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Mixed Integer Rounding Cuts and Master Group Polyhedra

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Abstract

We survey recent research on mixed-integer rounding (MIR) inequalities and a generalization, namely the two-step MIR inequalities defined by Dash and Günlük (2006). We discuss the master cyclic group polyhedron of Gomory (1969) and discuss how other subadditive inequalities, similar to MIR inequalities, can be derived from this polyhedron. Recent numerical experiments have shed much light on the strength of MIR inequalities and the closely related Gomory mixed-integer cuts, especially for the MIP instances in the MIPLIB 3.0 library, and we discuss these experiments and their outcomes. Balas and Saxena (2007), and independently, Dash, Günlük and Lodi (2007), study the strength of the MIR closure of MIPLIB instances, and we explain their approach and results here. We also give a short proof of the well-known fact that the MIR closure of a polyhedral set is a polyhedron. Finally, we conclude with a survey of the complexity of cutting-plane proofs which use MIR inequalities.

This survey is based on a series of 5 lectures presented at the Seminaire de mathematiques superieures, of the NATO Advanced Studies Institute, held in the University of Montreal, from June 19-30, 2006.

1 Introduction

Over the last 10-15 years, cutting planes have emerged as a vital tool in mixed-integer programming. We call a linear inequality satisfied by all integer points in a polyhedron P a *cutting plane* (or *cut*) for P. Most commercial software which solve mixed-integer programs, such as ILOG-CPLEX [63] or XPRESS-MP [89], use sophisticated algorithms to find cutting planes and combine them with linear programming based branch-and-bound in a *branch-and-cut* system. There is a lot of literature on problem-specific cutting planes; in the context of the traveling salesman problem (TSP) for example, comb inequalities are very useful in solving TSP instances to optimality. In this survey, we will mainly discuss cutting planes for general (mixed) integer programs. That is, we will not assume any underlying combinatorial structure. In such cases, the *mixed-integer rounding* (MIR) inequalities (or MIR cut) and the closely related *Gomory mixed-integer* (GMI) cuts form the most important class of cutting planes.

The GMI cut was derived by Gomory in 1960 [54]. After some initial limited experimentation, these cuts were hardly used to solve mixed-integer programs for a long time. One exception is

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the paper of Geoffrion and Graves [51] in 1974 who described a successful combination of GMI cuts with LP based branch-and-bound to solve MIPs. In particular, they used a "hybrid branch-and-bound/cutting-plane approach" where "the cuts employed are the original mixed integer cuts proposed by Gomory in 1960, and are applied to each node problem in order to strengthen the LP bounds". They used the above ideas to solve a pure 0-1 integer program with several hundred binary variables. Some of the implementation ideas in this paper anticipate the (independent) work of Balas, Ceria, Cornuéjols and Natraj [14], who performed a systematic study of GMI cuts and popularized them as a tool for general mixed-integer programs in the 1990s. See [29] for additional historical information. Subsequent computational studies [18] confirmed the usefulness of the GMI cut for practical mixed-integer programs.

Nemhauser and Wolsey [76, p.244] introduced mixed-integer rounding inequalities, or cutting planes that can be produced by what they call the *MIR procedure*. These authors later [77] strengthened and redefined the MIR procedure and the resulting inequality; see [39] for a discussion on the development of MIR inequalities. They also showed that the MIR inequality, the GMI cut, and the *split cut* were equivalent. Split cuts were defined by Cook, Kannan and Schrijver [28], and are a special case of the *disjunctive cuts* introduced by Balas [12]. Marchand and Wolsey [72] later showed that many cuts in the literature are special cases of the MIR cut. They also computationally established the usefulness of the MIR cut; the MIR cuts in their experiments are different from the GMI cuts used in [14, 18]. The definition of the MIR cut we use in this paper is equivalent to the one in [77], though our presentation is based on [88]. By the late 1990s, the importance of the GMI cut and the MIR cut in solving mixed-integer programs was clear, and these cuts have become a standard feature of major commercial MIP solvers.

Gomory [55] introduced group relaxations of integer programs. Some computational studies of group relaxations can be found in White [87], Shapiro [84], and Gorry, Northup and Shapiro [60]. The latter paper contains a study of a group relaxation based branch-and-bound algorithm. After some initial work on this topic, by the mid-1970s, group relaxations were not viewed as a central technique to solve integer programs (see [52]). In his important work on polyhedral properties of group relaxations, Gomory [56] studied *corner polyhedra* (the convex hull of solutions of group relaxations) and the related cyclic group polyhedra. He highlighted the role of subadditive functions in obtaining cutting planes for corner polyhedra and for MIPs, and derived the GMI cut for pure integer programs from a facet of the master cyclic group polyhedron (MCGP). GMI cuts for mixed-integer programs can similarly be derived from the mixed-integer extension of the MCGP; see Gomory and Johnson [57]. The MCGP has a rich polyhedral structure and it is natural to ask if it has other facets that would lead to useful cutting planes for mixed-integer programs. We call such cuts *group cuts*. Following recent work on this topic by Gomory, Johnson and co-authors in [8] and [59], cyclic group polyhedra and group cuts are an active area of study; see [35]–[41].

An active research topic is the computational study of the strength of *closures* (or *elementary closures*) of different families of cutting planes. For a family of cutting planes $\mathcal{F}$, and a polyhedron P, the closure is the set of all points in P satisfying the cutting planes in $\mathcal{F}$. Chvátal [24] rediscovered Gomory cuts (we call them Gomory-Chvátal cuts (GC) in this paper) and initiated the study of the *Chvátal closure*, i.e., the closure with respect to Gomory-Chvátal cuts. A computational study of the Chvátal closure can be found in a recent paper by Fischetti and Lodi [47]. They framed the problem of separating a point from the Chvátal closure, or equivalently, of finding a violated GC cut, as an MIP (such an MIP can also be found in Bockmayr and Eisenbrand [19]). See [62] for related work on verifying that an inequality has Chvátal rank 1 or 2. Fischetti and Lodi used a standard MIP solver to find violated GC cuts and approximately optimized over the Chvátal closure of MIP instances in the MIPLIB 3.0 problem set. This procedure is computationally intensive, but yields strong bounds for many of the pure integer instances in MIPLIB, in contrast to using Gomory cuts derived from optimal simplex tableau rows, as attempted in earlier papers on this topic. Independently, Bonami and Minoux [21] approximately optimized over the closure with respect to lift-and-project cuts for 0-1 MIP instances from MIPLIB 3.0. Recently Bonami et. al. [20] optimized over the closure of *projected GC cuts* for the mixed-integer instances from MIPLIB using an MIP solver to find violated cuts. GC cuts and lift-and-project cuts are both special cases of split/MIR cuts. Motivated by the above work, Dash, Günlük and Lodi [39] and independently, Balas and Saxena [16] approximately optimize over the MIR closure of MIP instances, and show that a large fraction of the integrality gap can be closed in this way for MIPLIB 3.0 instances. However, it takes (in these papers) much more time to optimize over the closures above than to solve the original MIP (with few exceptions, as noted in [47], [16]). This is not surprising as it is NP-hard to separate an arbitrary point from the Chvátal closure [42], or from the split closure [23] (though one can solve an LP to find a violated lift-and-project cut).

Besides their usefulness in practice, cutting planes have some very interesting theoretical properties. Many statements with a combinatorial flavor, such as "the maximum size clique in a specific graph G has fewer than 10 nodes", can be proved by a *Gomory-Chvátal cutting-plane proof*, or a sequence of GC cuts, where each one is obtained from some original constraints modeling the combinatorial problem and the previous GC cuts. This is a consequence of Gomory's result [53] on the finite termination of his cutting plane algorithm for solving integer programs. Pudlák [78] showed that for 0-1 integer programs without integer solutions, GC cutting-plane proofs certifying their infeasibility have exponentially many cuts in the worst case. A similar result for MIR cutting plane proofs was recently proved in [33].

In this paper, we survey recent work on MIR cuts and cyclic group polyhedra. We start off with a simple extension of MIR cuts, namely the two-step MIR cuts in Dash and Günlük [36], a parametric family of group cuts. We then discuss fundamental properties of Gomory's cyclic group polyhedra, including Gomory's characterization of the convex hull of non-trivial facets of these polyhedra in Section 3. We further discuss some important facets of these polyhedra, and their relationship to subadditive functions, and to valid inequalities for mixed-integer programs. We move on (in Section 4) to a discussion of the MIR closure of a polyhedral set, and present a short proof that the MIR closure is a polyhedron, based on (but different from) a recent proof in Dash, Günlük, and Lodi [39]. This result follows from the result of Cook, Kannan and Schrijver [28] that the split closure of a polyhedral set is a polyhedron. We also present the MIP model from [39] to find violated MIR cuts. In Section 5, we discuss computational work on using cyclic group polyhedra to solve integer programs. We discuss recent studies on the computational effectiveness of *two-step MIR* cuts in Dash, Günlük and Goycoolea [35], and *interpolated group cuts* in Fischetti and Saturni [49]. We also discuss computational studies of the strength of the MIR closure. Finally, in Section 6, we discuss a recent result from [33] that shows that an MIR cutting plane proof certifying that an infeasible integer program has no integral solutions can have exponentially many cuts in the worst case.

2 The MIR inequality and extensions

Consider the set

$$P = \{v \in \mathbb{R}^l, x \in \mathbb{Z}^n : Cv + Ax = d, \quad v, x \geq 0, \ x \text{ integer}\}.$$

By a mixed-integer program, we mean the problem of minimizing a linear function $gv + hx$ subject to $(v, x) \in P$. We denote the continuous relaxation of P by P^{LP}, and the convex hull of P by $conv(P)$. We will study valid inequalities for P, or (equivalently) cutting planes for P^{LP}. We assume that all numerical data is rational. Let

$$Q \ = \ \Big\{v \in \mathbb{R}^{|J|}, x \in \mathbb{Z}^{|I|} \ : \ \sum_{j \in J} c_j v_j + \sum_{i \in I} a_i x_i \ = \ b, \quad v, \ x \geq 0\Big\},$$

where the equation defining Q is obtained as a linear combination of the equations defining P, and I and J are index sets of the integer/non-integer variables. In other words, $a = \lambda^T A$, $c = \lambda^T C$ and $b = \lambda^T d$ for some real vector λ of appropriate dimension. As $P \subseteq Q$, valid inequalities for Q yield valid inequalities for P. For a valid inequality derived in this way, we will refer to $\sum_{j \in J} c_j v_j + \sum_{i \in I} a_i x_i = b$ as its *base* equation. Finally, for a number v, let $\hat{v}$ stand for $v - \lfloor v \rfloor$, the fractional part of v.

2.1 The MIR inequality

There are many equivalent ways of defining the MIR inequality; see [39]. The presentation here is based on the presentation in the book [88] by Wolsey. Define

$$Q^1 = \big\{v \in \mathbb{R}, \ z \in \mathbb{Z} \ : \ v + z \ \geq \ b, \quad v \geq 0\big\}.$$

The *basic mixed-integer* inequality, defined in [88] as

$$v + \hat{b}z \geq \hat{b}\lceil b \rceil, \tag{1}$$

is valid and facet-defining for Q^1.

Lemma 1 $conv(Q^1) = \big\{v, z \in R \ : \ v + z \ \geq \ b, \quad v + \hat{b}z \geq \hat{b}\lceil b \rceil, \quad v \geq 0\big\}$.

Proof The result is trivial if $\hat{b} = 0$; so assume $\hat{b} \neq 0$. Let Q' be the set on the right-hand side of the equation in Lemma 1. Let $(v, z) \in Q^1$. If $z \geq \lceil b \rceil$, then $v \geq 0$ implies that $v + \hat{b}z \geq \hat{b}\lceil b \rceil$. If $z \leq \lfloor b \rfloor$, then $v + z \geq b$ implies that

$$v \ \geq \ \hat{b} + \lfloor b \rfloor - z \ \geq \ \hat{b} + \hat{b}(\lfloor b \rfloor - z) \ = \ \hat{b}(\lceil b \rceil - z).$$

Therefore, (1) is a valid inequality for Q^1 and $conv(Q^1) \subseteq Q'$. On the other hand, the extreme points of Q' are $(0, \lceil b \rceil)$ and $(\hat{b}, \lfloor b \rfloor)$, given by the intersections of the first and third inequalities with the second inequality. Both these points lie in Q^1, and thus $Q' \subseteq conv(Q^1)$. ∎

In Figure 1(a), we depict the points in Q^1 by horizontal lines. In Figure 1(b), the half-plane above the dashed line represents (1), and contains the shaded regions which are $Q^1 \cap \{z \leq \lfloor b \rfloor\}$ and $Q^1 \cap \{z \geq \lceil b \rceil\}$. If b is not integral (i.e., $\hat{b} \neq 0$), then the point $(\bar{v}, \bar{z}) = (0, b)$ violates the basic mixed-integer inequality.

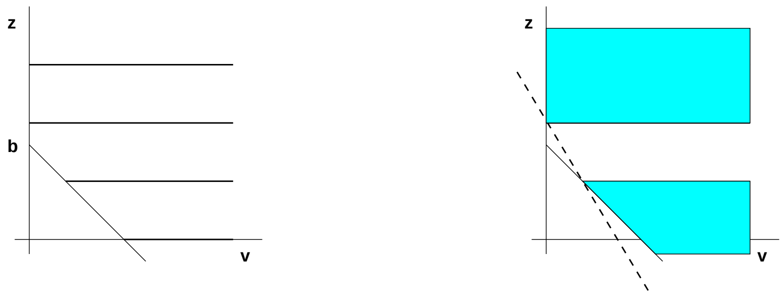


Figure 1: The basic mixed-integer inequality

An approach to deriving valid inequalities for more general sets defined by a single inequality is to combine variables to get a structure resembling Q^0. For example, consider Q, and a set $S \subseteq I$. We can relax the equation defining Q by rounding up the coefficients of x_i for $i \in I \setminus S$, and dropping continuous variables with negative coefficients to obtain

$$\sum_{c_j > 0} c_j v_j + \sum_{i \in S} a_i\, x_i + \sum_{i \in I \setminus S} \lceil a_i \rceil\; x_i \geq b, \tag{2}$$

as a valid inequality for Q. Writing $a_i = \lfloor a_i \rfloor + \hat{a}_i$ for $i \in S$, and re-arranging terms, we get

$$\Big(\sum_{c_j > 0} c_j v_j + \sum_{i \in S} \hat{a}_i\, x_i \Big) + \Big(\sum_{i \in S} \lfloor a_i \rfloor\; x_i + \sum_{i \in I \setminus S} \lceil a_i \rceil\; x_i \Big) \geq b. \tag{3}$$

The first part of this inequality is non-negative, and the second part is integral for all $(v, x) \in Q$. Therefore, the basic mixed-integer inequality implies that

$$\Big(\sum_{c_j > 0} c_j v_j + \sum_{i \in S} \hat{a}_i\, x_i \Big) + \hat{b} \Big(\sum_{i \in S} \lfloor a_i \rfloor\; x_i + \sum_{i \in I \setminus S} \lceil a_i \rceil\; x_i \Big) \geq \hat{b} \lceil b \rceil. \tag{4}$$

is valid for Q. The coefficients of x_i in this inequality are $\hat{b}\lfloor a_i \rfloor + \hat{a}$ if $i \in S$, and $\hat{b}\lceil a_i \rceil$ if $i \in I \setminus S$. Therefore, $S = \{i \in I : \hat{a}_i \leq \hat{b}\}$ gives the strongest inequality of this form, which is

$$\sum_{c_j > 0} c_j v_j + \sum_{i \in I : \hat{a}_i \leq \hat{b}} (\hat{b} \lfloor a_i \rfloor + \hat{a}_i) x_i + \sum_{i \in I : \hat{a}_i > \hat{b}} \hat{b} \lceil a_i \rceil x_i \geq \hat{b} \lceil b \rceil. \tag{5}$$

We call (5) the *mixed-integer rounding* inequality for Q. Note that the coefficients of x_i can be written as $\hat{b}\lfloor a_i \rfloor + \min\{\hat{a}_i, \hat{b}\}$. Finally, as observed by Cornuéjols, Li, and Vandenbussche [30], scaling the the equation defining Q by a rational number t before writing the MIR inequality, can be useful in some circumstances. We call such an inequality a *t-scaled MIR* inequality.

Assume $\hat{b} \neq 0$. Subtracting $\hat{b}$ times $\sum_{j \in J} c_j v_j + \sum_{i \in I} a_i\, x_i = b$ from the MIR inequality, and dividing by $(1 - \hat{b})$, we get the equivalent inequality:

$$\sum_{j \in J : c_j > 0} c_j v_j - \sum_{j \in J : c_j < 0} \frac{\hat{b}\, c_j}{1 - \hat{b}}\, v_j + \sum_{i \in I : \hat{a}_i \leq \hat{b}} \hat{a}_i x_i + \sum_{i \in I : \hat{a}_i > \hat{b}} \frac{\hat{b}(1 - \hat{a}_i)}{1 - \hat{b}}\, x_i \geq \hat{b}, \tag{6}$$

This is the *Gomory mixed-integer cut*, as originally defined in [54].

To define the MIR inequality for P, we start off with some multiplier vector λ and define $c = \lambda^T C, a = \lambda^T A$, and $b = \lambda^T d$. Then $cv + ax = b$ is a valid inequality for P, and we call (5) an MIR inequality for P. Define the *MIR rank* of an inequality valid for P^{LP} to be 0. For an inequality valid for P, define the MIR rank to be a positive number t if it does not have MIR rank $t-1$ or less, but is a non-negative linear combination of MIR inequalities derived from inequalities with MIR rank $t-1$. All cutting planes for a pure integer program have finite MIR rank, but this is not true in the case of mixed-integer programs [28].

A linear inequality $gv + hx \geq q$ is called a *split cut* for P if it is valid for both $P^{LP} \cap \{\bar{\alpha}x \leq \bar{\beta}\}$ and $P^{LP} \cap \{\bar{\alpha}x \geq \bar{\beta} + 1\}$, where $\bar{\alpha}$ and $\bar{\beta}$ are integral. The inequality $gv + hx \geq h$ is said to be derived from the *disjunction* $\bar{\alpha}x \leq \bar{\beta} \vee \bar{\alpha}x \geq \bar{\beta} + 1$. All points in P satisfy any split cut for P. The basic mixed-integer inequality is a split cut for Q^1 derived from the disjunction $z \leq \lfloor b \rfloor$ and $z \geq \lceil b \rceil$, when $\hat{b} \neq 0$. Therefore, the MIR inequality (5) is a split cut for P. Nemhauser and Wolsey [77] showed that every split cut for P is also an MIR inequality.

The derivation of the MIR inequality can be viewed as a special case of the following approach. Consider a linear transformation

$$\mathcal{T} : Q \rightarrow Q' = \{v, z \in \mathbb{R} : \begin{pmatrix} v' \\ z' \end{pmatrix} = U \begin{pmatrix} v \\ x \end{pmatrix} + l\} \subseteq Q^1,$$

where U is a matrix, l is a vector. If $\alpha^T \binom{v'}{z'} \geq \beta$ is valid for Q', then $\alpha^T U \binom{v}{x} + \alpha^T l \geq \beta$ is valid for Q. In the case of the MIR inequality, $l = 0$, and the coefficients in the first and second rows of U are defined, respectively, by the coefficients in the first and second bracketed expressions in (3). This approach is used in *local cuts* for the TSP by Applegate et. al. [7] with varying linear transformations into varying sets, and not a fixed set such as Q^1. The transformations in [7] are defined by the operation of shrinking sets of nodes to single nodes, and thus have a combinatorial nature, unlike the somewhat abstract transformations in the case of the MIR inequality.

Subsequent to the work of Wolsey [88], and Marchand and Wolsey [72], deriving (or explaining) valid inequalities for mixed-integer programs in the above manner, starting from valid inequalities for very simple polyhedral sets became an active research topic. For example, let $\sum_{i \in I} \alpha_i x_i \geq \beta$ be a valid inequality for P. Then $\sum_{i \in I} \lceil \alpha_i \rceil x_i \geq \lceil \beta \rceil$ is valid for P, and is called a *Gomory-Chvátal* cut if P has no continuous variables, and a *projected Chvátal-Gomory* cut [20] otherwise. The above inequality can be derived by mapping P into the set $Q^0 = \{z \in \mathbb{Z} : z \geq \beta\}$ via the mapping $z = \sum_{i \in I} \lceil \alpha_i \rceil x_i$, and then using the only facet-defining inequality of Q^0, namely $z \geq \lceil \beta \rceil$. Günlük and Pochet [61] define the *mixing-mir* inequalities from facets of the set $\{(x, z) \in \mathbb{R} \times \mathbb{Z}^n : x + z_i \geq b_i, \text{ for } i = 1, \ldots, n, \; x \geq 0\}$. The underlying motivation behind such research in the context of mixed-integer programming and in local cuts for the TSP is that strong (facet-defining) inequalities for simple sets defined on few variables yield useful inequalities for problems with many variables. The hard part is to find the appropriate small sets and linear transformations. In recent work, Espinoza [44] generated linear transformations dynamically, in a manner similar to the work in [7], to find useful cutting planes for Q, though with moderate success.

2.2 Two-step MIR inequalities

In the relaxation of Q in (2), the coefficient a_i of an integer variable x_i is either unchanged or rounded up to $\lceil a_i \rceil$. One can get different cuts for Q by increasing a_i to a number less than $\lceil a_i \rceil$, say $\lfloor a_i \rfloor + \alpha$ where $0 < \alpha < 1$, thereby obtaining a stronger relaxation than (2).

Dash and Günlük [36] obtained such cuts from a simple mixed-integer set with three variables:

$$Q^2 = \{v \in \mathbb{R},\ y, z \in \mathbb{Z} \,:\, v + \alpha y + z \ \geq\ \beta,\ \ v, y \geq 0\},$$

where $\alpha, \beta \in R$ are parameters that satisfy $1 > \beta > \alpha > 0$, and $\lceil \beta/\alpha \rceil > \beta/\alpha$. Though β is required to be less than 1 in Q^2, the fact that z can take on negative values makes the set fairly general. We do not know of an explicit description of the convex hull of points in Q^2; however one can obtain the inequalities describing the convex hull in polynomial time using results in [11] and [3]. Dash and Günlük showed that the following inequalities are valid for Q^2, and facet defining under some conditions:

$$v + \alpha y + \beta z \ \geq\ \beta, \tag{7}$$

$$\big(1/(\beta - \alpha \lfloor \beta/\alpha \rfloor)\big) v + y + \lceil \beta/\alpha \rceil z \ \geq\ \lceil \beta/\alpha \rceil . \tag{8}$$

Lemma 2 *[36] The inequality (7) is valid and facet-defining for* Q^2*. If* $1/\alpha \geq \lceil \beta/\alpha \rceil$*, then the inequality (8) is valid and facet-defining for* Q^2*.*

Proof The inequality (7) can be obtained by treating $v + \alpha y$ as a continuous variable and applying the basic mixed-integer inequality (1) to $(v + \alpha y) + z \ \geq\ \beta$. To see that inequality (8) is valid, notice that the inequalities

$$\begin{aligned} (1/\alpha)v \ + \ y \ + \ \ (\beta/\alpha)z \ &\geq \beta/\alpha, \\ (1/\alpha)v \ + \ y \ + \ \ (1/\alpha)z \ &\geq \beta/\alpha, \end{aligned}$$

are valid for Q^2. Therefore, for any $\gamma \in R$ satisfying $1/\alpha \geq \gamma \geq \beta/\alpha$, the inequality

$$(1/\alpha)v \ + \ y \ + \ \ \gamma z \ \ \geq \beta/\alpha$$

is also valid as it can be obtained as a convex combination of valid inequalities. If $1/\alpha \geq \lceil \beta/\alpha \rceil$, then

$$v/\alpha + y + \lceil \beta/\alpha \rceil z \ \ \geq\ \ \beta/\alpha \tag{9}$$

is valid for Q^2. Applying (1) to the previous inequality with v/α treated as a continuous variable and $y + \lceil \beta/\alpha \rceil z$ treated as an integer variable gives inequality (8). Consider the following points in Q^2:

$$p_1 = (0, 0, 1),\ \ p_2 = (0, \lceil \beta/\alpha \rceil, 0),\ \ p_3 = (\beta - \alpha \lfloor \beta/\alpha \rfloor, \lfloor \beta/\alpha \rfloor, 0),\ \ p_4 = (\beta, 0, 0).$$

As depicted in Figure 2, the points p_1, p_3 and p_4 are affinely independent and tight for (7). Also, the affinely independent points $p1, p2$ and $p3$ are tight for (8), and thus these two inequalities are facet-defining for Q^2. ∎

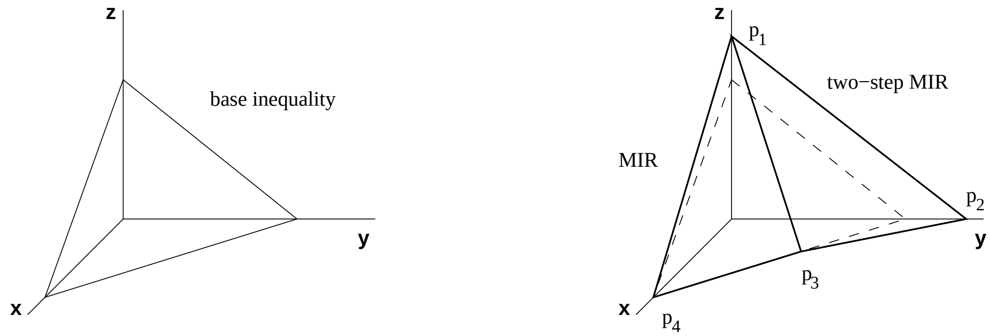


Figure 2: Some facets of Q^2

We call (8) the *two-step MIR* inequality for Q^2; it is obtained by applying (1) twice, and has MIR rank at most two. The inequalities (7) and (8) are not necessarily sufficient to describe the convex hull of Q^2. However, let $Q^{2+} = Q^2 \cap \{z \geq 0\}$; (7) and (8), along with the inequalities $v, x, z \geq 0$, define the convex hull of integer solutions of Q^{2+} [36]. The additional restriction $1/\alpha \geq \lceil \beta/\alpha \rceil$ is not required. Conforti and Wolsey [25] characterized the convex hull of a generalization of Q^{2+} (and of Q^{2+}) which they call X^{2DIV}. They assume all variables are non-negative and thus their results do not imply Lemma 2. We note that if $1/\alpha \leq \lceil \beta/\alpha \rceil$ ($1/\alpha = \lceil \beta/\alpha \rceil$), inequality (8) is an $(1/\alpha)$-scaled MIR inequality for Q^{2+} (Q^2). Thus, it is only when $1/\alpha > \lceil \beta/\alpha \rceil$ that the two-step MIR inequality can have MIR rank 2.

We illustrate the use of inequality (8) in an example. Consider the equation $3.35x_1 + 2.5x_2 - 1.2x_3 = 4.7$ with all variables non-negative and integral. This equals $.35x_1 + .5x_2 + .8x_3 + w = .7$, where $w = 3x_1 + 2x_2 - 2x_3 - 4$. Here $\beta = 0.7$; let $\alpha = 0.4$. Then $1/\alpha = 2.5 > \lceil \beta/\alpha \rceil = 2$. As $x_1, x_3 \geq 0$, the inequality $(.1x_2) + .4(x_1 + x_2) + (x_3 + w) \geq .7$ is a relaxation of the previous equation. Of the three terms in brackets, the first two are non-negative, and the last two are integral; we can thus apply inequality (8) to obtain the valid inequality: $(1/2)x_1 + (2/3)x_2 + x_3 + w \geq 1$.

We now formalize this procedure to obtain valid inequalities for Q. Let $\beta = \hat{b}$, and choose $\alpha \in (0, \hat{b})$ such that $1/\alpha \geq \lceil \hat{b}/\alpha \rceil$. Define $\tau = \lceil \hat{b}/\alpha \rceil \geq 2$, and define $\rho = \hat{b} - \alpha \lfloor \hat{b}/\alpha \rfloor$. Let k_i, l_i be integers such that $k_i \leq \lfloor \hat{a}_i/\alpha \rfloor$, and $l_i \geq \lceil \hat{a}_i/\alpha \rceil$, for $i \in I$. Let I_0, I_1 and I_2 be sets which form a partition of I, and let $w = \sum_{i \in I} \lfloor a_i \rfloor x_i - \lfloor b \rfloor$. We can relax the equation in Q to obtain

$$\sum_{c_j > 0} c_j v_j + \sum_{i \in I_1} (k_i \alpha + (\hat{a}_i - k_i \alpha))\, x_i + \sum_{i \in I_2} l_i \alpha\, x_i + \sum_{i \in I_0} x_i + w \quad \geq \quad \hat{b},$$

which can be rewritten as

$$\sum_{c_j > 0} c_j v_j + \sum_{i \in I_1} (\hat{a}_i - k_i \alpha)\, x_i + \alpha \Big(\sum_{i \in I_1} k_i\, x_i + \sum_{i \in I_2} l_i\, x_i \Big) + \Big(\sum_{i \in I_0} x_i + w \Big) \quad \geq \quad \hat{b},$$

We can map points in Q to points in Q^2 by equating the first variable in Q^2 with the sum of the first two terms in the equation above, and equating the second and third variables with the first and second bracketed expressions, respectively. Applying inequality (8) and substituting for w leads to inequality

$$\sum_{c_j > 0} c_j v_j + \sum_{i \in I} \gamma_i x_i + \rho\, \tau\, z \geq \rho\, \tau\, \lceil b \rceil, \tag{10}$$

$$\text{where } \gamma_i = \rho\,\tau\ \lfloor a_i \rfloor + \begin{cases} \rho\,\tau & \text{if } i \in I_0, \\ k_i\rho\ + \hat{a}_i - k_i\alpha & \text{if } i \in I_1, \\ l_i\rho & \text{if } i \in I_2. \end{cases}$$

By inspection, the strongest inequality of this form is obtained by setting $k_i = k_i^* = \lfloor \hat{a}_i/\alpha \rfloor$ and $l_i = l_i^* = \lceil \hat{a}_i/\alpha \rceil$ for $i \in I$, and letting

$$I_0 = \{i \in I\ :\ \hat{a}_i \geq \hat{b}\},$$

$$I_1 = \{i \in I \setminus I_0\ :\ \hat{a}_i - k_i^*\alpha < \rho\}, \qquad I_2 = \{i \in I \setminus I_0\ :\ \hat{a}_i - k_i^*\alpha \geq \rho\}.$$

In other words,

$$\gamma_i = \rho\,\tau\ \lfloor a_i \rfloor + \min\{\rho\,\tau\ ,\ k_i^*\rho\ + \hat{a}_i - k_i^*\alpha\ ,\ l_i^*\rho\ \}.$$

As shown in [36], these inequalities imply the *strong fractional cuts* of Letchford and Lodi [70]. They can be separated in polynomial time, under some restrictions.

Theorem 3 *(Dash, Goycoolea, Günlük [35]) Given a point $(v^*, x^*) \in Q^{LP}$, the most violated two-step MIR inequality can be found in polynomial time, assuming $\tau \leq k$ for some fixed k.*

Here, the violation of (10) with respect to (v^*, x^*) is given by its right-hand side minus the left-hand side evaluated at (v^*, x^*). The algorithm given in [35] is very simple; just write down the inequality (10) for all feasible choices of α from the set $\{\hat{a}_i/t : i \in I, t \in \mathbb{N}, \hat{a}_i/t \geq \hat{b}/k\} \cup \{1/t : t \in \mathbb{N}, t\hat{b} \leq k\}$, and compute the violation.

Kianfar and Fathi [64] generalize the two-step MIR inequalities and generate *n-step MIR* inequalities for Q. These inequalities have MIR rank at most n.

3 Master polyhedra

Gomory [56] developed the concepts of corner polyhedra and master polyhedra as tools to generate cutting planes for general integer programs, and to solve asymptotic integer programs. We do not discuss the latter aspect here. The *corner polyhedron* (*strengthened* corner polyhedron) for a vertex of P^{LP} is the convex hull of integer points satisfying only the n linearly independent constraints which define the vertex (all constraints tight at the vertex). Clearly, valid inequalities for a corner polyhedron of P yield valid inequalities for P. The central element of Gomory's approach is the fact that many different corner polyhedra can be viewed as faces of a much smaller number of *master polyhedra*, which are much more 'regular' and amenable to analysis. Gomory, and later Gomory and Johnson [57] showed how to obtain facets of master polyhedra, which yield valid inequalities for corner polyhedra and thereby for P. In this section, we describe in detail master cyclic group polyhedra, or master polyhedra associated with corner polyhedra for single constraint (plus non-negativity of variables) systems, such as the one defining Q. Many results are available for such polyhedra, but master polyhedra for multiple constraint systems are less well-understood, beyond the initial results of Gomory. We briefly discuss some recent research on this topic later.

For the set Q, assume that $\hat{a}_i$ ($i \in I$) and $\hat{b}$ are rational numbers with a common denominator n, and let $\hat{b} = r/n$, where $0 < r < n$. Rewrite Q as

$$Q \;=\; \Big\{ v \in \mathbb{R}^{|J|}, x \in \mathbb{Z}^{|I|} \;:\; (\sum_{i \in I} \lfloor a_i \rfloor x_i - \lfloor b \rfloor) + \sum_{j \in J} c_j v_j + \sum_{i \in I} \hat{a}_i\, x_i = \hat{b}, \quad v, x \geq 0 \Big\}$$

Let $I_k = \{ i \in I \,:\, \hat{a}_i = k/n \}$ and define the mapping

$$\begin{aligned} w_k &= \textstyle\sum_{i \in I_k} x_i, \qquad z = -(\sum_{i \in I} \lfloor a_i \rfloor x_i - \lfloor b \rfloor) \\ v_+ &= \textstyle\sum_{c_j > 0} c_j v_j, \quad v_- = -\sum_{c_j < 0} c_j v_j, \end{aligned} \tag{11}$$

that maps each point (v, x) in Q to a point (v_+, v_-, w) in the polyhedron

$$P'(n, r) = conv\{ v_+, v_- \in \mathbb{R},\; w \in \mathbb{Z}^{n-1} \;:\; v_+ - v_- + \sum_{i=1}^{n-1} \frac{i}{n}\, w_i - z = \frac{r}{n}, \;\; v_+, v_-, w \geq 0, z \in \mathbb{Z} \}.$$

(If I_k is empty for some k, set w_k to zero.) If Q has no continuous variables, then (11) maps points in Q to points in the *master cyclic group polyhedron* of Gomory:

$$P(n, r) = conv\{\; w \in \mathbb{Z}^{n-1} \;:\; \sum_{i=1}^{n-1} \frac{i}{n}\, w_i - z = \frac{r}{n}, \;\; w \geq 0, z \in \mathbb{Z} \}. \tag{12}$$

We view $P'(n, r)$ as the mixed-integer extension of $P(n, r)$. For $P(n, r)$, z can be assumed to be non-negative, but not for $P'(n, r)$. Note that the constraint defining $P(n, r)$ can be written as $\sum_{i=1}^{n-1} i w_i \equiv r \,(\mathrm{mod}\, n)$. In [54], Gomory first derived the GMI cut as a valid inequality for points satisfying a similar equation.

Recently, Dash, Fukasawa and Günlük [34] studied the polyhedron

$$K(n, r) = conv\{\; (x, y) \in \mathbb{Z}^n \times \mathbb{Z}^n : \sum_{i=1}^{n} i x_i - \sum_{i=1}^{n} i y_i = r, \; x, \, y \geq 0 \} \tag{13}$$

where $n, r \in \mathbb{Z}$ and $0 < r \leq n$, and characterized the convex hull of its non-trivial facets. $K(n, r)$ was first defined by Uchoa [85] in a slightly different form. Replacing z in (12) by y_n, and multiplying the defining constraint by n, we see that $P(n, r)$ defines a face of $K(n, r)$. Facets of $P(n, r)$ can be lifted to obtain facets of $K(n, r)$, but not all facets can be obtained this way [34].

3.1 Basic properties

Both $P(n, r)$ and $P'(n, r)$ are full-dimensional, unbounded polyhedra, and their characteristic (recession) cones are $\mathbb{R}^{n-1}_+$ and $\mathbb{R}^{n+1}_+$ respectively. Also, the inequalities $x_i \geq 0$ for $i = 1, \ldots, n-1$ define facets for both $P(n, r)$ and $P'(n, r)$, and the inequalities $v_- \geq 0$ and $v_+ \geq 0$ are facet-defining for $P'(n, r)$. Therefore, if $\eta^T w \geq \eta_o$ is a facet defining inequality of $P(n, r)$, then $\eta, \eta_o \geq 0$. Further, for any *nontrivial* facet (i.e., not defined by the non-negativity inequalities), $\eta_o > 0$. We will assume in what follows that η_o is scaled to be 1. Finally, any nontrivial facet of $P(n, r)$ has the following property: there is a set of $n - 1$ linearly independent integer points χ^i in $P(n, r)$ satisfying $\chi^i_i \geq 1$, for $i = 1, \ldots, n - 1$. Gomory characterized the convex hull of nontrivial facets of $P(n, r)$.

Theorem 4 *[56] If $r \neq 0$, then $\sum_{i=1}^{n-1} \eta_i w_i \geq 1$ is a non-trivial facet of $P(n,r)$ if and only if $\eta = (\eta_i)$ is an extreme point of the inequalities*

$$\begin{aligned}
\eta_i + \eta_j &\geq \eta_{(i+j) \bmod n} \quad \forall i,j \in \{1,\ldots,n-1\}, && (14)\\
\eta_i + \eta_j &= \eta_r \quad \forall i,j \text{ such that } r = (i+j) \bmod n, && (15)\\
\eta_j &\geq 0 \quad \forall j \in \{1,\ldots,n-1\}, && (16)\\
\eta_r &= 1. && (17)
\end{aligned}$$

The property (14) is called *subadditivity.*

Gomory actually proved a more general result. Consider an integer $k > 0$, and let $r, n \in \mathbb{Z}_+^k$, with each component of n greater than 1, and $0 < r_l < n_l$, for $l = 1,\ldots,k$. Let $G \subseteq \mathbb{Z}_+^k$ consist of all vectors with the lth component contained in $\{0,\ldots,n_l - 1\}$, and let $G_+ = G \setminus \mathbf{0}$. In the defining constraint for $P(n,r)$, replace $\sum_{i=1}^{n-1} iw_i \equiv r \pmod n$ by $\sum_{i \in G_+} iw_i \equiv r \pmod n$, where a vector equals another modulo n if and only if the component-wise modular equations hold. Then Theorem 4 is true if r, n are defined as above, the indices i, j are elements of G_+, and we replace $\{1,\ldots,n-1\}$ by G_+. The elements of G, along with the operation of addition modulo n, form the abelian group $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$, which is cyclic when $k = 1$. When $k > 1$, some valid inequalities (e.g., scaled MIR inequalities) when applied to the individual constraints of $P(n,r)$ define facets of $P(n,r)$. For $k = 2$ (equivalently, $G = \mathbb{Z}_{n1} \times \mathbb{Z}_{n2}$), in limited shooting experiments (see Section 3.3) we observe that taking integral linear combinations of the first and second constraints (with small multipliers, e.g., 1,1) and writing a scaled MIR (or two-step MIR) inequality on the resulting constraint also yields facets. Dey and Richard studied facet-defining inequalities in the case $k >= 2$ (though for an "infinite group" [58] variant of $P(n,r)$); see [40], [41]. When $k >= 2$, the relaxation of $P(n,r)$ obtained by removing the integrality of w is studied in [6] and [31] (in a different form).

Optimizing a linear function over $P(n,r)$ is similar to solving a knapsack problem and can be done in pseudo-polynomial time (in n) via dynamic programming; therefore, one can also separate a point from $P(n,r)$ in pseudo-polynomial time. The theorem above implies that separation can in fact be done by obtaining a basic optimal solution of the constraints in the theorem. It also yields a way of showing that a particular inequality defines a facet of $P(n,r)$.

Gomory and Johnson [57] showed that the polyhedral structure of $P'(n,r)$ can be analyzed completely by just studying $P(n,r)$.

Theorem 5 *[57] An inequality defines a nontrivial facet of $P'(n,r)$ if and only if it has the form*

$$n\eta_1 v_+ + n\eta_{n-1} v_- + \sum_{i=1}^{n-1} \eta_i w_i \geq 1, \tag{18}$$

where $(\eta_1,\ldots,\eta_{n-1})$ defines a facet of $P(n,r)$.

Gomory and Johnson also described the convex hull of non-trivial facets of $P'(n,r)$, as in Theorem 4, when r is non-integral, but for our purpose Theorem 5 suffices. Valid inequalities for

$P'(n,r)$ yield valid inequalities for Q. Given a facet (18) of $P'(n,r)$,

$$n\eta_1(\sum_{c_j\geq 0} c_j v_j) + n\eta_{n-1}(\sum_{c_j<0} c_j v_j) + \sum_{i\in I} f(\hat{a}_i)x_i \geq 1, \tag{19}$$

is a valid inequality for Q, where $f(\hat{a}_i) = \eta_k$ if $\hat{a}_i = k/n$. We call such inequalities *group cuts* for Q. We will see that the GMI cut can be derived as a group cut. Let $(v', x') \in Q^{LP} \setminus conv(Q)$, and let (v', x') be mapped to (v'_+, v'_-, w') via (11). Then (v', x') satisfies all group cuts (and $(v'_+, v'_-, w') \in P'(n,r)$) if and only if the *separation LP*

$$\min\Big\{(nv'_+)\eta_1 + (nv'_-)\eta_{n-1} + \sum_i w'_i\eta_i \; : \; \eta \text{ satisfies } (14) - (17)\Big\}.$$

has optimum value at least 1. If some group cut is violated by (v', x'), then the optimum solution of the separation LP yields a most violated group cut with objective value less than 1.

For $P(n,r)$, let e_i ($i = 1,\ldots,n-1$) be the unit vectors in $\mathbb{R}^{n-1}$ with ones in the ith component and zeros elsewhere.

Lemma 6 *If $\eta \in \mathbb{R}^{n-1}$ satisfies the constraints (15) - (17) and $\eta^T x \geq 1$ is a valid inequality for $P(n,r)$, then η satisfies the subadditivity constraints (14).*

Proof Let $j, k \in \{1,,\ldots,n-1\}$, and let $i = j + k \bmod n$. Let $\chi = e_j + e_k + e_{r-i}$. Then $\chi \in P(n,r)$, and $\eta^T\chi = \eta_j + \eta_k + \eta_{r-i} \geq 1$. As $\eta_i + \eta_{r-i} = 1$, it follows that $\eta_j + \eta_k \geq \eta_i$. ∎

Proof of Theorem 4 Let $\gamma^T w \geq 1$ define a non-trivial facet of $P(n,r)$. We argued earlier that $\gamma \geq 0$. We next show that γ satisfies (15) and (17). Observe that e_r, $e_i + e_{r-i} \in P(n,r)$ and thus $\gamma_r \geq 1$, and $\gamma_i + \gamma_{r-i} \geq 1$. Further, there are integral points χ^1, χ^2 and χ' in $P(n,r)$ lying on this facet such that $\chi^1_i \geq 1$ and $\chi^2_{r-i} \geq 1$ and $\chi'_r \geq 1$. Then $\gamma^T(\chi^1 + \chi^2) = 2$. But

$$\chi = \chi^1 + \chi^2 - e_i - e_{r-i} \in P(n,r) \Rightarrow \gamma^T\chi = 2 - \gamma_i - \gamma_{r-i} \geq 1.$$

Therefore, $\gamma_i + \gamma_{r-i} \leq 1$. Similarly $\gamma^T\chi' = 1$, and

$$\bar{\chi} = 2\chi' - e_r \in P(n,r) \Rightarrow \gamma^T\bar{\chi} = 2 - \gamma_r \geq 1 \Rightarrow \gamma_r \leq 1.$$

Therefore γ satisfies (15) - (17). By Lemma 6, it also satisfies the subadditivity constraints.

Let η satisfy (14) - (17). For any integral point $w^* \in P(n,r)$

$$\sum_i \eta_i w^*_i \geq \sum_i \eta_{(iw^*_i)} \geq \eta_{(\sum_i iw^*_i)} = \eta_r = 1,$$

where the subscripts inside the brackets are computed modulo n. Therefore, $\eta^T w \geq 1$ defines a valid inequality for $P(n,r)$. Therefore, a nontrivial facet of $P(n,r)$ is an extreme point of (14) - (17); otherwise it would be a convex combination of solutions of this system, each of which defines a valid inequality for $P(n,r)$. Let $P(n,r) = \{w \in \mathbb{R}^{n-1} : Aw \geq \mathbf{1},\ w \geq 0\}$, where $Aw \geq \mathbf{1}$ represents the non-trivial facets of $P(n,r)$. Therefore, if A_j stands for the jth column of A, then for any index $i \neq r$, $A_i + A_{r-i} = A_r = \mathbf{1}$. Observe that $e_r \in P(n,r)$ and $\eta^T e_r = \eta_r = 1$. Therefore

$$\min\{\eta^T w : Aw \geq \mathbf{1},\ w \geq 0\} = 1,$$

and an optimal dual vector y of the above LP satisfies

$$y^T A \leq \eta, \quad y^T \mathbf{1} = 1, \quad y \geq 0.$$

If $y^T A_i < \eta_i$ for any index $i \neq r$, then

$$1 = \eta_i + \eta_{r-i} > y^T A_i + y^T A_{r-i} = y^T \mathbf{1} = 1.$$

Therefore $y^T A_i = \eta_i$ for $i = 1, \ldots, n-1$. This implies that η is a convex combination of non-trivial facets $\{\eta^j\}$ of $P(n,r)$, which in turn are solutions of (14) - (17). Thus, if η is an extreme point (14) - (17), $\eta^T x \geq 1$ defines a facet of $P(n,r)$, and the vector y is a unit vector. ∎

As the defining equation for $P(n,r)$ has the same form as Q, we can apply the MIR or two-step MIR inequalities to $P(n,r)$. We define the t-scaled (two-step) MIR inequality for $P(n,r)$ for rational $t > 0$ as the inequality obtained by applying the t-scaled (two-step) MIR inequality to $Q_p = \{w_i, z \in \mathbb{Z} : \sum_{i=1}^{n-1} \frac{i}{n} w_i - z = \frac{r}{n}, w \geq 0\}$ and then substituting out z. We focus on t-scaled inequalities for integral t. For an integer $n > 0$ and integers t and i, define $(ti)_n = ti \bmod n$, where $k \bmod n$ stands for $k - n\lfloor k/n \rfloor$. For an integer t, the t-scaled MIR inequality for $P(n,r)$ becomes

$$\sum_{(ti)_n < (tr)_n} \frac{(ti)_n}{(tr)_n} x_i + \sum_{(ti)_n \geq (tr)_n} \frac{n - (ti)_n}{n - (tr)_n} x_i \geq 1. \tag{20}$$

Given an integer $t \neq n$, it is shown in [37] that the $(-t)$-scaled MIR inequality is the same as the t-scaled MIR inequality and also the $(n-t)$-scaled MIR inequality.

Definition 7 *An inequality $\sum_{i=1}^{n-1} \eta_i w_i \geq 1$ is a* two-slope *inequality if $\eta_i - \eta_{i-1}$ equals either η_1 or $-\eta_{n-1}$ for $i = 2, \ldots, n-1$.*

Theorem 8 *(Gomory, Johnson [57]) Every two-slope inequality for $P(n,r)$ which satisfies the constraints (14) - (17) defines a facet for $P(n,r)$.*

Proof By definition, for $i = 2, \ldots, n-1$ either $\eta_1 + \eta_{i-1} = \eta_i$ or $\eta_{n-1} + \eta_i = \eta_{i-1}$. The constraints above are just the subadditivity constraints (14) (satisfied as equalities) when $i \neq r, r+1$, and constraints of the form (15) otherwise. These, along with the constraint $\eta_r = 1$, define $n-1$ linearly independent constraints from (14) - (17). ∎

Let $\eta^T w \geq 1$ stand for the MIR inequality for $P(n,r)$. By definition $\eta_i = i/r$ if $i < r$, and $\eta_i = (n-i)/(n-r)$ otherwise. It clearly is a two-slope inequality, and satisfies (15) - (17). As it is a valid inequality for $P(n,r)$, Lemma 6 implies that it satisfies the subadditivity constraints, and therefore the requirements of Theorem 8. This implies the following result of Gomory.

Corollary 9 *[56] The MIR inequality for $P(n,r)$ defines a facet of $P(n,r)$.*

The following result can be proved in a similar manner.

Corollary 10 *If an integer $t > 0$ is a divisor of n and tr is not a multiple of n, then the t-scaled MIR inequality defines a facet of $P(n,r)$.*

We see later that for every non-zero integer t, such that tr is not a multiple of n, the t-scaled MIR inequality (20) defines a facet of $P(n,r)$ [36]. We also note that t-scaled MIR inequalities for $P(n,r)$ for some non-integral $t > 0$ define facets of $P(n,r)$. The two-step MIR inequalities also yield facets of $P(n,r)$.

Theorem 11 *(Dash and Günlük [36]) Let $\Delta \in \mathbb{Z}^+$ be such that $r > \Delta > 0$, and $n > \Delta \lceil r/\Delta \rceil > r$. The two-step MIR inequality for $P(n,r)$, obtained by applying (10) to Q_p with $\alpha = \Delta/n$ and $b = r/n$ defines a facet of $P(n,r)$.*

We call a facet in the above result a 1-scaled two-step MIR facet of $P(n,r)$ with parameter Δ. Note that $\alpha = \Delta/n$ satisfies the conditions of Lemma 2: (i) $r > \Delta > 0 \Rightarrow \hat{b} = r/n > \alpha > 0$, and (ii) $n > \Delta \lceil r/\Delta \rceil > r \Rightarrow n/\Delta > \lceil r/\Delta \rceil > r/\Delta \Rightarrow 1/\alpha > \lceil \hat{b}/\alpha \rceil > \hat{b}/\alpha$. Therefore the parameter α is assigned valid values. The proof of Theorem 11 is identical to the proof of Corollary 9. Further, if tr is not a multiple of n, then for appropriate choices of α, the t-scaled two-step MIR inequalities define facets of $P(n,r)$ [36]. If t is a divisor of n, the proof of this result is similar to that of Corollary 9. The two-step MIR facets contain the *2slope* facets in Araoz et. al. [8]. Other facet classes can be found in [8] (*3slope* facets), and also in [74].

An automorphism ϕ is a bijection from $\{0, 1, \ldots, n-1\}$ to itself such that

$$\phi((a+b) \bmod n) = (\phi(a) + \phi(b)) \bmod n.$$

A bijection ϕ is an automorphism if and only if $\phi(i) = (ti)_n$ where t is coprime with n (n and t have no common divisors). The inverse of an automorphism is also an automorphism; if $\phi(i) = (ti)_n$, then $\phi^{-1}(i) = (ui)_n$ where u satisfies $tu \equiv 1 \pmod n$ (such a u exists as t and n are coprime).

Theorem 12 *(Gomory [56]) Let r be an integer such that $0 < r < n$. Let ϕ be an automorphism defined by $\phi(i) = (ti)_n$, and let $s = \phi(r)$. If $\sum_i \eta_i w_i \geq 1$ is a non-trivial facet of $P(n,r)$, then $\sum_i \eta_i w_{\phi(i)} \geq 1$ is a non-trivial facet of $P(n,s)$. Equivalently, $\sum_i \eta_{\phi^{-1}(i)} w_i \geq 1$ is a non-trivial facet of $P(n,s)$.*

Theorem 12 says that for an automorphism ϕ, the facets of $P(n,r)$ are identical to facets of $P(n,\phi(r))$ after permuting facet coefficients, i.e., $P(n,r)$ and $P(n,\phi(r))$ have the same polyhedral structure (are isomorphic polyhedra). Therefore, for a given n, we need only study $P(n,r)$ where r is a divisor of n in order to understand $P(n,r)$ for all r.

Example 13 *Let $w^* = (w_1^*, w_2^*, w_3^*, w_4^*)$ be a non-negative integer vector satisfying*

$$1w_1 + 2w_2 + 3w_3 + 4w_4 \equiv 3 \bmod 5.$$

Note that 2 is co-prime with 5, and $2 \times 3 \equiv 1 \bmod 5$. Clearly w^* also satisfies $2(1w_1 + 2w_2 + 3w_3 + 4w_4) \equiv 2 \times 3 \equiv 1 \bmod 5$, or $2w_1 + 4w_2 + 1w_3 + 3w_4 \equiv 1 \bmod 5$. Note that the above equation is precisely the defining equation of $P(5,1)$, except that the variable indices are different from their coefficients. Now any solution w' of the last equation satisfies $3(2w_1 + 4w_2 + 1w_3 + 3w_4) \equiv 3 \bmod 5$, which is the same as the first modular equation. Therefore, $P(5,3)$ and $P(5,1)$ have the same polyhedral structure.

3.2 Subadditivity

A function $f : \mathbb{R} \to \mathbb{R}$ is subadditive if for all $u, v \in \mathbb{R}$, $f(u) + f(v) \geq f(u+v)$. Given a subadditive function f, it is easy to see that a non-negative integral solution x of $\sum_{i \in I} a_i x_i = b$ satisfies $\sum_{i \in I} f(a_i) x_i \geq f(b)$. More generally, if the coefficients of Q are multiples of $1/n$, then

$$nf(1/n) \sum_{c_j > 0} c_j v_j - nf(1 - 1/n) \sum_{c_j < 0} c_j v_j + \sum_{i \in I} f(a_i) x_i \geq f(b)$$

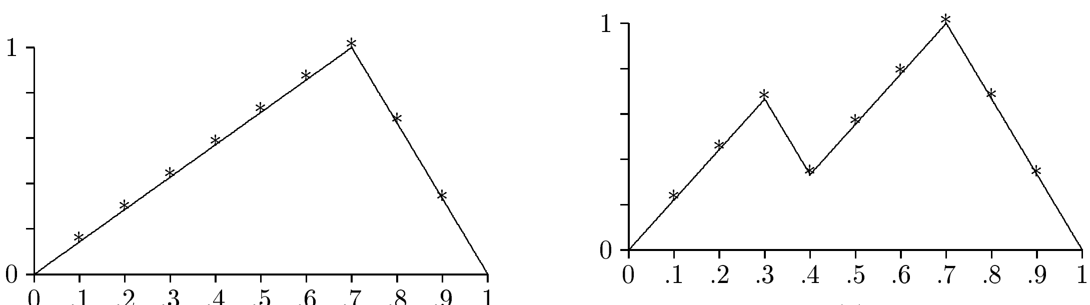
is a valid inequality for Q. If we dont know n, we can replace $nf(1/n)$ by $\lim_{v \to 0+} f(v)/v$ and $nf(1-1/n)$ by $\lim_{v \to 0+} f(1-v)/v$. Thus, subadditive functions yield valid inequalities for integer programs and for $P(n,r)$.

Gomory and Johnson [57] showed how to obtain subadditive functions from facets of $P(n,r)$ using the property (14). Let $\sum_{i=1}^{n-1} \eta_i w_i \geq 1$ define a non-trivial facet of $P(n,r)$. Define a function $f(v)$ over the domain $[0,1]$ as follows:

$$f(0) := 0, \;\; f(i/n) := \eta_i, \forall i = 1, \ldots, n-1, \tag{21}$$

$$f((i+\delta)/n) := (1-\delta) f(i/n) + \delta f((i+1)/n)), \forall i = 0, \ldots, n-1 \text{ and } \delta \in (0,1). \tag{22}$$

Then define $f(v)$ for all $v \in \mathbb{R}$ by $f(v) := f(\hat{v})$. Gomory and Johnson proved that f is subadditive. We call f a *facet-interpolated* function (FIF); such a function is piecewise linear and continuous. In Figure 3, we plot the coefficients of two facet-defining inequalities for $P(10,7)$, and depict the corresponding facet-interpolated functions. Note that the coefficients of the GMI cut (6), when divided by the right hand side $\hat{b}$, are given by the function in Figure 3(i). If f stands for this function, and a_i is the coefficient of x_i in Q, then the coefficient of x_i in the GMI cut is given by $\hat{b}$ times $f(\hat{a}_i)$; if a continuous variable v_j has a positive (negative) coefficient c_j, then $\hat{b}c_j$ times the slope of f at the origin (at 1) gives its cut coefficient.



(a)

Figure 3: (i) The MIR facet for $P(10,7)$; (ii) A two-step MIR facet for $P(10,7)$

An interesting aspect of FIFs is that one can use a facet of $P(n', r')$ to obtain a valid inequality for $P(n,r)$, where n, r are completely unrelated to n', r'. For FIFs derived from some simple facets of $P(n', r')$, such as t-scaled MIR facets where t is a divisor of n', the corresponding valid inequality for $P(n,r)$ is dominated by the t-scaled MIR inequality for $P(n,r)$ [37]. A similar statement is true for two-scaled MIR FIFs, when Δ in Theorem 11 satisfies some additional conditions.

FIFs form a somewhat restricted subclass of all subadditive functions; they only take non-negative values and they are periodic. Dash, Fukasawa and Günlük [34] show how to obtain more general subadditive functions by interpolating coefficients of facets of $K(n,r)$, and give examples of such functions which take on nonnegative values.

3.3 Shooting experiments

Gomory and Johnson [57] showed that $P'(n,r)$ has exponentially many facets (in n). Is there a way of determining which facets are more "important" and yield more important group cuts?

Gomory proposed using the solid angle subtended at the origin by a facet as a measure of the importance of the facet. Gomory, Johnson and Evans [59] estimate the solid angle subtended at the origin by a facet of $P(n,r)$ by generating vectors uniformly distributed over the unit sphere and computing the frequency with which different facets are hit by these directions. Given a direction $d \geq 0$ (assume d has norm 1), we say that d hits a non-trivial facet $\eta^T w \geq 1$ of $P(n,r)$ if it is the last facet intersected by the ray $\{td : t \geq 0\}$. In Figure 3.3, we depict d by the arrow. The ray

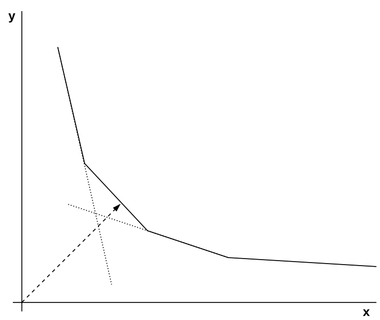


Figure 4: Facet hit by a direction vector

$\{td : t \geq 0\}$ intersects an inequality $\eta^T w \geq 1$ when $\eta^T(td) = 1$. Therefore the facet hit by d is given by

$$\max\{t : \eta^T(td) = 1, \eta \text{ is a facet}\} \equiv \max\{\frac{1}{\eta^T d} : \eta \text{ is a facet}\} \equiv \min\{\eta^T d : \eta \text{ is a facet}\}.$$

Therefore, for any $d \in R_+^{n-1}$, a basic optimal solution of the linear program

$$\min\{d^T \eta : \eta \text{ satisfies } (14) - (17)\}$$

gives the facet hit by d. The facet hit by d can be interpreted as the one most violated by d. Similar shooting experiments were first performed by Kuhn in the 1950s in the context of the TSP, see [67, 68].

An important question about the shooting experiment is whether its results depend on the r in $P(n,r)$? We next give a result from [37] which says that the classes of facets discussed earlier are invariant under automorphisms. For example, the scaled MIR facets of $P(10,1)$ are isomorphic to the scaled MIR facets of $P(10,3)$. In this section, we only discuss integer scaling factors.

Theorem 14 *[37] Let n, r be integers with $0 < r < n$. Let k be an integer co-prime with n, and let $\phi(i) = ki \bmod n$. Then the scaled MIR and two-step MIR facets of $P(n,r)$ are isomorphic,* *respectively, to the scaled MIR facets and scaled two-step MIR facets of $P(n, \phi(r))$. Further, if t is the largest divisor of n and tr is not a multiple of n, then the t-scaled MIR facet of $P(n,r)$ is isomorphic to the t-scaled MIR facet of $P(n, \phi(r))$.*

Proof We only prove the result for scaled MIR facets. It is not difficult to see that the t-scaled MIR inequality in (20) can be written as

$$\sum_{i=1}^{n-1} h^{tr/n}(ti/n)\, w_i \geq 1,$$

where $h^b(v) = \hat{v}/\hat{b}$ if $\hat{v} < \hat{b}$, and $h^b(v) = (1-\hat{v})/(1-\hat{b})$ otherwise (the function in Figure 3(i)). The values of $h^b(v)$ depend only on $\hat{b}$ and $\hat{v}$, the fractional parts of b and v, respectively. Thus if b' and v' are numbers such that $b - b'$ and $v - v'$ are integral, then $h^b(v) = h^{b'}(v')$. Let k and ϕ be defined as in the theorem. From (20), the t-scaled MIR inequality for $P(n,r)$ is isomorphic to the following inequality of $P(n, \phi(r))$:

$$\sum_{i=1}^{n-1} h^{tr/n}(ti/n)\, w_{\phi(i)} \geq 1. \tag{23}$$

Let s be the unique integer such that $sk \bmod n = t$. We will show that (23) is the s-scaled MIR inequality for $P(n, \phi(r))$. If j is any integer between 1 and $n-1$, then $s\phi(j) \bmod n = skj \bmod n = tj \bmod n \Rightarrow s\phi(j)/n - tj/n$ is integral. Therefore (23) is the same as

$$\sum_{i=1}^{n-1} h^{s\phi(r)/n}(s\phi(i)/n)\, w_{\phi(i)} \geq 1,$$

which is the s-scaled MIR inequality for $P(n, \phi(r))$. We conclude that the t-scaled MIR inequality defines a facet of $P(n,r)$ if and only if the s-scaled MIR inequality defines a facet of $P(n, \phi(r))$. Finally, if t is the largest divisor of n, and $sk \bmod n = t$, then $s = t$, and the theorem follows. ∎

Corollary 15 *[36] If an integer $t > 0$ is not a divisor of n and tr is not a multiple of n, then the t-scaled MIR inequality defines a facet of $P(n,r)$.*

Proof Assume s is the largest common divisor of t and n. Then $k = t/s$ and n have no common divisors. Define ϕ using k as in Theorem 14; from its proof we know that the s-scaled MIR inequality for $P(n, \phi(r))$ is isomorphic to the t-scaled MIR inequality for $P(n,r)$. This is because $sk \bmod n = t$. Now s is a divisor of n, and sr is not a multiple of n (otherwise tr would also be a multiple of n). Corollary 10 implies that the s-scaled MIR inequality for $P(n, \phi(r))$ defines a facet, and so does the t-scaled MIR inequality of $P(n,r)$. ∎

An argument similar to the one above can be used to show that t-scaled two-step MIR inequalities define facets of $P(n,r)$.

Gomory, Johnson and Evans [59] observe that a relatively small number of facets of $P(n,r)$ absorb most of the hits and the most important facets of $P(n,r)$ are related to the MIR inequality (they are t-scaled MIR facets [36]). Evans [46] reports that the 2slope facets [8] constitute another important class of facets. The experiments in [59] and [46] are performed on $P(n,r)$ with $n \leq 30$. Dash and Günlük [37] extend these experiments to $P(n,r)$ for n up to 200, and measure the importance of additional facet classes.

Table 1 contains results for selected master polyhedra from [37]. The second and third columns give the number of (integrally) scaled MIR facets and two-step MIR facets. The fourth column gives the number of *distinct* facets hit in 100,000 shots. The fifth, sixth, and seventh columns give, respectively, the hit frequencies for the scaled MIR facets, the scaled two-step MIR facets, and most frequently hit facet from these classes. For example, for $P(50,1)$ we see that the 378

	Number of facets			% of shots hit		
The group	MIRs	2-step MIRs	total hit	MIRs	2-step MIRs	top facet
P(42,1)	21	186	46,407	23.5	6.9	9.4
P(42,21)	11	236	53,754	11.8	15.4	9.6
P(50,1)	25	353	65,346	14.6	5.1	8.1
P(50,25)	13	452	68,202	8.9	11.1	8.1
P(72,1)	36	673		11.6	2.2	5.0
P(72,18)	27	616		2.3	2.2	0.9
P(100,1)	50	1534		4.8	0.5	3.2
P(200,1)	100	6584		0.5	0.0	0.4

Table 1: Shots absorbed by MIR based facets for different $P(n,r)$

scaled MIR and two-step MIR facets (out of 65,346 facets) absorb almost 20% of all hits, and that a single facet from this class absorbs 8.1% of all hits. Thus, a very few facets absorb a large fraction of all hits, and are mostly scaled MIR and two-step MIR facets. In addition, neither class is uniformly more important than the other. For example, for $P(42,1)$ the scaled MIR facets are more important, whereas the scaled two-step MIR facets are more important for $P(42,21)$. For the examples in Table 1, the facet in column 7 turns out to be a t-scaled MIR facet with t the largest divisor of n for which the t-scaled MIR inequality is valid and facet-defining for $P(n,r)$. For $P(50,1)$, the facet in column 7 is the 25-scaled MIR facet, absorbing 8.1% of all hits whereas all remaining scaled MIR facets combined absorb only 6.5% of the shots. For $P(72,18)$, the 36-scaled MIR does not define a facet as 36×18 is a multiple of 72. Instead, the most important facet is the 18-scaled MIR.

We believe that *if t is the largest divisor of n, then the t-scaled MIR facet (when it exists) is the most important facet of $P(n,r)$.* This is the case for all $P(n,r)$ studied by Dash and Günlük. However, we do not expect such a role for other divisors, independent of r, as Theorem 14 only guarantees the invariance of the t-scaled MIR – where t is the largest divisor of n – over isomorphic master polyhedra. For example, for $P(100,4)$, the most important MIR facet is the 10-scaled MIR facet (neither 25 nor 50 are valid scaling parameters). This facet of $P(100,4)$ is isomorphic to the 30-scaled MIR of $P(100,28)$, which is therefore the most important MIR facet of $P(100,28)$. See Cornuéjols, Li and Vandenbussche [30], who first suggested the possibility that the 1-scaled MIR inequality need not be uniformly superior to other scaled MIR inequalities.

The practical implications of the shooting results are not clear. First, most practical MIPs have upper and lower bounds on variables; at least one of these is ignored when deriving group cuts from corner polyhedra relaxations. Secondly, continuous variables are often present; then the 1-scaled MIR seems to have a special role. Recall from (18) that in any facet η of $P'(n,r)$, the coefficients of v_+ and v_- are $n\eta_1$ and $n\eta_{n-1}$, respectively. From (14), it follows that for any index i between 1 and $n-1$ $i\eta_1 \geq \eta_i$ and $i\eta_{n-1} \geq \eta_{n-i}$. Using $\eta_r = 1$, this implies that

$$\eta_1 \geq \frac{1}{r} \quad \text{and} \quad \eta_{n-1} \geq \frac{1}{n-r}. \tag{24}$$

For the 1-scaled MIR facet, $\eta_1 = 1/r$ and $\eta_{n-1} = 1/(n-r)$; therefore, the 1-scaled MIR inequality has the smallest possible coefficients for the continuous variables among all group cuts for Q.

Gomory, Johnson and Evans [59] proposed using FIFs based on important (as measured by shooting) facets of $P(n,r)$ with small n as valid inequalities for Q. Dash and Günlük [37] argue that one can trivially find inequalities dominating the ones generated by the approach above. The reason is that the important facets of $P(n,r)$ are scaled MIR and two-step MIR facets, and the valid inequalities for Q obtained from many of the corresponding FIFs are dominated by the scaled MIR and two-step MIR inequalities for Q. See Section 3.2.

4 MIR closure

In this section, we discuss properties of the MIR closure of a polyhedral set $P = \{v \in \mathbb{R}^l,\ x \in \mathbb{Z}^n : Cv + Ax = b,\ v, x \geq 0\}$ with m constraints. We define the *MIR closure* of P as the set of points in P^{LP} which satisfy all MIR cuts for P, and denote it by P^{MIR}. Nemhauser and Wolsey's result [77] showing the equivalence of split cuts and MIR cuts for P implies that the *split closure* of P – defined as the set of points in P^{LP} satisfying all split cuts for P – equals its MIR closure. Cook, Kannan and Schrijver [28] showed that the split closure of P is a polyhedron. Andersen, Cornuéjols and Li [5], Vielma [86], and Dash, Günlük and Lodi [39] give alternative proofs that the split closure of a polyhedral set is a polyhedron. The latter proof is in terms of the MIR closure of P, and we discuss it below. Caprara and Letchford [23] studied the separation problem for split cuts, i.e., the problem of finding a violated split cut given a point $(v^*, x^*) \in P^{LP}$ or proving that no such cut exists. They proved that this separation problem is NP-hard.

For a vector w, let w^+ stand for $\max\{w, \mathbf{0}\}$, where the maximum is taken component-wise. In this section, we assume that A, C and b have integral components (this is without loss of generality, as we assumed earlier that they were rational matrices). Let

$$\begin{aligned}
\Pi = \Big\{(\lambda, c^+, \check{\alpha}, \bar{\alpha}, \check{\beta}, \bar{\beta}) \in \quad & \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^n \times \mathbb{Z}^n \times \mathbb{R} \times \mathbb{Z} : \\
& c^+ \geq \lambda C, \qquad\qquad c^+ \geq 0, \\
& \check{\alpha} + \bar{\alpha} \geq \lambda A, \qquad 1 \geq \check{\alpha} \geq 0, \\
& \check{\beta} + \bar{\beta} \leq \lambda b, \qquad 1 \geq \check{\beta} \geq 0\}.
\end{aligned}$$

Note that for any $(\lambda, c^+, \check{\alpha}, \bar{\alpha}, \check{\beta}, \bar{\beta}) \in \Pi$,

$$c^+ v + (\check{\alpha} + \bar{\alpha})x \geq \check{\beta} + \bar{\beta} \tag{25}$$

is valid for P^{LP} as it is a relaxation of $(\lambda C)v + (\lambda A)x = \lambda b$. Furthermore, using the basic mixed-integer inequality (1), we infer that

$$c^+ v + \check{\alpha} x + \check{\beta} \bar{\alpha} x \geq \check{\beta}(\bar{\beta} + 1) \tag{26}$$

is a valid inequality for P. We call (26) a *relaxed MIR inequality* derived from the *base inequality* (25). (We use the notation $\check{\alpha}$ and $\check{\beta}$ because for a fixed λ, the best choice for $\check{\beta}$ equals $\widehat{\lambda b}$, and the best choices for components of $\check{\alpha}$ are 0 or fractional parts of components of λA, see the next paragraph.) If $\check{\beta} = 0$, then the relaxed MIR inequality is trivially satisfied by all points in P^{LP}. If $\check{\beta} = 1$, then (26) is identical to its base inequality (25) and is satisfied by all points in P^{LP}. Further, (26) is a split cut for P derived from the disjunction $\bar{\alpha} x \leq \bar{\beta} \vee \bar{\alpha} x \geq \bar{\beta} + 1$, and is therefore violated by $(v^*, x^*) \in P^{LP}$ only if $\bar{\beta} < \bar{\alpha} x^* < \bar{\beta} + 1$. This implies the following lemma.

Lemma 16 *A relaxed MIR inequality (26) violated by $(v^*, x^*) \in P^{LP}$ satisfies (i) $0 < \check{\beta} < 1$, (ii) $0 < \Delta < 1$, where $\Delta = \bar{\beta} + 1 - \bar{\alpha} x^*$.*

It is easy to see that the MIR inequality (5) for P is also a relaxed-MIR inequality. For a given multiplier vector λ, let α denote λA. Further, set $c^+ = (\lambda C)^+$, $\bar{\beta} = \lfloor \lambda b \rfloor$ and $\check{\beta} = \lambda b - \lfloor \lambda b \rfloor$. Also, define $\check{\alpha}$ and $\bar{\alpha}$ as follows: if $\alpha_i - \lfloor \alpha_i \rfloor < \check{\beta}$ then $\check{\alpha}_i = \alpha_i - \lfloor \alpha_i \rfloor$ and $\bar{\alpha}_i = \lfloor \alpha_i \rfloor$, otherwise $\check{\alpha}_i = 0$ and $\bar{\alpha}_i = \lceil \alpha_i \rceil$. Clearly, $(\lambda, c^+, \check{\alpha}, \bar{\alpha}, \check{\beta}, \bar{\beta}) \in \Pi$ and the corresponding relaxed MIR inequality (26) is the same as the MIR inequality (5). Therefore, every split cut is also a relaxed MIR inequality.

Recall the discussion in Section 2, where we observe that the MIR inequality is the strongest inequality of the form (3). This property and Lemma 16 are used by Dash, Günlük and Lodi [39, Lemma 6] to show that a point in P^{LP} satisfies all MIR inequalities, if and only if it satisfies all relaxed MIR inequalities. Therefore the MIR closure of P can be defined as

$$P^{MIR} = \Big\{ (v,x) \in P^{LP} \;:\; c^+ v + \check{\alpha} x + \check{\beta} \bar{\alpha} x \geq \check{\beta}(\bar{\beta} + 1) \text{ for all } (\lambda, c^+, \check{\alpha}, \bar{\alpha}, \check{\beta}, \bar{\beta}) \in \Pi \Big\}.$$

Therefore, for a given point $(v^*, x^*) \in P^{LP}$, one can test if $(v^*, x^*) \in P^{MIR}$ by solving the non-linear integer program (MIR-SEP):

$$\begin{array}{ll} \max & \check{\beta}(\bar{\beta} + 1) - (c^+ v^* + \check{\alpha} x^* + \check{\beta} \bar{\alpha} x^*) \\ \text{s.t.} & (\lambda^T, c^+, \check{\alpha}, \bar{\alpha}, \check{\beta}, \bar{\beta}) \; \in \; \Pi. \end{array}$$

If the optimal value of MIR-SEP is zero or less, then $(v^*, x^*) \in P^{MIR}$. On the other hand, if the optimal value is positive, the optimal solution gives a most violated MIR inequality. For a point (v^*, x^*), we define the violation of a relaxed MIR inequality to be its right-hand side minus its left-hand side evaluated at (v^*, x^*). The violation is bounded above (and so is the optimal value of MIR-SEP) by $\check{\beta}(\bar{\beta} + 1 - \bar{\alpha} x^*)$ which is strictly less than 1 for a violated MIR inequality.

Observe that MIR-SEP becomes an LP if $\bar{\alpha}$ and $\bar{\beta}$ are fixed to some integral values. Let ϕ be an optimal solution of MIR-SEP . Fix the values of $\bar{\alpha}$ and $\bar{\beta}$ in MIR-SEP to the corresponding values in ϕ, say $\bar{\alpha}_\phi$ and $\bar{\beta}_\phi$, respectively. The resulting LP has a basic optimal solution ϕ' with the same objective value as ϕ. The LP constraints (other than the variable bounds) can be written as

$$\begin{bmatrix} A^T & -I & & \\ C^T & & -I & \\ b^T & & & -1 \end{bmatrix} \begin{pmatrix} \lambda \\ \check{\alpha} \\ c^+ \\ \check{\beta} \end{pmatrix} \begin{matrix} \leq \\ \leq \\ \geq \end{matrix} \begin{pmatrix} \bar{\alpha}_\phi \\ 0 \\ \bar{\beta}_\phi \end{pmatrix}.$$

This implies the following result.

Theorem 17 *[39] If there is an MIR inequality violated by the point (v^*, x^*), then there is another MIR inequality violated by (v^*, x^*) for which $\check{\beta}$ and the components of $\lambda, \check{\alpha}$ are rational numbers with denominator equal to a subdeterminant of $[A\,C\,b]$.*

We will now argue that Theorem 17 implies that the MIR closure of P is a polyhedron. This argument is different from the one in [39]; there Theorem 17 is used in a different manner. Consider a polyhedron in $\mathbb{R}^k$ defined by $Gx + Hy \leq b$, let Φ be the maximum absolute value of subdeterminants of $[G\,H\,b]$. The convex hull of its x-integral points is a polyhedron, with extreme points whose x-coefficients have magnitude bounded by $(k+1)\Phi$, and other coefficients have encoding size bounded by a polynomial function of k and Φ. See Theorems 16.1 and 17.1 in [83]. By Lemma 16, every violated relaxed-MIR inequality satisfies $0 < \check{\beta} < 1$. Therefore, Theorem 17 implies that $\check{\beta}$ can be assumed to lie in the set $\mathcal{B} = \{r/s \in (0,1) : r, s \in \mathbb{Z},\, s \text{ is a sub-determinant of } [A\,C\,b]\}$. For any fixed $\check{\beta}$, MIR-SEP becomes a mixed-integer program; call this MIR-SEP($\check{\beta}$). The convex hull of its $(\bar{\alpha}, \bar{\beta})$-integral solutions is a polyhedron, say $P'(\check{\beta})$. The set $\mathcal{M}$ formed by the union of the extreme points of $P'(\check{\beta})$ for $\check{\beta} \in \mathcal{B}$ is clearly finite. Further, the relaxed-MIR inequalities defined by this set define the MIR closure of P. After all, if a point (v^*, x^*) is not contained in the MIR closure of P, by Theorem 17, it is violated by some inequality with $\check{\beta}$ in $\mathcal{B}$ and therefore by some inequality in $\mathcal{M}$.

The above argument also implies that the relaxed-MIR cuts defining the MIR closure of P can be derived from disjunctions contained in a bounded set; more precisely $(\bar{\alpha}, \bar{\beta})$ are contained in the set $D = [-(n+l+1)\Phi^2, (n+l+1)\Phi^2]^{n+1}$, where Φ is the maximum absolute value of subdeterminants of $[A\,C\,b]$. Therefore, one can derive using the equivalence of split cuts and MIR cuts that the split closure of P equals

$$\bigcap_{(c,d)\in D} conv(P^{LP} \cap \{cx \leq d\}\} \cup P^{LP} \cap \{cx \geq d+1\}\}).$$

Further, the vector of multipliers λ also has bounded coefficients. This result is similar to [39, Lemma 21], where a bound on the magnitude of λ needed for non-redundant MIR cuts is given; the latter bound is sharper as the proof in [39] does not use such general arguments.

Lemma 18 *[39, Lemma 21] Assume that the coefficients in $Cv + Ax = b$ are integers. If there is an MIR inequality violated by the point (v^*, x^*), then there is another MIR inequality violated by (v^*, x^*) with $\lambda_i \in (-m\Psi, m\Psi)$, where m is the number of rows in $Cv + Ax = b$, and Ψ is the largest absolute value of subdeterminants of C.*

If P has no continuous variables, it is easier to show that the MIR closure of P is a polyhedron; one can show that $\lambda \in (0,1)^m$. Caprara and Letchford [23, Lemma 1] show (in a different form) that $\lambda \in (-1,1)^m$, when P has no continuous variables; this yields a short proof that the split closure of P is a polyhedron.

Assume (in this paragraph) that P has no continuous variables, i.e., $C = 0$. The problem of obtaining a violated Gomory-Chvátal cut be framed as a mixed-integer program, see Bockmayr and Eisenbrand [19] and Fischetti and Lodi [47]. One can then infer, as in the discussion on the MIR closure above, that the Chvátal closure of P is a polyhedron. Bockmayr and Eisenbrand study this mixed-integer program and use the fact that its integral hull has polynomially many extreme points in fixed dimension to show that the Chvátal closure of P in fixed dimension has polynomially many facets. In fixed dimension, the number of extreme points of $P'(\check{\beta})$ can be shown to be bounded by a polynomial function of the encoding size of $Cv + Ax = b$. If one can show that $\mathcal{M}$ is the union of $P'(\check{\beta})$ for polynomially many choices of $\check{\beta}$, then one could give a positive answer to Eisenbrand's question [43]: Does the MIR (or split) closure of P have polynomially many facets in fixed dimension? Dash, Günlük and Lodi show that the separation problem for MIR cuts can be framed as a mixed-integer program (see the discussion below). Unfortunately, the number of variables in their separation MIP depends on the encoding size of $[A\,C\,b]$ and one cannot use the technique of Bockmayr and Eisenbrand to give a positive answer to Eisenbrand's question.

We next describe the approximate MIR separation model in [39] obtained by approximately linearizing the product $\check{\beta}(\bar{\beta}+1-\bar{\alpha}x^*)$ in the objective function of MIR-SEP. Define a new variable Δ that stands for $(\bar{\beta}+1-\bar{\alpha}x)$. Let $\tilde{\beta} \le \check{\beta}$ be an approximation of $\check{\beta}$ which is representable over some $\mathcal{E} = \{\epsilon_k : k \in K\}$. Let a number δ be *representable* over $\mathcal{E}$ if $\delta = \sum_{k\in\bar{K}} \epsilon_k$ for some $\bar{K} \subseteq K$. One can write $\tilde{\beta}$ as $\sum_{k\in K} \epsilon_k \pi_k$ using binary variables π_k, and approximate $\check{\beta}\Delta$ by $\tilde{\beta}\Delta$. This last term can be written as $\sum_{k\in K} \epsilon_k \pi_k \Delta$. This results in the following approximate MIP model APPX-MIR-SEP for the separation of the most violated MIR inequality:

$$\max \quad \sum_{k\in K} \epsilon_k \Delta_k - (c^+ v^* + \check{\alpha} x^*) \tag{27}$$

$$\text{s.t.} \quad (\lambda, c^+, \check{\alpha}, \bar{\alpha}, \check{\beta}, \bar{\beta}) \in \Pi \tag{28}$$

$$\check{\beta} \ge \sum_{k\in K} \epsilon_k \pi_k \tag{29}$$

$$\Delta = (\bar{\beta}+1) - \bar{\alpha} x^* \tag{30}$$

$$\Delta_k \le \Delta \qquad \forall k \in K \tag{31}$$

$$\Delta_k \le \pi_k \qquad \forall k \in K \tag{32}$$

$$\pi \in \{0,1\}^{|K|} \tag{33}$$

Let z^{sep} and $z^{apx-sep}$ denote the optimal values of MIR-SEP and APPX-MIR-SEP, respectively. For any integral solution of APPX-MIR-SEP, $(\lambda, c^+, \check{\alpha}, \bar{\alpha}, \check{\beta}, \bar{\beta}) \in \Pi$ and

$$\sum_{k\in K} \epsilon_k \Delta_k \le \sum_{k\in K} \epsilon_k \Delta \pi_k \le \check{\beta}\Delta,$$

implying that $z^{sep} \ge z^{apx-sep}$. In other words, if the approximate separation problem finds a solution with objective function value $z^{apx-sep} > 0$, the corresponding MIR cut is violated by at least as much. In the computational experiments of Dash, Günlük and Lodi with APPX-MIR-SEP, they use $\mathcal{E} = \{2^{-k} : k = 1, \ldots, \bar{k}\}$ for some small number $\bar{k}$ (between 5 and 7). They prove that with this choice of $\mathcal{E}$, APPX-MIR-SEP yields a violated MIR cut provided that there is an MIR cut with a "large enough" violation. More precisely, they prove [39, Theorem 8] that $z^{apx-sep} > z^{sep} - 2^{-\bar{k}}$.

If the relaxed-MIR inequality $\mathcal{I}$ with maximum violation has a value of $\check{\beta}$ which is representable over $\mathcal{E}$, one can choose $\pi \in \{0,1\}^{\bar{k}}$ such that $\check{\beta} = \sum_{k \in K} \epsilon_k \pi_k$. Set $\Delta = \bar{\beta} + 1 - \bar{\alpha} x^*$. Set $\Delta_k = 0$ if $\pi_k = 0$, and $\Delta_k = \Delta$ if $\pi_k = 1$. Then $\Delta_k = \pi_k \Delta$ for all $k \in K$, and $\check{\beta}\Delta = \sum_{k \in K} \epsilon_k \Delta_k$. Therefore, the relaxed-MIR inequality $\mathcal{I}$ yields an optimal solution of APPX-MIR-SEP whose objective function value equals the violation of the $\mathcal{I}$. In other words, $z^{sep} = z^{apx-sep}$. Theorem 17 implies that $\check{\beta}$ in a violated MIR cut can be assumed to be a rational number with a denominator equal to a subdeterminant of $[A\,C\,b]$. This implies the following result.

Theorem 19 *[39] Let Φ be the least common multiple of all subdeterminants of $[A\,C\,b]$, $K = \{1, \ldots, log\Phi\}$, and $\mathcal{E} = \{\epsilon_k = 2^k/\Phi, \forall k \in K\}$. Then APPX-MIR-SEP is an exact model for finding violated MIR cuts.*

Caprara and Letchford [23], and more recently, Balas and Saxena [16], present optimization models for finding a violated split cut for P. In both papers, the authors use two sets of multipliers that guarantee that the split cut is valid for both sides of the disjunction; see equations (8)-(13) in [23] and equation (SP) in [16]. It is argued in [39] that the separation model in Caprara and Letchford (equations (8)-(13)) actually finds the most violated MIR cut (the objective function equals four times the objective function of MIR-SEP). In other words, an optimal solution of MIR-SEP is also optimal for the Caprara-Letchford model and vice-versa. Similarly, the Balas-Saxena model (equation (2.1) or (PMILP) in [16]) is equivalent to MIR-SEP, and has the same objective function. Consequently, the models in [23] and [16] are also equivalent to each other.

Caprara and Letchford do not perform any computational tests with their model. As for Balas and Saxena, instead of bounding $\check{\beta}$ by $\tilde{\beta}$ and linearizing $\tilde{\beta}\Delta$ as in [39], they fix the term corresponding to $1 - \check{\beta}$ in their model to specific values between 0 and 1/2, and for each value, solve an MIP to obtain a violated split cut. We will discuss some of the computational results in [39] and [16] in the next section.

5 Computational issues

Balas, Ceria, Cornuéjols and Natraj [14] showed that GMI cuts (MIR cuts with simplex tableau rows as base inequalities), when added in *rounds* (all violated GMI cuts for the current optimal tableau are added simultaneously), are very useful in solving the general mixed-integer programs in MIPLIB 3.0. Bixby et. al. [18] extended this observation to larger problem sets, and performed additional experiments confirming the usefulness of GMI cuts relative to other cuts in the CPLEX solver. Marchand and Wolsey [72] proposed a different, effective way of generating MIR inequalities; their heuristic aggregates constraints of the original formulation to obtain base inequalities which are different from simplex tableau rows. Despite their effectiveness, GMI cuts often cause numerical difficulties, and this aspect limits their use. In some cases, simple implementations yield invalid cuts [73]; in other cases, adding too many GMI cuts makes the resulting LP hard to solve. The issue of invalid cuts is addressed in a recent paper by Cook et. al. [26] who generate *provably valid* GMI cuts with negligible reduction in performance (as measured by computing time and quality of bounds).

After Gomory [55] introduced group relaxations in 1965, White [87] and Shapiro [84] showed that for a number of small IPs, group relaxations yielded strong bounds on optimal values. For 12 of the 14 problems in the latter paper where the IP optimal is known and is different from the LP relaxation value, the group relaxation bound equals the optimal value. Gorry, Northup and Shapiro [60] performed a detailed study of a group relaxation based branch-and-bound algorithm, and solved problems with up to 176 rows and 2385 columns. In the above papers, group relaxations were solved via dynamic programming algorithms. See Salkin [82, Chapter 9] for a discussion on early computational work on this topic. In general it is NP-hard to solve the group relaxation problem or, equivalently, to optimize over an arbitrary corner polyhedron [69] (or strengthened corner polyhedra [48]). In a recent computational study, Fischetti and Monaci [48] optimize over the corner polyhedra (also strengthened corner polyhedra) associated with optimal vertices of LP relaxations of some MIPLIB 3.0 and MIPLIB 2003 instances by solving MIPs and show that the average integrality gap closed is 23.61% (34.48%) as opposed to 25.32% with GMI cuts. The gap closed using corner polyhedra relaxations is worse than that with GMI cuts; this is because throwing away the non-active bounds on variables often results in weak bounds on optimum values for MIPLIB instances, especially those which have binary variables.

Two interesting directions of research extending the above work consist of (a) generating valid inequalities other than MIR cuts – especially group cuts – from the same base inequalities, and (b) aggregating constraints of the original formulation to obtain base inequalities different from simplex tableau rows or those generated by Marchand and Wolsey. The difference between recent work on (a) and earlier work in the previous paragraph is that the corner polyhedron is used to generate cutting planes, rather than as a relaxation by itself.

For the mixed-integer knapsack polyhedron, Atamtürk [9] developed valid inequalities (via lifting) which use upper and lower bounds on variables, unlike the MIR cut in (5) which uses only one of the bounds. For a collection of randomly generated multiple knapsack instances [10], his inequalities along with MIR cuts close a significantly larger fraction of the integrality gap than MIR cuts alone. He uses scaled constraints of the original formulation as base inequalities. Fischetti and Saturni [49] and Dash, Goycoolea and Günlük [35] study the effectiveness of group cuts derived from simplex tableau rows relative to GMI cuts. The first paper contains a study of group cuts derived via interpolation; violated cuts of this type are obtained by solving an LP. The second paper presents a heuristic to generate violated two-step MIR inequalities, and shows that they are useful for the randomly generated instances of Atamtürk. For the unbounded instances (variables are nonnegative and not bounded above) in Atamtürk's data set, two-step MIR inequalities derived from rows of the initial optimal simplex tableau combined with GMI cuts (derived from the same tableau rows) close 78.65% of the integrality gap, whereas GMI cuts alone close only 56.25% of the integrality gap. For these instances, two-step MIR inequalities seem to be as effective as the lifted knapsack cuts of Atamtürk, and more effective than K-cuts [30]. Fischetti and Saturni show that K-cuts for $K = 1, \ldots, 50$ (or 1-50 scaled GMI cuts) close 75.99% of the integrality gap. Interestingly, the integrality gap closed via (strengthened or otherwise) corner polyhedra relaxations [48] is 79.43%, which is only slightly better than the gap closed using two-step MIR inequalities. These experiments suggest that the shooting experiments discussed earlier yield useful information for problems which resemble master cyclic group polyhedra in that the integer variables can take values from a large interval, and the constraints have general (not 0-1) coefficients. For the bounded Atamtürk instances, we note that the lifted knapsack cuts are more (about 5-6%) effective in closing the integrality gap than two-step MIR inequalities.

The authors in [35] observe that for the problems in MIPLIB 3.0, the gap closed by one round of two-step MIR cuts + GMI cuts is essentially the same as that closed by one round of GMI cuts alone. Interpolated group cuts, and 1-50 scaled GMI cuts also seem to behave similarly [49]. Motivated by this observation, Dash and Günlük [38] demonstrate that for a collection of practical instances (from MIPLIB 3.0, MIPLIB 2003, MILPLib [75], and instances from [71]), after GMI cuts derived from the initial optimal tableau rows are added, the solution of the resulting relaxation satisfies all non-GMI group cuts derived from the initial tableau rows for 35% of the instances. In other words, additional group cuts beyond the GMI cut derived from the initial optimal tableau rows are not useful at all for these instances. On the other hand, for 82% of the remaining instances (which potentially have violated group cuts), one can find violated two-step MIR inequalities. Thus, unlike the Atamtürk instances, group cuts from single tableau rows do not seem to be very useful for the above instances. However, two-step MIR inequalities seem to be important relative to other group cuts. Fukasawa and Goycoolea [50] extend the above experiments for MIPLIB instances to show that no other valid inequalities derived from the mixed-integer knapsacks defined by initial optimal tableau rows and bounds on variables improve the integrality gap by a significant margin. The above experiments suggest that for MIPLIB instances, in order to obtain cutting planes which improve the integrality gap closed by GMI cuts, it is important to use information from multiple constraints simultaneously. Some recent computational work in this direction can be found in Espinoza [45].

Bonami and Minoux [21] approximately optimize over the lift-and-project closure of 0-1 mixed integer programs via the equivalence of optimization and separation; in this context the separation problem can be framed as a linear program. Lift-and-project cuts are split cuts derived from disjunctions of the form $x_i \leq 0 \vee x_i \geq 1$ for some integral variable x_i. Balas and Perregard [15] proposed a method to generate *strengthened lift-and-project* cuts from simplex tableau rows, and their method was implemented by Balas and Bonami [13] with encouraging results. Independently, Fischetti and Lodi [47] show that for many practical MIPs, one can separate points from the Chvátal closure of pure integer programs in reasonable time by formulating the separation problem as an MIP and solving it with a general MIP solver. They apply their separation algorithm to approximately optimize over the Chvátal closures of MIPLIB instances and obtain tight bounds on optimal solution values for many instances. Bonami et. al. [20] extend the definition of Gomory-Chvátal cuts to mixed integer programs (*projected Gomory-Chvátal cuts*) and use a similar MIP based separation procedure to obtain bounds for mixed-integer programs in MIPLIB 3.0.

Optimizing over the split closure of MIPs should lead to stronger bounds than in the papers above. Balas and Saxena [16] approximately optimize over the split closure of MIPLIB instances to obtain strong bounds on their optimal values, and so do Dash, Günlük and Lodi [39], who combine APPX-MIR-SEP with some heuristics to find violated MIR cuts. In MIPLIB 3.0, 62 of the 65 instances have a non-zero integrality gap, other than *dsbmip, enigma* and *noswot*. Balas and Saxena show that for these instances (other than *arki001* where they use cuts which potentially have MIR rank 2), at least 71.3% of the integrality gap can be closed by optimizing over the split closure (the computation time is quite high though). For these 62 instances, Dash, Günlük and Lodi show that at least 59.3% of the integrality gap can be closed in one hour of computation time; a round of GMI cuts closes only 28.4% of the integrality gap. For the 21 pure IPs, the gaps closed by Dash, Günlük and Lodi, Fischetti and Lodi (GC cuts), Balas and Saxena, and GMI cuts are on the average 59.2%, 56.5%, 76.0% and 29.8%, respectively. For the remaining 41 MIPs, the corresponding numbers (with projected-GC cuts instead of GC cuts) are, respectively, 59.3, 28.8, 68.9, and 27.7. Note that in Bonami et. al. [20], the bounds for projected-GC cuts are obtained with only 20 minutes of computation, whereas the bounds with GC cuts in [47] are obtained after 3 hours of computation, on the average.

6 Proof complexity

If $NP \neq coNP$, then it cannot be true that for arbitrary $Ax \leq b$ without 0-1 solutions, there is a polynomial-size (in the encoding size of A, b) certificate of the absence of 0-1 solutions. We next discuss a recent result in [33] which proves the existence of a family of inequality systems without 0-1 solutions for which MIR cutting-plane proofs of 0-1 infeasibility have exponential length.

A boolean circuit can be viewed as a description of the elementary steps in an algorithm via a directed acyclic graph with three types of nodes: input nodes – nodes with no incoming arcs, a single output node – the only node with no outgoing arcs, and computation nodes (also called gates), each of which is labelled by one of the boolean functions $\wedge, \vee$, and $\neg$. For nodes i and j, an arc ij means that the value computed at i is used as an input to the gate at node j. A computation is represented by placing 0-1 values on the input gates, and then recursively applying the gates to inputs on incoming arcs, till the function at the output node is evaluated. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *monotone* if for x, y in $\mathbb{R}^n$, $x \leq y$ implies $f(x) \leq f(y)$. *Monotone operations* are monotone functions with one or two inputs; some examples are

$$tx, \quad r + x, \quad x + y, \quad \lfloor x \rfloor, \quad thr(x, 0)$$

where t is a non-negative constant, x and y are real variables, and r is a real constant; $thr(x, 0)$ is a *threshold function* which returns 0, if $x < 0$, and 1 otherwise. The functions $\wedge$ and $\vee$ are monotone operations over the domain $\{0, 1\}$. The function $f(x, y) = x - y$, where $x, y \in \mathbb{R}$, is not monotone. A *monotone boolean circuit* uses only $\wedge$ gates and $\vee$ gates; a *monotone real circuit* is one with arbitary monotone operations as gates.

Consider $CLIQUE_{k,n}$ (say k is a fixed function of n), the function which takes as input n-node graphs (represented by incidence vectors of their edges) and returns 1 if the graph has a clique of size k or more, and 0 otherwise. This function is monotone, as adding edges to a graph (changing some zeros to ones in the incidence vector) causes the maximum clique size to increase.

Every monotone boolean function can be computed by a monotone boolean circuit. Razborov [80] showed that any monotone boolean circuit solving $CLIQUE_{k,n}$ (for appropriate k) has a super-polynomial number of gates, and Alon and Boppana [4] strengthened his bound to an exponential lower bound. Pudlák [78], and independently, Cook and Haken [27], proved that the above bounds hold for monotone real circuits.

Theorem 20 *Let C_n be a monotone real circuit which takes as input graphs on n nodes (given as incidence vectors of edges), and returns 1 if the input graph contains a clique of size $k = \lfloor n^{2/3} \rfloor$, and 0 if the graph contains a coloring of size $k-1$ (and returns 0 or 1 for all other graphs). Then $|C_n| \geq 2^{\Omega((n/\log n)^{1/3})}$.*

Pudlák [78] presented a set of linear inequalities $\mathcal{I}$ related to the problem of Theorem 20 such that if $\mathcal{I}$ has a 0-1 solution, then there is a graph on n nodes which has both a clique of size k and a coloring of size $k-1$. He proved that given a Gomory-Chvátal cutting plane proof $\mathcal{P}$ with L cuts which proves that $\mathcal{I}$ has no 0-1 solution, one can construct a monotone real circuit with $O(poly(size(\mathcal{P})))$ gates solving $CLIQUE_{k,n}$. Therefore L is exponential in n. Pudlák used the following properties of Gomory-Chvátal cuts in mapping short cutting-plane proofs to small monotone real circuits:
(1) If $gx + hy \leq d$ is a Gomory-Chvátal cut for $Ax + By \leq c$, then for any 0-1 vector z, $gx \leq d - hz$ is a Gomory-Chvátal cut for $Ax \leq c - Bz$;
(2) if $gx + hy \leq d$ is a Gomory-Chvátal cut for $Ax \leq e,\ By \leq f$, then there are numbers r and s such that $gx \leq r$ is a Gomory-Chvátal cut for $Ax \leq e$, and $hy \leq s$ is a Gomory-Chvátal cut for $By \leq e$, and $r + s \leq d$.
(3) The number r (or s) can be computed from A, e (or B, f) with polynomially many monotone operations.

Property (1) is easy to prove. Consider property (2). As $gx + hy \leq d$ is a Gomory-Chvátal cut for $Ax \leq e,\ By \leq f$, we can assume that g, h, d are integral and there are multiplier vectors $\lambda, \mu \geq 0$ such that $g = \lambda A,\ h = \mu B,\ d = \lfloor \lambda e + \mu f \rfloor$. Clearly, $gx \leq \lfloor \lambda e \rfloor$ is a Gomory-Chvátal cut for $Ax \leq e$, and so is $hy \leq \lfloor \mu f \rfloor$, and $\lfloor \lambda e \rfloor + \lfloor \mu f \rfloor \leq \lfloor \lambda e + \mu f \rfloor$. Property (3) also follows from this; the number $\lfloor \lambda e \rfloor$ can be computed from e via polynomially many monotone operations (the coefficients of λ are treated as non-negative constants).

In general, for a class of cutting planes $\mathcal{C}$, if we can prove properties (1)-(3) in the previous paragraph for $\mathcal{C}$, then given a $\mathcal{C}$-proof of the fact that Pudlák's inequality system $\mathcal{I}$ has no 0-1 solution, we can construct a monotone real circuit solving $CLIQUE_{k,n}$ with polynomially many gates (in the size of the $\mathcal{C}$-proof). This would yield an exponential worst-case lower bound on the size of $\mathcal{C}$-proofs certifying that $\mathcal{I}$ has no 0-1 solution (of course, additional details have to be verified). In general, for many classes of cutting planes properties (1) and (2) hold, e.g., the matrix cuts of Lovász and Schrijver (cuts based on the N and N_+ operators; see [79], [32]. Property (3) is often hard to prove, and is not known to hold for matrix cuts. We prove in [33] that slight variants of properties (2) and (3) hold for MIR cuts, and thereby obtain an exponential worst-case lower bound on the complexity of MIR cuts. We state this result below. For completeness, we explicitly give the inequality system $\mathcal{I}$.

Let $k = \lfloor n^{2/3} \rfloor$. Let z be a vector of $n(n-1)/2$ 0-1 variables, such that every 0-1 assignment to z corresponds to the incidence vector of a graph on n nodes (assume nodes are numbered from $1, \ldots, n$). Let x be the 0-1 vector of variables $(x_i \,|\, i = 1, \ldots, n)$ and let y be the 0-1 vector of variables $(y_{ij} \,|\, i = 1, \ldots, n, j = 1, \ldots, k-1)$. Consider the inequalities

$$\sum_i x_i \geq k, \tag{34}$$

$$x_i + x_j \leq 1 + z_{ij}, \quad \forall\, i, j \in N, \text{ with } i < j, \tag{35}$$

$$\sum_{j=1}^{k-1} y_{ij} = 1, \quad \forall i \in N, \tag{36}$$

$$y_{is} + y_{js} \leq 2 - z_{ij}, \quad \forall\, i, j \in N \text{ with } i < j, \text{ and } \forall s \in \{1, \ldots, k-1\}. \tag{37}$$

Then, in any 0-1 solution of the above inequalities, the set of nodes $\{i \,|\, x_i = 1\}$ forms a clique of size k or more, and for all $j \in \{1, \ldots, k-1\}$, the set $\{i \,|\, y_{ij} = 1\}$ is a stable set. Thus, the variables y_{ij} define a mapping of nodes in a graph to $k-1$ colors in a proper colouring. Let $Ax + Cz \leq e$ stand for the inequalities (34) and (35), along with the bounds $0 \leq x \leq 1$. Let $By + Dz \leq f$ stand for the inequalities (36) and (37), along with the bounds $0 \leq y \leq 1$ and $0 \leq z \leq 1$. Then any 0-1 solution of $Ax + Cz \leq e$ and $By + Dz \leq f$ corresponds to a graph which has both a clique of size k, and a coloring of size $k-1$. Clearly, no such 0-1 solution exists. Note that the above inequalities have $O(n^3)$ variables and constraints.

Theorem 21 *[33] Every MIR cutting-plane proof of* $\mathbf{0}x + \mathbf{0}y + \mathbf{0}z \leq -1$ *from* $Ax + Cz \leq e$ *and* $By + Dz \leq f$ *has exponential length.*

Many families of inequalities are either special cases of MIR cuts (e.g., Gomory-Chvátal cuts, lift-and-project cuts), or can be obtained as rank k MIR cuts for some fixed number k. For example, the two-step MIR inequalities have MIR rank 2 or less. Therefore, one trivially obtains exponential worst-case lower bounds for the complexity of cutting plane proofs for all such families of cutting planes.

The technique of deriving a polynomial size monotone circuit from a proof of infeasibility is called *monotone interpolation*, and was proposed by Krajíček [65, 66] to establish lower bounds on the lengths of proofs in different proof systems. Razborov [81], and Bonet, Pitassi, and Raz [22], first used this idea to prove exponential lower bounds for some proof systems.

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