u, w)$ and $a_2 = (v, t)$ are two arcs in A_1'' . Let P_1 and P_2 be the two paths in C between u and v . Let us add the arc $a_3 = (t, w)$. Let C_1 and C_2 be the cycles defined by a_1, P_1, a_2, a_3 and a_1, P_2, a_2, a_3 respectively. One of them is even, C_1 say. We can apply the labeling procedure to C_1 and then remove a_3 and set the labels $l(w) = l(t) = 0$. Also any arc $(r, s) \in A''$ such that r has a label, receives the label $-l(r)$. We set to zero the labels for all remaining nodes and arcs. These labels define a new vector that satisfies with equality all constraints that were satisfied with equality by z .

The other cases are treated in a similar way. \square

Lemma 31. $|\dot{C}^+| \leq 1$.

Proof. Let u and v be two nodes in \dot{C} , suppose that $z(u) > 0$, $z(v) > 0$, and $A' = \emptyset$. Let P_1 and P_2 be the two paths in C between u and v . We add a new node t and the arcs $a_1 = (t, u)$ and $a_2 = (t, v)$. Let C_1 and C_2 be the cycles defined by a_1, P_1, a_2 and a_1, P_2, a_2 respectively. One of them is even, C_1 say. We apply the labeling procedure to C_1 . Then we remove t, a_1, a_2 . Any arc $(r, s) \in A''$ such that r has a label, receives the label $-l(r)$. We set to zero the labels for all remaining nodes and arcs. Again we obtain a new vector that leads to a contradiction.

Now assume that u and v are two nodes in \tilde{C} , suppose that $z(u) > 0$, $z(v) > 0$, and $A' = \{(w, u)\}$. From the preceding lemma we have $|A'| \leq 1$. We add an arc (v, w) and the rest of the proof is as above. \square

Lemma 32. *If $|\dot{C}^+| = 1$ then $|A_1''| = 0$.*

Proof. Let $v \in \dot{C}$ with $z(v) > 0$ and $u \in \tilde{C} \cup \dot{C}$, $u \neq v$. Assume that the arc $a_1 = (u, t)$ is in A_1'' . We add a node s and the arcs (t, s) and (s, v) . The rest of the proof is like in the preceding lemmas.

If $v \in \dot{C}$ with $z(v) > 0$ and the arc $a_1 = (v, t)$ is in A_1'' , we give the label -1 to v , the label $+1$ to one of the arcs in C incident to v , we extend the labels around C , and give the label -1 to the arc (v, t) . Any arc $(r, s) \in A''$ such that r has a label, receives the label $-l(r)$. We set to zero the labels for all remaining nodes and arcs. These labels define a new vector that leads to a contradiction. \square

5.3. Remainder of the proof of Theorem 1. We assume that z is a fractional extreme point of $P_p(G)$ for an extended odd Y -cycle G , where C is the Y -cycle in G . In the preceding section we have seen that several configurations can be eliminated. The remaining cases are:

- (1) $A' = \{(u, v)\}$. The lemmas above imply $A_1'' = \emptyset$ and $\dot{C}^+ = \{v\}$.
- (2) $A_1'' = \{(u, v)\}$. The lemmas above imply $A' = \emptyset$ and $\dot{C}^+ = \emptyset$.
- (3) $\dot{C}^+ = \{v\}$. The lemmas above and Case (1) imply $A' \cup A_1'' = \emptyset$.
- (4) $\dot{C}^+ = \emptyset$. This implies $A' = \emptyset$. Case (2) implies $A_1'' = \emptyset$.

We have seen in Sub-section 2.3 that an extreme point of $P_p(G)$ is a convex combination of two extreme points of $P(G)$. Let \tilde{z} and \hat{z} be these two vectors, we use them to study some of these cases. Lemma 26 implies that $\tilde{z}(u, v) = \tilde{z}(v)$ and $\hat{z}(u, v) = \hat{z}(v)$ for all $(u, v) \in A(C)$. So Lemmas 27, 28 and 29 apply to both \tilde{z} and \hat{z} .

Case (1). $A' = \{(u, v)\}$, $A_1'' = \emptyset$, $\dot{C}^+ = \{v\}$.

Lemma 33. *If \tilde{z} is fractional and \hat{z} is integral then z does not exist.*

Proof. Lemma 27 implies that $\tilde{z}(v) = 0$ and $\tilde{z}(u, v) = 0$. Since $z(u, v) > 0$ we have $\hat{z}(u, v) = 1$. This implies $\hat{z}(u) = 0$ and $\hat{z}(v) = 1$. Let a be one of the arcs in C incident to v ; we have $\hat{z}(a) = 0$. We can continue setting the values of the components of \hat{z} around the cycle, based on the following equations:

$$\begin{aligned} \hat{z}(s, t) &= \hat{z}(t) \text{ for every arc } (s, t) \in A(C), \\ \hat{z}(s) &= 0, \text{ if } s \in \dot{C}, s \neq v, \\ \hat{z}(t, s) &= 1 - \hat{z}(t, w) \text{ if } t \in \dot{C}, t \neq v, (t, s), (t, w) \in A(C), \\ \hat{z}(s, t) &= 1 - \hat{z}(s), \text{ if } s \in \tilde{C}, (s, t) \in A(C). \end{aligned}$$

This is similar to the labeling procedure, we just have to identify the value one with the label +1 and the value zero with the label -1, except for the nodes in \dot{C} that keep the value zero. To stress this analogy we proceed as follows. Add a node v' . Let (v, r) and (v, n) be the two arcs in C incident to v , replace (v, n) by (v, v') and (v', n) . Let C' be this new cycle. It is even, so we can give the label -1 to (v, r) and extend the labels around the cycle.

Consider now the convex combination $z = \alpha\tilde{z} + (1 - \alpha)\hat{z}$. We obtain $z(u) = \alpha$, $z(v) = 1 - \alpha$, and for all other nodes in C we have the value $\alpha/2$ if its label is -1, $1 - \alpha/2$ if its label is +1, and 0 if its label is zero. Let S^+ be the set of nodes with the label +1, not including the node v' . Let S^- be the set of nodes with the label -1. We have that $|S^-| - |S^+| = 1$.

Thus $\sum_{r \in V} z(r) = q + \alpha/2$, where q is an integer. Since $\sum_{r \in V} z(r)$ should be an integer, we have that $\alpha/2$ should be an integer, thus $\alpha = 0$, a contradiction. \square

Lemma 34. *If \tilde{z} and \hat{z} are both integral then z does not exist.*

Proof. Suppose that $\tilde{z}(v) = 1$, then $\tilde{z}(a_1) = \tilde{z}(a_2) = 0$, where a_1 and a_2 are the arcs incident to v in C . We should have $\hat{z}(v) = 0$, this implies $\hat{z}(a_1) = 1$, say, and $\hat{z}(a_2) = 0$. Therefore $z(a_2) = 0$, a contradiction. \square

Lemma 35. *If \tilde{z} and \hat{z} are both fractional then z does not exist.*

Proof. Lemma 27 implies $z(v) = z(u, v) = 0$, a contradiction. \square

Case (2). $A'_1 = \{(u, v)\}$, $A' = \emptyset$, $\dot{C}^+ = \emptyset$.

Lemma 36. *If \tilde{z} is fractional and \hat{z} is integral then z does not exist.*

Proof. Lemma 28 implies $\tilde{z}(u, v) = 0$. Since $z(u, v) > 0$, we have $\hat{z}(u, v) = 1$. This implies $\hat{z}(u) = 0$ and $\hat{z}(v) = 1$. Let \bar{a} be an arc whose tail is u in C ; we have $\hat{z}(\bar{a}) = 0$. We can continue setting the values of the components of \hat{z} around the cycle, based on the equations:

$$\begin{aligned}\hat{z}(s, t) &= \hat{z}(t) \text{ for every arc } (s, t) \in A(C), \\ \hat{z}(s) &= 0, \text{ if } s \in \dot{C}, \\ \hat{z}(t, s) &= 1 - \hat{z}(t, w) \text{ if } t \in \dot{C}, \quad (t, s), (t, w) \in A(C), \\ \hat{z}(s, t) &= 1 - \hat{z}(s), \text{ if } s \in \tilde{C}, \quad (s, t) \in A(C).\end{aligned}$$

Again this is similar to the labeling procedure. Let (u, t) be an arc incident to u in C . We add a node u' and replace (u, t) by (u, u') and (u', t) . Then we give the label -1 to (u', t) and extend the labels.

Let S^+ be the set of nodes with the label $+1$, not including the node u' . Let S^- be the set of nodes with the label -1 . We have that $|S^-| - |S^+| = 1$. The rest of the proof is as in Lemma 33, we have $\sum_{r \in V} z(r) = q + \alpha/2$, where q is an integer. Since $\sum_{r \in V} z(r)$ should be an integer we have $\alpha = 0$. \square

Lemma 37. *If \tilde{z} and \hat{z} are both integral then z does not exist.*

Proof. Assume that $u \in \tilde{C}$. Suppose that $\tilde{z}(u, v) = 1$, then $\tilde{z}(a) = \tilde{z}(u) = 0$, where a is the arc whose tail is u in C . We should have $\hat{z}(u) = 1$, this implies $\hat{z}(a) = 0$. Therefore $z(a) = 0$, a contradiction. The proof when $u \in \dot{C}$ is similar. \square

Lemma 38. *If \tilde{z} and \hat{z} are both fractional then z does not exist.*

Proof. Lemma 28 implies $z(u, v) = 0$, a contradiction. \square

Case (3). $\dot{C}^+ = \{v\}$, $A' \cup A'_1 = \emptyset$.

Lemma 39. *If \tilde{z} is fractional and \hat{z} is integral then z does not exist.*

Proof. Lemma 27 implies $\tilde{z}(v) = 0$. Since $z(v) > 0$, we have $\hat{z}(v) = 1$. Let \bar{a} be one of the arcs in C incident to v ; we have $\hat{z}(\bar{a}) = 0$. We can continue setting the values of the components of \hat{z} around the cycle, based on the equations:

$$\begin{aligned}\hat{z}(s, t) &= \hat{z}(t) \text{ for every arc } (s, t) \in A(C), \\ \hat{z}(s) &= 0, \text{ if } s \in \dot{C}, \quad s \neq v, \\ \hat{z}(t, s) &= 1 - \hat{z}(t, w) \text{ if } t \in \dot{C}, \quad t \neq v, \quad (t, s), (t, w) \in A(C), \\ \hat{z}(s, t) &= 1 - \hat{z}(s), \text{ if } s \in \tilde{C}, \quad (s, t) \in A(C).\end{aligned}$$

Again we can use the labeling procedure as follows. We add the node v' . Let (v, r) and (v, n) be the two arcs in C incident to v , we replace (v, r) by (v, v') and (v', r) . Let C' be this new cycle. It is even, so we can give the label -1 to (v', r) and extend the labels around the cycle.

Consider now the convex combination $z = \alpha\tilde{z} + (1 - \alpha)\hat{z}$. We obtain $z(v) = 1 - \alpha$, and for all other nodes in C we have the value $\alpha/2$ if its label is -1 , $1 - \alpha/2$ if its label is $+1$,

and 0 if its label is zero. Let S^+ be the set of nodes with the label +1, not including the node v' . Let S^- be the set of nodes with the label -1. We have that $|S^-| - |S^+| = 1$.

Thus $\sum_{r \in V} z(r) = q - \alpha/2$, where q is an integer. Since $\sum_{r \in V} z(r)$ should be an integer, we have that $\alpha/2$ should be an integer, thus $\alpha = 0$, a contradiction. \square

Lemma 40. *If \tilde{z} and \hat{z} are both integral then z does not exist.*

Proof. Suppose that $\tilde{z}(v) = 1$, then $\tilde{z}(a_1) = \tilde{z}(a_2) = 0$, where a_1 and a_2 are the arcs incident to v in C . We should have $\hat{z}(v) = 0$, this implies $\hat{z}(a_1) = 1$, say, and $\hat{z}(a_2) = 0$. Therefore $z(a_2) = 0$, a contradiction. \square

Lemma 41. *If \tilde{z} and \hat{z} are both fractional then z does not exist.*

Proof. Lemma 27 implies $z(v) = 0$, a contradiction. \square

Case (4). $\hat{C}^+ = \emptyset$, $A' = \emptyset$, $A'' = \emptyset$.

In this case we have a vector that satisfies the hypothesis of Lemma 29. This implies $z(v) = 1/2$ for all $v \in \hat{C} \cup \tilde{C}$. Then we have that $\sum_{v \in V} z(v)$ is a fractional number, a contradiction. This completes the proof of Theorem 1.

6. THE BIPARTITE CASE

Now we assume that V is partitioned into V_1 and V_2 , $A \subseteq V_1 \times V_2$, and we deal with the system

$$(17) \quad \sum_{v \in V_2} y(v) = p,$$

$$(18) \quad \sum_{(u,v) \in A} x(u,v) = 1 \quad \forall u \in V_1,$$

$$(19) \quad x(u,v) \leq y(v) \quad \forall (u,v) \in A,$$

$$(20) \quad y(v) \geq 0 \quad \forall v \in V_2,$$

$$(21) \quad y(v) \leq 1 \quad \forall v \in V_2,$$

$$(22) \quad x(u,v) \geq 0 \quad \forall (u,v) \in A.$$

Let $\Pi_p(G)$ be the polytope defined by (17)-(22), in this section we characterize the bipartite graphs for which $\Pi_p(G)$ is an integral polytope.

Let \bar{V}_1 be the set of nodes $u \in V_1$ with $|\delta^+(u)| = 1$. Let \bar{V}_2 be the set of nodes in V_2 that are adjacent to a node in \bar{V}_1 . It is clear that the variables associated with nodes in \bar{V}_2 should be fixed, i.e., $y(v) = 1$ for all $v \in \bar{V}_2$. Let \bar{G} be the graph induced by $V \setminus \bar{V}_2$.

Let H be a graph with node set $\{u_1, u_2, u_3, v_1, v_2, v_3, v_4\}$ and arc set

$$\{(u_1, v_1), (u_2, v_2), (u_3, v_3), (u_1, v_4), (u_2, v_4), (u_3, v_4)\}.$$

If the graph \bar{G} contains H as a subgraph then we can construct a fractional extreme point as in Section 3. If \bar{G} contains an odd cycle and one extra node in $V_2 \setminus \bar{V}_2$, we can also construct a fractional extreme point. Now we prove that these are only configurations that should be forbidden in order to have an integral polytope.

Theorem 42. *The polytope $\Pi_p(G)$ is integral if and only if*

- (i) \bar{G} does not contain the graph H as a subgraph, and

- (ii) \bar{G} does not contain an odd cycle C and one extra node in $V_2 \setminus \bar{V}_2$.

So let G be a graph such that \bar{G} does not contain these two configurations. We assume that z is a fractional extreme point of $\Pi_p(G)$. As before, we can assume that $z(u, v) > 0$ for every arc $(u, v) \in A$.

Lemma 43. *We can assume that $z(u, v) = z(v)$ for each arc (u, v) such that $v \in V_2 \setminus \bar{V}_2$.*

Proof. Suppose that $z(u, v) < z(v)$ for an arc (u, v) and $v \in V_2 \setminus \bar{V}_2$. We can add the nodes u', v' , the arcs (u', v') , (u, v') and remove the arc (u, v) . Then define $z'(u', v') = z(v') = 1$, $z'(u, v') = z(u, v)$, and $z'(s, t) = z(s, t)$, $z'(w) = z(w)$, for all other nodes and arcs. Let G' be the new graph. Then z' is an extreme point of $\Pi_{p+1}(G')$. The graph G' satisfies the hypothesis of Theorem 42. \square

The proof of Theorem 42 is divided into the following three cases:

- (1) \bar{G} does not contain an odd cycle nor the graph H .
- (2) \bar{G} does not contain H and contains an odd cycle C that includes all nodes in $V_2 \setminus \bar{V}_2$, and $|V_2 \setminus \bar{V}_2| \geq 5$.
- (3) \bar{G} does not contain H and contains an odd cycle C that includes all nodes in $V_2 \setminus \bar{V}_2$, and $|V_2 \setminus \bar{V}_2| = 3$.

We treat these three cases below.

6.1. \bar{G} does not contain an odd cycle nor the graph H .

Lemma 44. *For all $u \in V_1$ we have $|\delta^+(u)| \leq 2$.*

Proof. Since \bar{G} has no odd cycle, the polytope defined by (18)-(22) is integral, this was proved in [2]. This and Lemma 5 show that z is a convex combination of two integral vectors. Therefore $|\delta^+(u)| \leq 2$. \square

Now we build an auxiliary undirected graph G' whose node-set is $V_2 \setminus \bar{V}_2$. For each node $u \in V_1$ such that $\delta^+(u) = \{(u, s), (u, t)\}$, $\{s, t\} \subseteq V_2 \setminus \bar{V}_2$, we have an edge in G' between s and t . This could create parallel edges. Notice that any node v in G' is adjacent to at most two other nodes. If v was adjacent to three other nodes, we would have the sub-graph H in \bar{G} .

Lemma 43 implies that if $z(v) = 1$ for $v \in V_2 \setminus \bar{V}_2$, then v is not adjacent to any other node in G' . A node $v \in V_2 \setminus \bar{V}_2$ is called fractional if $0 < z(v) < 1$. So G' consists of a set of isolated nodes, and a set of cycles and paths. We have to study the four cases below.

- If G' contains a cycle, it should be even, because \bar{G} has no odd cycle. For a cycle in G' we can label the nodes with $+1$ and -1 so that adjacent nodes in the cycle have opposite labels. This labeling translates into a labeling in G as follows: If s and t have the labels $+1$ and -1 respectively, and the arcs (u, s) and (u, t) are in G , then (u, s) receives the label $+1$ and (u, t) receives the label -1 . If s has the label $l(s)$ and the arcs (u, s) and (u, t) are in G with $t \in \bar{V}_2$, then (u, s) receives the label $l(s)$ and (u, t) receives the label $-l(s)$. All other nodes and arcs receive the label 0. This defines a new vector that satisfies with equality the same constraints that z satisfies with equality.
- If there is a path with an even number of fractional nodes we label them as before. This translates into a labeling in G as follows. If s and t have the labels $+1$ and -1 respectively, and the arcs (u, s) and (u, t) are in G , then (u, s) receives the

label $+1$ and (u, t) receives the label -1 . If s has the label $l(s)$ and the arcs (u, s) and (u, t) are in G with $t \in \bar{V}_2$, then (u, s) receives the label $l(s)$ and (u, t) receives the label $-l(s)$. All other nodes and arcs receive the label 0 . This defines a new vector that satisfies with equality the same constraints that z satisfies with equality.

- If G' has two paths with an odd number of fractional nodes then again we can label the fractional nodes in these two paths and proceed as before.
- It remains the case where G' contains just one path with an odd number of fractional nodes. Let v_1, \dots, v_{2q+1} be the ordered sequence of nodes in this path. We should have $z(v_i) = \alpha$ if i is odd, and $z(v_i) = 1 - \alpha$ if i is even, with $0 < \alpha < 1$. This implies $\sum_{v \in V_2} z(v) = r + \alpha$ where r is an integer. We have then a contradiction.

6.2. \bar{G} does not contain H and contains an odd cycle C that includes all nodes in $V_2 \setminus \bar{V}_2$, and $|V_2 \setminus \bar{V}_2| \geq 5$. Here we use several transformations to obtain a new graph \tilde{G} that satisfies conditions (i) and (ii) of Theorem 1, and we use the fact that $P_p(\tilde{G})$ is an integral polytope.

Lemma 45. *Let $u, v \in V(C)$, then there is no arc $(u, v) \notin A(C)$.*

Proof. If such an arc exists, then the graph H would be present. \square

Lemma 46. *A node $u \in (V_1 \setminus \bar{V}_1)$ cannot be adjacent to more than one node in \bar{V}_2 .*

Proof. Suppose that the arcs (u, v_1) and (u, v_2) exist with v_1 and v_2 in \bar{V}_2 . We can add and subtract ϵ to $z(u, v_1)$ and $z(u, v_2)$ to obtain a new vector that satisfies with equality the same constraints that z does. \square

Lemma 47. *We can assume that $(V_1 \setminus \bar{V}_1) \setminus V(C) = \emptyset$*

Proof. Consider a node $u \in (V_1 \setminus \bar{V}_1) \setminus V(C)$ and suppose that the arcs (u, v_1) and (u, v_2) exist, with $v_1, v_2 \in V(C)$. If both paths in C between v_1 and v_2 contain another node in V_2 , then there is an odd cycle in \bar{G} and an extra node in $V_2 \setminus \bar{V}_2$. Then we can assume that there is a node $w \in V(C)$ and $(w, v_1), (w, v_2) \in A(C)$. If there is another node $v_3 \in V(C)$ such that the arc (u, v_3) exists, then the graph H is present, this is because $|V_2 \setminus \bar{V}_2| \geq 5$. Thus u cannot be adjacent to any other node in $V(C)$. Lemma 43 implies

$$(23) \quad z(u, v_1) = z(w, v_1),$$

$$(24) \quad z(u, v_2) = z(w, v_2).$$

Then we remove the node u and study the vector z' that is the restriction of z to $G \setminus u$. If there is another vector z'' that satisfies with equality the same constraints that z' does, we can extend z'' using equations (23) and (24), to obtain a vector that satisfies with equality the same constraints that z does.

If there is a node $u \in (V_1 \setminus \bar{V}_1) \setminus V(C)$ that is adjacent to exactly one node $v \in V(C)$, then u is adjacent also to a node $w \in \bar{V}_2$. It follows from Lemma 46 that the node in \bar{V}_2 is unique. Lemma 43 implies

$$(25) \quad z(u, v) = z(v),$$

and we also have

$$(26) \quad z(u, v) + z(u, w) = 1.$$

Then we remove the node u and study the vector z' that is the restriction of z to $G \setminus u$. If there is another vector z'' that satisfies with equality the same constraints that z' does, we can extend z'' using equations (25) and (26), to obtain a vector that satisfies with equality the same constraints that z does.

The resulting graph does not contain H and contains the odd cycle C . \square

Now consider a node $u \in \bar{V}_1$ that is adjacent to $v \in \bar{V}_2$. We should have $z(u, v) = 1$ and $z(v) = 1$. We remove u from the graph and keep v with $z(v) = 1$.

Finally we add slack variables to the inequalities (21) for each node in $V_2 \setminus \bar{V}_2$. For that we add a node v' and the arc (v, v') , for each node $v \in V_2 \setminus \bar{V}_2$. Then we add the constraints

$$\begin{aligned} z(v) + z(v, v') &= 1, \\ z(v, v') &\leq z(v'), \\ z(v') &= 1, \\ z(v, v') &\geq 0. \end{aligned}$$

Let \tilde{G} be this new graph, and $\tilde{p} = p + |V_2 \setminus \bar{V}_2|$. It follows from Lemmas 45, 46 and 47 that \tilde{G} is a graph satisfying conditions (i) and (ii) of Theorem 1. Here we have a face of $P_{\tilde{p}}(\tilde{G})$; because $z(v) = 0$ for all $v \in V_1$. Since $P_{\tilde{p}}(\tilde{G})$ is an integral polytope, there is an integral vector \tilde{z} that satisfies with equality the same constraints that z does. From $\tilde{z} \in P_{\tilde{p}}(\tilde{G})$ one can easily derive $\tilde{z}' \in P_p(G)$ that satisfies with equality the same constraints that $z \in P_p(G)$ satisfies with equality.

6.3. \bar{G} does not contain H and contains an odd cycle C that includes all nodes in $V_2 \setminus \bar{V}_2$, and $|V_2 \setminus \bar{V}_2| = 3$. Let $p' = p - |\bar{V}_2|$. If $p' = 3$, we should have $z(v) = 1$ for all $v \in V_2$. Then it is easy to see that we have an integral polytope. So we assume that $p' \leq 2$. Let $V_2 \setminus \bar{V}_2 = \{v_1, v_2, v_3\}$.

Consider first $p' = 2$. If z is fractional, then at most one variable $z(v_i)$ can take the value one, so assume that

$$\begin{aligned} z(v_1) &= 1, \\ 1 > z(v_2) &> 0, \\ 1 > z(v_3) &= 1 - z(v_2) > 0. \end{aligned}$$

We give the label $l(v_2) = +1$ to v_2 , the label $l(v_3) = -1$ to v_3 , and $l(v) = 0$ for every other node in V_2 . Then for each arc (u, v) with $z(u, v) = z(v)$, we give it the label $l(u, v) = l(v)$. If there is a node $u \in V_1$ that has only one arc (u, v) incident to it that is labeled, pick another arc (u, w) with $z(u, w) > 0$ and give it the label $l(u, w) = -l(u, v)$. For all the other arcs give the label 0. These labels define a new vector that satisfies with equation the same constraints that z does.

Now suppose that

$$\begin{aligned} 1 > z(v_1) &> 0, \\ 1 > z(v_2) &> 0, \\ 1 > z(v_3) &> 0. \end{aligned}$$

Then for every node $u \in V_1$ there is at most one arc (u, v) such that $z(u, v) = z(v)$. Otherwise there is a node $w \in V_2 \setminus \bar{V}_2$ with $z(w) = 1$. Let us define a new vector z' as follows. Start with $z' = 0$. Set $z'(v_1) = z'(v_2) = 1$, $z'(v_3) = 0$, and $z'(v) = 1$ for all $v \in \bar{V}_2$.

Then for each arc (u, v_1) with $z(u, v_1) = z(v_1)$ set $z'(u, v_1) = 1$. Also for each arc (u, v_2) with $z(u, v_2) = z(v_2)$ set $z'(u, v_2) = 1$. For each node u with $\sum_{(u,v) \in \delta^+(u)} z'(u, v) = 0$, pick an arc (u, v) with $v \neq v_3$ and set $z'(u, v) = 1$. This new vector satisfies with equality all the constraints that z does.

Finally suppose $p' = 1$ and

$$\begin{aligned} z(v_1) &> 0, \\ z(v_2) &> 0, \\ z(v_3) &> 0. \end{aligned}$$

We define a new vector z' as below. We set $z'(v_1) = 1$, $z'(v_2) = z'(v_3) = 0$, and $z'(v) = 1$ for $v \in \bar{V}_2$. For each node $u \in V_1$, if the arc (u, v_1) exists, we set $z'(u, v_1) = 1$; otherwise there is a node $v \in \bar{V}_2$ such that the arc (u, v) exists, we set $z'(u, v) = 1$. We set $z'(s, t) = 0$ for every other arc. Every constraint that is satisfied with equality by z is also satisfied with equality by z' .

7. RECOGNITION OF THIS CLASS OF GRAPHS

Now we discuss how to decide in polynomial time if a graph satisfies conditions (i) and (ii) of Theorem 1.

Clearly condition (i) can be tested in polynomial time. So the only question is how to find an odd Y -cycle if there is any. For that we first split every pendent node as in Lemma 8, and then we look for an odd cycle in the new graph. We split all pendent nodes to avoid obtaining a cycle that contains a pendent node. An algorithm for finding an odd cycle has been given in [2].

Conditions (i) and (ii) of Theorem 42 can be treated in a similar way.

REFERENCES

- [1] P. AVELLA AND A. SASSANO, *On the p -median polytope*, Math. Program., 89 (2001), pp. 395–411.
- [2] M. BAÏOU AND F. BARAHONA, *On the integrality of the uncapacitated facility location polytope*, Technical Report RC24101, IBM Watson Research Center, 2006.
- [3] ———, *On the p -median polytope of Y -free graphs*. To appear in Discrete Optimization, 2007.
- [4] L. CÁNOVAS, M. LANDETE, AND A. MARÍN, *On the facets of the simple plant location packing polytope*, Discrete Appl. Math., 124 (2002), pp. 27–53. Workshop on Discrete Optimization (Piscataway, NJ, 1999).
- [5] D. C. CHO, E. L. JOHNSON, M. PADBERG, AND M. R. RAO, *On the uncapacitated plant location problem. I. Valid inequalities and facets*, Math. Oper. Res., 8 (1983), pp. 579–589.
- [6] D. C. CHO, M. W. PADBERG, AND M. R. RAO, *On the uncapacitated plant location problem. II. Facets and lifting theorems*, Math. Oper. Res., 8 (1983), pp. 590–612.
- [7] G. CORNUEJOLS, M. L. FISHER, AND G. L. NEMHAUSER, *Location of bank accounts to optimize float: an analytic study of exact and approximate algorithms*, Management Sci., 23 (1976/77), pp. 789–810.
- [8] G. CORNUEJOLS AND J.-M. THIZY, *Some facets of the simple plant location polytope*, Math. Programming, 23 (1982), pp. 50–74.
- [9] I. R. DE FARIAS, JR., *A family of facets for the uncapacitated p -median polytope*, Oper. Res. Lett., 28 (2001), pp. 161–167.
- [10] M. GUIGNARD, *Fractional vertices, cuts and facets of the simple plant location problem*, Math. Programming Stud., (1980), pp. 150–162. Combinatorial optimization.
- [11] S. NESTOROV, S. ABITEBOUL, AND R. MOTWANI, *Extracting schema from semistructured data*, in SIGMOD '98: Proceedings of the 1998 ACM SIGMOD international conference on Management of data, New York, NY, USA, 1998, ACM Press, pp. 295–306.
- [12] F. TOUMANI, *Personal communication*. 2002.
- [13] A. VIGNERON, L. GAO, M. J. GOLIN, G. F. ITALIANO, AND B. LI, *An algorithm for finding a k -median in a directed tree*, Inform. Process. Lett., 74 (2000), pp. 81–88.

(M. Baïou) CNRS, LIMOS COMPLEXE SCIENTIFIQUE DES CZEAX, 63177 AUBIERE CEDEX, FRANCE.

E-mail address, M. Baïou: `Mourad.BAIOU@univ-bpclermont.fr`

(F. Barahona) IBM T. J. WATSON RESEARCH CENTER, YORKTOWN HEIGHTS, NY 10589, USA

E-mail address, F. Barahona: `barahon@us.ibm.com`