RC24603 (W0807-057) July 9, 2008
Mathematics

IBM Research Report

A New Proof of the Sylvester-Gallai Theorem

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In 1893 J. J. Sylvester [8] posed the following celebrated problem: Given a finite collection of points in the affine plane, not all lying on a line, show that there exists a line which passes through precisely two of the points. Sylvester's problem was reposed in this Monthly by Erdős in 1944 [4] and then later that year a proof was given by Gallai [6]. Since then, many proofs of the Sylvester-Gallai Theorem have been found. Of these proofs, that given by Kelly (as communicated by Coxeter in [2] and [3]) and that attributed to Melchior (as implied in [7][1]) are particularly elegant. Kelly's proof uses a simple distance argument while Melchior considers the dual collection of lines and applies Euler's formula. For more extensive treatments of the Sylvester-Gallai Theorem and its relatives, see [1] and [5].

Given a collection of points, a line passing through just two of the points is commonly referred to as an **ordinary line**. As in Melchior [7], one can use projective duality to obtain a fully equivalent dual formulation of the theorem, namely that given a collection of n lines in the real projective plane, not all passing through a common point, there must be a point of intersection of just two lines. Such a point of intersection is generally referred to as an **ordinary point**. In what follows I provide a new and particularly simple non-metric proof of the theorem in this dual form.

Theorem 1 (Gallai, 1944) *Given a finite collection of n lines in the real projective plane, not all passing through a common point, there exists an ordinary point.*

Proof. We consider the problem in the projective plane modeled as the Euclidean plane, together with "points at infinity" corresponding to each possible real slope, and a "line at infinity" containing all of the points at infinity. By virtue of the fact that not all lines meet at a common point we may pick out some three lines which do not meet in a common point. Call these lines i, j and k with intersection points $u = i \cap j$, $v = j \cap k$ and $w = i \cap k$. The lines i, j and k partition the (projective) plane into four triangular regions. If u is ordinary we are done, so assume that it is not ordinary and that a third line, ℓ_0, passes through u in addition to i and j. ℓ_0 passes into two of the four triangular regions formed by i, j and k. Pick either one of these projective triangular regions and call it T. ℓ_0 meets the edge (v,w) opposite u at a vertex v_0 partitioning T into two subtriangles $R = \triangle(u, v_0, v)$ and $S = \triangle(u, v_0, w)$. See Figure 1. If v_0 is ordinary we are done. Otherwise there is a third line ℓ_1 through v_0 entering the interior of either R or S at v_0 and meeting an opposite edge at a new vertex v_1. Suppose the triangle entered is R. We may then throw away S and continue with the same argument as in the previous step inside the triangle R. Since at each step we arrive at a new vertex v_i, and there are only a finite number of such vertices, we must eventually encounter an ordinary point. □

[1] See also [5].

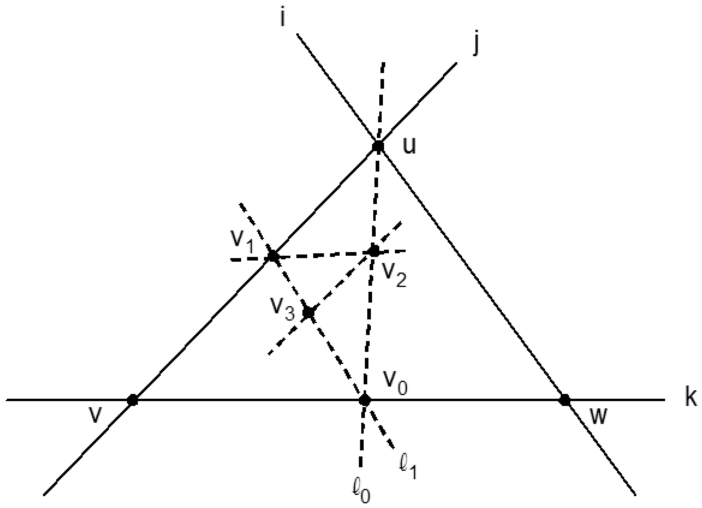

Figure 1. The proof in a picture.

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