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Difference Triangle Set Constructions

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Abstract: We review difference triangle set constructions by Robinson and Bernstein [2], Kløve [1], Ling [3] and Chen, Fan and Jin [4]. We find extensions and improvements allowing us to construct some difference triangle sets of smaller size than the best previously known.

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Following Kløve [1], an (n, k) difference triangle set, T, is a set of integers, $\{a_{ij} | 1 \leq i \leq n, 0 \leq j \leq k\}$, such that all of the differences, $\{a_{i\ell} - a_{ij} | 1 \leq i \leq n, 0 \leq \ell \neq j \leq k\}$, are distinct. Let $m = m(T)$ be the maximum difference. We are interested in constructing (n, k) difference triangle sets, T, with $m(T)$ as small as impossible. Let $M(n, k) = \min \{ m(T) \mid T \text{ is an } (n, k) \text{ difference triangle set} \}$. In this paper we review earlier constructions ([2],[1],[3],[4]) and show how they can be improved and extended.

An important special case occurs when $n = 1$. A k mark Golomb ruler may be defined as a set of integers, $\{a_i | 1 \leq i \leq k\}$, with $0 = a_1 < a_2 < \ldots < a_k$ such that all of the differences $\{a_j - a_i | 1 \leq i < j \leq n\}$ are distinct. a_k is the length of the ruler. Clearly k mark Golomb rulers correspond to $(1, k-1)$ difference triangle sets. More generally $(n, k-1)$ difference triangle sets correspond to collections of n k mark Golomb rulers with no common differences.

The constructions we will study are based on the modular version of Golomb rulers. We say a (v, k) modular Golomb ruler is a set of integer residues modulo v $\{a_1, a_2, \ldots, a_k\}$ such that all of the differences $\{a_i - a_j | 1 \leq i \neq j \leq k\}$ are distinct and nonzero modulo v. We will generally assume the residues are chosen from the set $\{0, 1, \ldots, v-1\}$.

Generally we are interested in dense modular Golomb rulers. So for fixed k we want v as small as possible. Note by simple counting $v \geq k(k-1) + 1$.

Singer [5] showed that for every prime power q there is a $(q^2+q+1, q+1)$ modular Golomb ruler.

Bose [6] showed that for every prime power q there is a (q^2-1, q) modular Golomb ruler. The missing differences are the $q-1$ multiples of $q+1$.

Ruzca [7] showed for every prime p there is a $(p^2-p, p-1)$ modular Golomb ruler. The missing differences are the $2p-2$ multiples of p or $p-1$.

The first few examples of each of these constructions are listed in tables 1, 2 and 3.

Table 1: Singer Construction

q	v	k														
2	7	3	0	1	3											
3	13	4	0	1	4	6										
4	21	5	0	2	7	8	11									
5	31	6	0	1	4	10	12	17								
7	57	8	0	4	5	17	19	25	28	35						
8	73	9	0	2	10	24	25	29	36	42	45					
9	91	10	0	1	6	10	23	26	34	41	53	55				
11	133	12	0	2	6	24	29	40	43	55	68	75	76	85		
13	183	14	0	4	6	20	35	52	59	77	78	86	89	99	122	127

Table 2: Bose Construction

q	v	k												
3	8	3	0	1	3									
4	15	4	0	1	3	7								
5	24	5	0	1	4	9	11							
7	48	7	0	5	7	18	19	22	28					
8	63	8	0	2	8	21	22	25	32	37				
9	80	9	0	1	12	16	18	25	39	44	47			
11	120	11	0	1	4	9	23	30	41	43	58	68	74	
13	168	13	0	3	11	38	40	47	62	72	88	92	93	105	111

If $\{a_1, \ldots, a_k\}$ is a (v, k) modular Golomb ruler and m and b are integers with m relatively prime to v then $\{ma_1 + b, \ldots, ma_k + b\}$ (where the arithmetic is modulo v) is also a (v, k) modular Golomb ruler. So the above constructions can be put into alternative forms.

Let $\{a_1 < a_2 < \ldots < a_k\}$ be a (v, k) modular Golomb ruler. Then $\{(a_1 - a_1) < (a_2 - a_1) < \ldots < (a_k - a_1)\}$ is also a (v, k) modular Golomb ruler. So we may assume $a_1 = 0$. Then $\{0 = a_1 < a_2 < \ldots < a_k\}$ is a k mark Golomb ruler since if the differences are distinct modulo v they are also distinct as integers. Note we can add v to obtain a $(k+1)$ mark Golomb ruler or we can shorten the ruler by deleting some

Table 3: Ruzca Construction

q	v	k												
3	6	2	0	1										
5	20	4	0	1	3	9								
7	42	6	0	1	3	11	16	20						
11	110	10	0	13	16	17	25	31	52	54	59	78		
13	156	12	0	1	3	10	18	32	38	43	59	89	93	112

marks. Furthermore Golomb rulers can be pulled apart to obtain difference triangle sets. Let $a_{11} < \ldots < a_{1h} \leq a_{21} < \ldots < a_{2h} \leq \ldots \leq a_{l1} < \ldots < a_{lh}$ be a collection of l subsequences of length h (possibly sharing endpoints only) of a Golomb ruler. Then $\{(a_{ij} - a_{i1}) | 1 \leq i \leq l, 1 \leq j \leq h\}$ is a $(l, h-1)$ difference triangle set.

Robinson and Bernstein [2] used the above idea to construct (n, k) difference triangle sets for certain small values of n and k starting with the Singer construction. Atkinson and Hassenklover noted [8] that in some cases using the Bose construction produces better results. Kløve [1] (tables II and III) gave extensive tables of the sizes of the best difference triangle sets that can be obtained in this way. This requires a computer search to determine the best choices for the form of the modular Golomb ruler construction and the way it is pulled apart into a difference triangle set. Apparently there was an error in Kløve's program as I was able to find smaller difference triangle sets in some cases (some of these are near the edges of Kløve's table and are explained as derived from larger modular Golomb rulers than he considered but others can not be explained in this way). Table 4 lists the parameters in Kløve's table II for which I found better constructions for (I, J) difference triangle sets with the above method and the respective sizes. Note even these improved constructions may not be the best known. This idea can also be used with the Ruzca construction but it does not produce any improvements for these parameters.

Ling [3] found another way to construct difference triangle sets from modular Golomb rulers. Suppose we have a (v, k) modular Golomb ruler, $X = \{a_1 < a_2 < \ldots < a_k\}$. Let $v = pw$. Then we may split X into p modular Golomb rulers $\{X_1, \ldots, X_p\}$ depending on which residue class modulo p the elements of X lie in. Let $X_i = \{a_{i1} < \ldots < a_{ik_i}\}$ for $1 \leq i \leq p$. Since the elements of X_i are congruent modulo p this means the differences will all be divisible by p. Hence we may define $\{Y_1, \ldots, Y_p\}$ where $Y_i = \{(a_{i1} - a_{i1})/p < \ldots < (a_{ik_i} - a_{i1})/p\} = \{0 = b_{i1} < \ldots < b_{ik_i}\}$ for $1 \leq i \leq p$. Clearly Y_i is a (w, k_i) modular Golomb ruler and the differences for all the $\{Y_i\}$ are disjoint. So if $h \leq \min \{k_i\}$ then $\{b_{ij} | 1 \leq i \leq k, 1 \leq j \leq h\}$ is a $(k, h-1)$ difference triangle set. Again a computer program can find the best difference triangle sets that can be constructed in this way. Note this construction will work best when the residue classes modulo p are equal (or nearly equal) in size so that h is as large as possible. Ling applied his method to the Bose modular Golomb ruler construction

Table 4: Improvements on Kløve Table II

I	J	Shearer	Kløve	I	J	Shearer	Kløve
2	23	919	921	6	40	9408	9416
2	40	2889	2928	6	41	9822	10005
2	44	3521	3526	7	6	235	242
2	49	4377	4420	7	13	1124	1138
3	8	164	166	7	19	2474	2486
3	24	1549	1551	7	23	3608	3621
3	36	3588	3589	7	32	7089	7112
3	43	5145	5217	7	35	8415	8508
3	46	5990	6017	8	12	1095	1124
4	34	4419	4444	8	29	6641	6691
4	42	6678	6751	8	30	6945	7074
4	47	8358	8386	8	31	7457	7826
4	49	9226	9283	9	5	216	220
5	9	376	379	9	11	1052	1068
5	16	1191	1210	9	14	1745	1748
5	49	11654	11723	9	27	6551	6677
5	50	12016	12316	10	5	241	246
6	9	458	462	10	19	3635	3656
6	32	5871	5977	10	24	5667	5820
6	36	7569	7606	10	25	6264	6493

only. The method does not work very well starting with the Singer construction because $v = p^2 + p + 1$ has fewer small factors and when v does have small factors the residue classes tend to have unequal sizes. However the method does work well starting with the Ruzca construction.

For example use the Ruzca construction with $p = 97$ to construct the following modular Golomb ruler with 96 elements mod 9312.

140	265	559	799	842	1018	1047	1057	1308	1390
1511	1547	1790	1791	1797	1863	1997	2067	2147	2265
2312	2514	2657	2677	2701	2848	2878	3100	3171	3475
3740	3793	3878	3896	4062	4137	4244	4292	4304	4354
4440	4443	4456	4477	4752	4867	4872	4912	4980	4989
5081	5138	5169	5243	5297	5451	5515	5578	5582	5634
5676	5775	5802	5813	5889	5908	6414	6431	6560	6660
6898	6933	6988	7002	7302	7343	7351	7483	7529	7607
7830	7960	7975	8150	8285	8397	8836	8858	9046	9125
9157	9190	9216	9218	9241	9299				

Multiply by 13 mod 9312 and reorder.

80	166	579	730	869	930	1019	1048	1073	1075
1243	1472	1487	1509	1610	1634	1806	1820	1848	1887
2013	2056	2060	2061	2120	2308	2329	2339	2443	2749
2772	2975	3052	3124	3410	3445	3518	3677	3854	3922
3975	4088	4159	4299	4429	4646	4659	4737	4746	4757
5273	5595	5679	5771	5854	5866	5904	6246	6321	6511
6605	6729	6865	6881	7036	7177	7218	7221	7267	7297
7330	7337	7382	7399	7464	7692	7726	7927	7984	8058
8064	8090	8247	8389	8604	8612	8670	8758	8868	8886
8985	9088	9107	9143	9236	9287				

Separate into residue classes mod 6.

930	1806	1848	2772	4746	5904	6246	7218	7464	7692
8058	8064	8604	8670	8868	8886				
1075	1243	2329	2443	2749	3445	4159	4429	6511	6865
7177	7267	7297	7399	7927	8389				
80	1472	1610	1634	1820	2060	2120	3410	3518	3854
4088	4646	7382	8090	8612	9236				
579	1509	1887	2013	2061	3975	4299	4659	4737	5595
5679	6321	6729	7221	8247	8985				
166	730	1048	2056	2308	3052	3124	3922	5854	5866
7036	7330	7726	7984	8758	9088				
869	1019	1073	1487	2339	2975	3677	4757	5273	5771
6605	6881	7337	9107	9143	9287				

Translate each residue class to include 0 and then divide the elements by 6. Now all the differences (positive and negative within each class) are distinct mod 1552.

0	146	153	307	636	829	886	1048	1089	1127
1188	1189	1279	1290	1323	1326				
0	28	209	228	279	395	514	559	906	965
1017	1032	1037	1054	1142	1219				
0	232	255	259	290	330	340	555	573	629
668	761	1217	1335	1422	1526				
0	155	218	239	247	566	620	680	693	836
850	957	1025	1107	1278	1401				
0	94	147	315	357	481	493	626	948	950
1145	1194	1260	1303	1432	1487				
0	25	34	103	245	351	468	648	734	817
956	1002	1078	1373	1379	1403				

Rotate the classes mod 1552 so that the largest gap is at the end.

0	193	250	412	453	491	552	553	643	654
687	690	916	1062	1069	1223				
0	59	111	126	131	148	236	313	646	674
855	874	925	1041	1160	1205				
0	118	205	309	335	567	590	594	625	665
675	890	908	964	1003	1096				
0	54	114	127	270	284	391	459	541	712
835	986	1141	1204	1225	1233				
0	2	197	246	312	355	484	539	604	698
751	919	961	1085	1097	1230				
0	6	30	179	204	213	282	424	530	647
827	913	996	1135	1181	1257				

The classes now form (considered as integers) a $(6, 15)$ difference triangle set with maximum difference 1257. Hence $M(6, 15) \leq 1257$.

Table 5 shows bounds on $M(I, J)$ found similarly.

Table 5: Upper bounds from Ling idea applied to Ruzca construction

p	I	J	$M(I,J)$	p	I	J	$M(I,J)$
97	6	15	1257	151	10	14	1879
113	7	15	1492	181	12	14	2272
113	8	13	1318	193	12	15	2627
127	9	13	1430				

Note since in the above example the original modular Golomb ruler did not have differences that are multiples of 96 this difference triangle set does not have any differences that are multiples of 16. This means we can add 16 times a Golomb ruler with 16 marks to the difference triangle set obtaining a $(7, 15)$ difference triangle set. This is not interesting in this example as the enlarged difference triangle set will have a maximum difference of at least 16*177=2832. However in some cases it is. For example Ling [5] used the Bose construction (with $p = 223$) to find a $(14, 14)$ difference triangle set with maximum difference 2630 (showing $M(14, 14) \leq 2630$). It turns out this difference triangle set has no differences which are multiples of 16. Thus we can add to it 16 times a 15 mark Golomb ruler. The smallest such ruler has length 151 so the multiple has length 16*151=2416. So we now have $M(15, 14) \leq 2630$. Furthermore we can reoptimize the $(14, 14)$ difference triangle set assuming we will replace one of the rulers with the multiple of the Golomb ruler obtaining $M(14, 14) \leq 2595$. Similarly the $M(12, 14) \leq 2272$ bound above leads to the bound

$M(13,14) \le 2272$ and the improved bound $M(12,14) \le 2265$ by adding 15 times the 15 mark Golomb ruler with length 151.

The above difference triangle set constructions are based on pulling apart a modular Golomb ruler. We can also build up difference triangle sets from a modular Golomb ruler. The following constructions are based on and extend a construction in [4]. We say an n by m integer matrix A has property S if $(a_{ik} - a_{il}) \neq (a_{jk} - a_{jl})$ for all $1 \le i < j \le n$ and $1 \le k < l \le m$. Let $B = \{b_1, ..., b_m\}$ be a (v, m) modular Golomb ruler. Let $c_{ij} = b_j + v \times a_{ij}$ for all $1 \le i \le n$ and $1 \le j \le m$. We claim the c_{ij} form a $(n, m-1)$ difference triangle set, C. For suppose $(c_{ik_1} - c_{il_1}) = (c_{jk_2} - c_{jl_2})$. Then $(b_{k_1} - b_{l_1}) + v \times (a_{ik_1} - a_{il_1}) = (b_{k_2} - b_{l_2}) + v \times (a_{jk_2} - a_{jl_2})$ Since B is a modular Golomb ruler we must have $k_1 = k_2 = k$ and $l_1 = l_2 = l$. So $(a_{ik} - a_{il}) \times v = (a_{jk} - a_{jl}) \times v$ or dividing by v $(a_{ik} - a_{il}) = (a_{jk} - a_{jl})$ Since A has property S this is a contradiction.

Note if the modular Golomb rulers are from the Bose or Ruzca constructions the difference triangle sets constructed above will not contain any differences which are multiples of $q + 1$ (Bose) or p or $p - 1$ (Ruzca). These difference triangle sets can be extended by adding suitable multiples of Golomb rulers.

Matrices A with property S can be generated as follows. Let p be a prime. Let $A = \{a_{ij} \equiv (i-1) \times (j-1) \bmod p | 1 \le i, j \le p\}$ Then A is a p by p matrix with property S. For $((a_{ik} - a_{il}) - (a_{jk} - a_{jl})) \equiv (i-j) \times (k-l) \bmod p$ which is $\neq 0$ unless $i = j$ or $k = l$. Furthermore we can add constants to any row or column of A modulo p or permute the rows and columns of A and A will still have property S. Finally any submatrix of A will also have property S. In this way a large number of matrices with property S can be generated.

Next we discuss how to heuristically choose matrices with property S which will work well in the above construction for difference triangle sets. We will assume the elements of the modular Golomb ruler have been sorted in increasing order. And we will assume A is chosen so that the minimum element in each row is 0. Then the maximum difference from that row will be between the positions of the rightmost maximum sized element in that row and the leftmost 0 in that row. So it will generally be better if the size of maximum element in A is as small as possible, and if the number of pairs consisting of 0 and an element of maximum size in the same row is as small as possible and if the columns of A have been permuted so that the maximum difference between the positions of the maximum size element and 0 is as large as possible.

For example computer searches with $p = 11, 13$ produced the following square arrays with property S heuristically optimized as discussed above.

$$
A11 = \begin{array}{ccccccccccc}
0 & 5 & 0 & 2 & 7 & 4 & 7 & 4 & 2 & 1 & 1 \\
0 & 7 & 4 & 7 & 2 & 0 & 5 & 1 & 1 & 4 & 2 \\
4 & 2 & 1 & 5 & 1 & 0 & 7 & 2 & 4 & 0 & 7 \\
1 & 1 & 2 & 7 & 4 & 4 & 2 & 7 & 0 & 0 & 5 \\
2 & 4 & 7 & 2 & 0 & 1 & 1 & 5 & 0 & 4 & 7 \\
7 & 0 & 5 & 1 & 0 & 2 & 4 & 7 & 4 & 1 & 2 \\
5 & 0 & 7 & 4 & 4 & 7 & 0 & 2 & 1 & 2 & 1 \\
7 & 4 & 2 & 0 & 1 & 5 & 0 & 1 & 2 & 7 & 4 \\
2 & 1 & 1 & 0 & 2 & 7 & 4 & 4 & 7 & 5 & 0 \\
1 & 2 & 4 & 4 & 7 & 2 & 1 & 0 & 5 & 7 & 0 \\
4 & 7 & 0 & 1 & 5 & 1 & 2 & 0 & 7 & 2 & 4
\end{array}
$$

$$
A13 = \begin{array}{ccccccccccccc}
0 & 1 & 8 & 5 & 3 & 7 & 8 & 1 & 0 & 4 & 7 & 3 & 5 \\
0 & 5 & 3 & 1 & 8 & 8 & 7 & 4 & 7 & 1 & 0 & 5 & 3 \\
7 & 3 & 5 & 4 & 7 & 3 & 0 & 1 & 8 & 5 & 0 & 1 & 8 \\
8 & 8 & 1 & 1 & 0 & 5 & 0 & 5 & 3 & 3 & 7 & 4 & 7 \\
3 & 7 & 4 & 5 & 0 & 1 & 7 & 3 & 5 & 8 & 8 & 1 & 0 \\
5 & 0 & 1 & 3 & 7 & 4 & 8 & 8 & 1 & 7 & 3 & 5 & 0 \\
1 & 0 & 5 & 8 & 8 & 1 & 3 & 7 & 4 & 0 & 5 & 3 & 7 \\
4 & 7 & 3 & 7 & 3 & 5 & 5 & 0 & 1 & 0 & 1 & 8 & 8 \\
1 & 8 & 8 & 0 & 5 & 3 & 1 & 0 & 5 & 7 & 4 & 7 & 3 \\
5 & 3 & 7 & 0 & 1 & 8 & 4 & 7 & 3 & 8 & 1 & 0 & 5 \\
3 & 5 & 0 & 7 & 4 & 7 & 1 & 8 & 8 & 3 & 5 & 0 & 1 \\
8 & 1 & 0 & 8 & 1 & 0 & 5 & 3 & 7 & 5 & 3 & 7 & 4 \\
7 & 4 & 7 & 3 & 5 & 0 & 3 & 5 & 0 & 1 & 8 & 8 & 1
\end{array}
$$

We illustrate these ideas by constructing a difference triangle set which shows $M(12,10) \leq 892$. Let $B = \{0,3,5,16,33,37,47,55,56,62,82\}$ which is a (120,11) modular Golomb ruler obtained by the Bose construction put into the form that works best. Note the differences don't contain any multiples of 12. Using $A11$ above we obtain.

0	603	5	256	873	517	887	535	296	182	202
0	843	485	856	273	37	647	175	176	542	322
480	243	125	616	153	37	887	295	536	62	922
120	123	245	856	513	517	287	895	56	62	682
240	483	845	256	33	157	167	655	56	542	922
840	3	605	136	33	277	527	895	536	182	322
600	3	845	496	513	877	47	295	176	302	202
840	483	245	16	153	637	47	175	296	902	562
240	123	125	16	273	877	527	535	896	662	82
120	243	485	496	873	277	167	55	656	902	82
480	843	5	136	633	157	287	55	896	302	562

Sorting the rows and translating so the first element is zero we obtain.

0	5	182	202	256	296	517	535	603	873	887
0	37	175	176	273	322	485	542	647	843	856
0	25	88	116	206	258	443	499	579	850	885
0	6	64	67	189	231	457	461	626	800	839
0	23	124	134	207	223	450	509	622	812	889
0	30	133	179	274	319	524	533	602	837	892
0	44	173	199	292	299	493	510	597	842	874
0	31	137	159	229	280	467	546	621	824	886
0	66	107	109	224	257	511	519	646	861	880
0	27	65	112	188	222	430	441	601	818	847
0	50	131	152	282	297	475	557	628	838	891

The differences don't include any multiples of 12. Therefore we can add 12 times the Golomb ruler $\{0, 1, 4, 13, 28, 33, 47, 54, 64, 70, 72\}$ obtaining finally.

0	5	182	202	256	296	517	535	603	873	887
0	37	175	176	273	322	485	542	647	843	856
0	25	88	116	206	258	443	499	579	850	885
0	6	64	67	189	231	457	461	626	800	839
0	23	124	134	207	223	450	509	622	812	889
0	30	133	179	274	319	524	533	602	837	892
0	44	173	199	292	299	493	510	597	842	874
0	31	137	159	229	280	467	546	621	824	886
0	66	107	109	224	257	511	519	646	861	880
0	27	65	112	188	222	430	441	601	818	847
0	50	131	152	282	297	475	557	628	838	891
0	12	48	156	336	396	564	648	768	840	864

This is a $(12, 10)$ difference triangle set with maximum element 892 which shows $M(12, 10) \leq 892$.

Table 6 lists new upper bounds on $M(I, J)$ obtained by this type of construction. Exhaustive searches over all possible constructions did not appear to be feasible so the A matrices were chosen heuristically as discussed above. Therefore more extensive searches might find improvements on these bounds. However I would expect any such improvements to be modest.

Difference triangle sets with no particular structure can be found by computer searches. A program for doing such searches (partial or exhaustive) was described in [9]. Additional searches using similar programs have found the bounds listed in Table 7. The searches for $M(3, 8)$ and $M(3, 9)$ were run by Doug Fortune using a program obtained from my website.

Table 6: Upper bounds from extended CFJ construction

I	J	$M(I,J)$	I	J	$M(I,J)$	I	J	$M(I,J)$	I	J	$M(I,J)$
10	9	631	12	9	746	13	11	1113	14	13	2205
11	8	532	12	10	892	13	12	1413	14	15	3082
11	9	672	12	11	1102	14	8	659	15	9	915
11	10	877	12	12	1402	14	9	906	15	11	1316
11	11	1097	13	8	604	14	10	998	15	12	1978
11	12	1392	13	9	755	14	11	1301	15	13	2220
12	8	589	13	10	989	14	12	1484	15	15	3092

Table 7: Upper bounds from computer search

I	J	$M(I,J)$	I	J	$M(I,J)$	I	J	$M(I,J)$	I	J	$M(I,J)$
2	10	153	3	11	309	5	11	527	6	11	638
2	12	244	4	6	94	5	12	660	6	12	797
3	8	134	4	9	253	6	5	95	8	12	1076
3	9	181	4	10	328	6	6	145	11	4	111
3	10	238	5	7	171	6	7	210	12	4	122

In some cases a modified version of this search program proved more effective. Starting with a good difference triangle set the program searches for additional good difference triangle sets by deleting some of the Golomb rulers constituting the difference triangle set and attempting to complete the difference triangle set in alternative ways. The procedure is then repeated on any additional good difference triangle sets found. Bounds found by the modified search program are listed in Table 8.

Table 8: Upper bounds from modified computer search

I	J	$M(I,J)$	I	J	$M(I,J)$	I	J	$M(I,J)$	I	J	$M(I,J)$
4	7	136	7	11	740	9	12	1213	13	5	214
4	8	187	8	5	129	10	5	164	13	6	324
5	6	119	8	6	197	10	6	248	13	7	464
5	8	238	7	10	587	9	10	761	12	6	299
5	9	317	8	7	282	9	11	956	12	7	431
5	10	412	8	8	395	10	7	352	13	4	133
6	8	288	8	9	521	10	8	497	14	5	231
6	9	386	8	10	672	10	10	840	14	6	352
6	10	492	8	11	844	10	11	1067	14	7	499
7	5	113	9	5	146	10	12	1348	15	4	154
7	6	171	9	6	222	11	5	181	15	5	250
7	7	251	9	7	318	11	7	393	15	6	374
7	8	339	9	8	447	11	13	1782	15	8	735
7	9	456	9	9	585	12	5	197	15	10	1268

References

[1] T. Kløve, "Bounds and Constructions for Difference Triangle Sets", *IEEE Transactions on Information Theory*, vol. It-35, pp. 879-886, 1989.

[2] J.P. Robinson and A. J. Bernstein, "A class of binary recurrent codes with limited error propagation", *IEEE Transactions on Information Theory*, vol. It-13, pp. 106-113, 1967.

[3] A. C. H. Ling, "Difference Triangle sets from affine planes", *IEEE Transactions on Information Theory*, vol. 48, 2399-2401, 2002.

[4] Z. Chen, P. Fan and F. Jin, "Disjoint Diffence sets, Difference Triangle Sets and Related Codes", *IEEE Transactions on Information Theory*, vol. It-38, pp. 518-522, 1992.

[5] J. Singer, "A theorem in projective geometry and some applications to number theory", *Transactions of the American Mathematical Society*, vol. 43, 377-385, 1938.

[6] R. C. Bose, "An affine analogue of Singer's theorem", *J. Ind. Math. Soc.*, vol. 6, 1-15, 1942.

[7] I. Z. Ruzca, "Solving a linear equation in a set of integers I", *Acta Arithmetica*, vol. 65, 259-282, 1993.

[8] M.D. Atkison and A. Hassenklover "Sets of Integers with Distinct Differences" School of Computer Science, Carlton University, Ottawa Ontario, Canada, Report SCS-TR-63, August 1984.

[9] J.B. Shearer, "Some New Difference Triangle Sets", *The Journal of Combinatorics Mathematics and Combinatorial Computing*, vol. 27, pp. 65-76, 1998.