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On Mixing Inequalities: Rank, Closure and Cutting Plane Proofs

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Abstract

We study the mixing inequalities which were introduced by Günlük and Pochet (2001). We show that a mixing inequality which mixes n MIR inequalities has MIR rank at most n if it is a type I mixing inequality and at most $n-1$ if it is a type II mixing inequality. We also show that these bounds are tight for $n = 2$.

Given a mixed-integer set $P_I = P \cap Z(I)$ where P is a polyhedron and $Z(I) = \{x \in \mathbb{R}^n : x_i \in \mathbb{Z}\ \forall i \in I\}$, we define mixing inequalities for P_I. We show that the elementary mixing closure of P with respect to I can be described using a bounded number of mixing inequalities, each of which has a bounded number of terms. This implies that the elementary mixing closure of P is a polyhedron.

Finally, we show that any mixing inequality can be derived via a polynomial length MIR cutting plane proof. Combined with results of Dash (2006) and Pudlák (1997), this implies that there are valid inequalities for a certain mixed-integer set that cannot be obtained via a polynomial-size mixing cutting-plane proof.

1 Introduction

Günlük and Pochet [6] study the polyhedral structure of the so-called *mixing* set

$$S \;=\; \Big\{ s \in \mathbb{R}_+, z \in \mathbb{Z}^n \;:\; s + z_k \geq b_k \quad \text{for } k = 1, \ldots, n \Big\}$$

where $0 < b_1 < b_2 < \ldots < b_n \leq 1$, and show that the following *n-term mixing inequalities* are valid for S:

$$s + b_1 z_1 + \sum_{k=2}^{n} (b_k - b_{k-1}) z_k \;\geq\; b_n, \tag{1}$$

$$s + (b_1 + 1 - b_n) z_1 + \sum_{k=2}^{n} (b_k - b_{k-1}) z_k \;\geq\; b_n. \tag{2}$$

Note that if $b_n = 1$, inequalities (1) and (2) are identical. These inequalities are called "mixing" inequalities as they combine, or *mix*, mixed-integer rounding (MIR) inequalities based on individual constraints, namely

$$s + b_k z_k \geq b_k, \tag{3}$$

for $k = 1, \ldots, n$. Notice that if $n = 1$, the mixing inequality (1) becomes the MIR inequality (3) and the mixing inequality (2) simply becomes the inequality defining S. MIR inequalities were introduced by Nemhauser and Wolsey [7] and they are obtained by applying a simple procedure to any implied (base) inequality. Nemhauser and Wolsey [8] also showed that split cuts, introduced by Cook, Kannan and Schrijver [2] are equivalent to MIR inequalities. The inequalities (3) are not the only MIR inequalities for S; see [5] for a recent study on MIR inequalities.

The original description of the mixing set and the mixing inequalities does not assume $0 < b_i \leq 1$ but for the sake of simplicity, and without loss of generality, we make this assumption. Notice that in the definition of S, the z_i variables do not have lower bounds and therefore if $b_i \notin (0, 1]$ for some i, it possible to simply "shift" variable z_i by $\lceil b_i \rceil - 1$ and then replace b_i with $b_i - \lceil b_i \rceil + 1$ to obtain $b_i \in (0, 1]$. The mixing inequalities (1) and (2) are identical to the ones presented in [6] after this simple transformation. We note that MIR inequalities are known to be invariant under "shifting", and more generally, under unimodular transformations [5].

Let $I' = \{i_1, \ldots, i_t\}$ be a subset of $\{1, \ldots, n\}$ and let $proj_{[I']}(S)$ denote the projection of S in the space of the s and z_i variables for $i \in I'$. In other words, $proj_{[I']}(S) = \{s \in \mathbb{R}_+, z \in \mathbb{Z}^{|I'|} : s + z_k \geq b_k, \ \forall k \in I'\}$. Clearly the mixing inequalities

$$s + b_{i_1} z_{i_1} \qquad + \sum_{k=2}^{|I'|} (b_{i_k} - b_{i_{k-1}}) z_{i_k} \ \geq \ b_{i_{|I'|}}, \tag{4}$$

$$s + (b_{i_1} + 1 - b_{i_{|I'|}}) z_{i_1} + \sum_{k=2}^{|I'|} (b_{i_k} - b_{i_{k-1}}) z_{i_k} \ \geq \ b_{i_{|I'|}} \tag{5}$$

for $proj_{[I']}(S)$ are also valid for S. We refer to inequality (4) as $mix1_{I'}$, and to inequality (5) as $mix2_{I'}$, and say that these are $|I'|$-term mixing inequalities, of type I and type II, respectively. Günlük and Pochet in [6] show that $mix1_{I'}$ and $mix2_{I'}$ are facet defining for the convex hull of S for all $I' \subseteq \{1 \ldots, n\}$, and these inequalities completely describe the convex hull of S.

One of our main contributions in this paper is to show how mixing inequalities can be obtained by repeatedly applying the MIR procedure. More precisely, we show that the inequality $mix1_{I'}$ has MIR-rank at most $|I'|$ and $mix2_{I'}$ has MIR-rank at most $|I'| - 1$. In addition, for $|I'| = 2$, we show that these bounds can be tight. We also show that the inequalities $mix1_{I'}$ and $mix2_{I'}$ have MIR cutting-plane proofs of length $O(|I'|^2)$ from the inequalities defining S. This result, when combined with a recent result by Dash [3], implies that cutting-plane proofs of infeasibility for mixed-integer sets using mixing inequalities have exponential worst-case complexity.

In addition, we define mixing inequalities for general mixed-integer sets and formulate the separation problem as a quadratic mixed-integer program. Using this formulation, we then study the elementary closure of mixing inequalities and show that it is polyhedral. More precisely, we show that the mixing closure of any given mixed integer set can be described using a bounded number of mixing inequalities each of which has a bounded number of terms.

The rest of the paper is organized as follows: we next define MIR inequalities and give some of their well-known properties. We present our results on the rank of mixing inequalities in Section 2. In Section 3 we define mixing inequalities for general sets and study some of their basic properties. In Section 4 we define the elementary closure of mixing inequalities for general sets and show that it is polyhedral. Finally, in Section 5, we give a polynomial length MIR cutting-plane proof of mixing inequalities.

1.1 Preliminaries

Wolsey [10] defines a two variable mixed-integer set $Q = \left\{s \in \mathbb{R},\ z \in \mathbb{Z}\ :\ s + z\ \geq\ b,\quad s \geq 0\right\}$ and shows that the *basic mixed-integer* inequality

$$s + \hat{b}z \geq \hat{b}\,\lceil b \rceil\,, \tag{6}$$

where $\hat{b} = b - (\lceil b \rceil - 1)$ is valid and facet-defining for Q. This observation can be used to generate valid inequalities for a general mixed-integer set $P_I = P \cap Z(I)$ where

$$P = \{x \in \mathbb{R}^n\ :\ Ax \geq b\ \}, \qquad Z(I) = \{x \in \mathbb{R}^n\ :\ x_i \in \mathbb{Z}\ \forall i \in I\},$$

and I is a subset of $\{1, \ldots, n\}$. We assume that any non-negativity constraints on some of the variables are included in the system $Ax \geq b$. Let $v = Ax - b$ and note that $v \geq 0$ for all $x \in P$. Assume A has m rows and let $\lambda \in \mathbb{R}^m$ be a row vector such that $(\lambda A)_i$ is (a) integral for $i \in I$, and (b) 0 for $i \notin I$. Define λ^+ by $\lambda_i^+ = \max\{\lambda_i, 0\}$. Then the equation $-\lambda v + (\lambda A)x = \lambda b$ is valid for P and so is the inequality

$$(-\lambda)^+ v + (\lambda A)x \geq \lambda b. \tag{7}$$

In addition, for all points in P_I, $(-\lambda)^+ v$ is non-negative and $(\lambda A)x$ is integral. Let $\beta = \lambda b$. The basic mixed-integer inequality implies that $(-\lambda)^+ v + \hat{\beta}(\lambda A)x \geq \hat{\beta}\,\lceil \beta \rceil$, or equivalently

$$(-\lambda)^+(Ax - b) + \hat{\beta}(\lambda A)x \geq \hat{\beta}\,\lceil \beta \rceil\,, \tag{8}$$

is a valid inequality for points in P_I (recall that $\hat{\beta} = \beta - (\lceil \beta \rceil - 1)$). This inequality is the *mixed-integer rounding (MIR)* inequality generated by λ. Inequality (7) is called the base inequality of the MIR inequality. See [5] for other ways of defining the MIR inequality, and the equivalence of (8) with the definition in [8].

Some well known properties of the MIR inequalities are the following:

1. The MIR inequality (8) generated by $-\lambda$ is equivalent to the one generated by λ.
2. If a point $x^* \in P$ violates the MIR inequality (8), then
   (a) x^* satisfies $\lfloor \beta \rfloor < (\lambda A)x^* < \lceil \beta \rceil$;
   (b) $\lambda^+ v^* < 1$, where $v^* = Ax^* - b$.

An inequality $cx \geq d$ is called a *split cut* for P_I if $cx \geq d$ is satisfied by points in $P \cap \{\alpha x \leq \beta\}$ and $P \cap \{\alpha x \geq \beta + 1\}$, where α, β are integral and $\alpha_i = 0$ for $i \notin I$. We say that $cx \geq d$ is derived using the *disjunction* $\alpha x \leq \beta \vee \alpha x \geq \beta + 1$. It is known that for $\hat{\beta} \neq 0$, the inequality (8) is a split cut for P_I derived using the disjunction $(\lambda A)x \leq \lfloor \beta \rfloor \vee (\lambda A)x \geq \lceil \beta \rceil$. In addition, every split cut for P_I is also an MIR inequality generated by some $\lambda \in \mathbb{R}^m$ [8].

2 MIR rank of mixing inequalities

The elementary MIR closure of P with respect to I, denoted by $P^{[1]}$, is the set of points in P that satisfy all MIR inequalities that can be generated using the inequalities defining P and the integrality of the variables x_i for $i \in I$. It is known that $P^{[1]}$ is a polyhedral set [2] and therefore it suffices to consider only a finite number of (un-dominated) MIR inequalities to obtain the MIR closure. For any integer $t \geq 2$, let $P^{[t]}$ be the MIR closure of $P^{[t-1]}$ with respect to I. Define $P^{[0]} = P$. We say that a valid inequality for P_I has MIR-rank t, for some integer $t \geq 1$, if it is valid for $P^{[t]}$, but not valid for $P^{[t-1]}$. If a valid inequality is implied by $Ax \geq b$, then it has MIR-rank 0.

2.1 MIR rank of type I mixing inequalities

We next study the MIR rank of type I mixing inequalities and show that the rank of the $|I'|$-term mixing inequality (4) is at most $|I'|$. For simplicity, we use $S^{[t]}$ to denote $S_{LP}^{[t]}$ where S_{LP} stands for the continuous relaxation of S. We start with analyzing inequality (1).

Theorem 2.1 *Inequality $mix1_{\{1,\ldots,n\}}$ is valid for $S^{[n]}$ and therefore it has MIR rank at most n.*

Proof. A mixing inequality with only one term (e.g., $mix1_{\{1\}}$) is just an MIR inequality and has MIR rank 1. Assume the theorem is true for mixing inequalities with $n-1$ terms. We next show that $mix1_{\{1,\ldots,n\}}$ is a split cut for $S^{[n-1]}$ derived from the disjunction:

$$(z_1 \geq z_2) \ \vee \ (z_1 \leq z_2 - 1).$$

As every split cut for $S^{[n-1]}$ is also an MIR cut for $S^{[n-1]}$, the theorem will follow.

For any point $(\bar{s}, \bar{z})$ in $S^{[n-1]}$ which satisfies $z_1 \geq z_2$,

$$\bar{s} + b_1\bar{z}_1 + \sum_{k=2}^{n}(b_k - b_{k-1})\bar{z}_k \geq \bar{s} + b_2\bar{z}_2 + \sum_{k=3}^{n}(b_k - b_{k-1})\bar{z}_k \geq b_n.$$

The second inequality above is true as points in $S^{[n-1]}$ satisfy $mix1_{\{2,\ldots,n\}}$.

We now consider a point $(\bar{s}, \bar{z})$ in $S^{[n-1]}$ which satisfies $z_1 \leq z_2 - 1$.
Case 1: Assume $(\bar{s}, \bar{z})$ satisfies $z_2 \leq z_k$ for all $k = 3, \ldots, n$. This fact, along with the inequality $z_1 \leq z_2 - 1$, implies that $\bar{z}_1 + 1 \leq \bar{z}_k$ for $k \geq 2$. Therefore

$$\bar{s} + b_1\bar{z}_1 + \sum_{k=2}^{n}(b_k - b_{k-1})\bar{z}_k \quad \geq \quad \bar{s} + b_1\bar{z}_1 + \sum_{k=2}^{n}(b_k - b_{k-1})(\bar{z}_1 + 1) = \bar{s} + b_n\bar{z}_1 + b_n - b_1$$

If $\bar{z}_1 \geq 0$, then using $s + b_1 z_1 \geq b_1$, we have

$$\bar{s} + b_n \bar{z}_1 + b_n - b_1 \geq \bar{s} + b_1 \bar{z}_1 + b_n - b_1 \geq b_n$$

and therefore $(\bar{s}, \bar{z})$ satisfies (1).

If, on the other hand, $\bar{z}_1 \leq 0$, then using $s + z_1 \geq b_1$, we have

$$\bar{s} + b_n \bar{z}_1 + b_n - b_1 = \bar{s} + \bar{z}_1 - (1 - b_n)\bar{z}_1 + b_n - b_1 \geq \bar{s} + \bar{z}_1 + b_n - b_1 \geq b_n$$

and therefore $(\bar{s}, \bar{z})$ satisfies (1).

Case 2: Assume that $(\bar{s}, \bar{z})$ satisfies $z_2 > z_k$ for some $k \in \{3, \ldots, m\}$, and let t be the smallest index in $\{3, \ldots, m\}$ for which this is true. Then $\bar{z}_k > \bar{z}_t$ for $k = 2, \ldots, t-1$. This implies that

$$\bar{s} + b_1 \bar{z}_1 + \sum_{k=2}^{n} (b_k - b_{k-1}) \bar{z}_k \quad \geq \quad \bar{s} + b_1 \bar{z}_1 + (b_t - b_1)\bar{z}_t + \sum_{k=t+1}^{n} (b_k - b_{k-1}) \bar{z}_k \geq b_n$$

as the second inequality follows from $mix1_{\{1,t,\ldots,n\}}$, which is satisfied by points in $S^{[n-1]}$. ■

We proved above that $mix1_{\{1,\ldots,n\}}$ is a split cut, and therefore an MIR cut, for a set S' defined by the inequalities $s \geq 0$, $s + z_1 \geq b_1$, $s + b_1 z_1 \geq b_1$, $mix1_{\{2,\ldots,n\}}$ and $mix1_{\{1,k,\ldots,n\}}$ for $k = 3, \ldots, n$ and contained in $S^{[n-1]}$.

We note that there are alternative derivations of $mix1_{\{1,\ldots,n\}}$ as a split cut for $S^{[n-1]}$. For example, one can replace inequalities $mix1_{\{1,k,\ldots,n\}}$ for $k = 3, \ldots, n$ with $mix1_{\{1,\ldots,k-1,k+1,\ldots,n\}}$ for $k = 2, \ldots, n-1$ in the definition of S' above. (To see this, modify the previous proof by assuming in *Case 2* that $\bar{z}_k > \bar{z}_{k+1}$ for some $k \in \{2, \ldots, n-1\}$, and by assuming in *Case 1* that $\bar{z}_2 \leq \ldots \leq \bar{z}_n$.) This derivation, however, leads to an exponential length MIR cutting plane proof of $mix1_{\{1,\ldots,n\}}$, whereas, the first one leads to a polynomial length MIR cutting plane proof (discussed in Section 5).

We next explicitly give the multipliers λ and show the derivation of $mix1_{\{1,\ldots,n\}}$ as an MIR inequality (8) from the inequalities defining S' above. Towards this end, we convert the above inequalities to equations by adding non-negative slack variables in the following manner:

$$s + \quad z_1 - v_1 \;=\; b_1, \qquad (M_1)$$

$$s + b_1 z_1 - v_2 \;=\; b_1, \qquad (M_2)$$

$$s + b_1 z_1 + (b_k - b_1) z_k + \sum_{i=k+1}^{n} (b_i - b_{i-1}) z_i - v_k = b_n \;\; (k = 3, \ldots, n), \qquad (M_k)$$

$$s + b_2 z_2 + \sum_{i=3}^{n} (b_i - b_{i-1}) z_i - v_{n+1} = b_n. \qquad (M_{n+1})$$

Note that (M_1)-(M_{n+1}) involve $n-1$ or fewer variables from $z_1, \ldots, z_n$.

We define a multiplier λ_k for inequality (M_k) for $k = 1, \ldots, n+1$ and use these multipliers to obtain a base inequality (7) such that the MIR inequality (8) equals inequality (1). First, let

$$\mu_k = \begin{cases} (b_2 - b_1)(\frac{1}{1 - b_1}) & \text{if } k = 1, \\ (b_2 - b_1)(\frac{1}{b_n - b_1} - \frac{1}{1 - b_1}) & \text{if } k = 2, \\ (b_2 - b_1)(\frac{1}{b_{k-1} - b_1} - \frac{1}{b_k - b_1}) & \text{for } k = 3, \ldots, n, \\ -1 & \text{if } k = n+1, \end{cases}$$

and note that

$$\sum_{k=3}^{p} \mu_k = (b_2 - b_1) \sum_{k=3}^{p} (\frac{1}{b_{k-1} - b_1} - \frac{1}{b_k - b_1}) = 1 - \frac{b_2 - b_1}{b_p - b_1}.$$

Furthermore

$$\sum_{k=1}^{n} \mu_k = \frac{b_2 - b_1}{1 - b_1} + (b_2 - b_1)(\frac{1}{b_n - b_1} - \frac{1}{1 - b_1}) + 1 - \frac{b_2 - b_1}{b_n - b_1} = 1.$$

Now consider $\sum_{k=1}^{n} \mu_k (M_k)$ and denote it by

$$\alpha_0 s + \sum_{k=1}^{n} \alpha_k z_k - \sum_{k=1}^{n} \mu_k v_k = \beta. \tag{9}$$

Note that $\alpha_0 = \sum_{k=1}^{n} \mu_k = 1$, and $\alpha_2 = 0$. In addition,

$$\alpha_1 = \mu_1 + b_1 \sum_{k=2}^{n} \mu_k = \mu_1(1 - b_1) + b_1 \sum_{k=1}^{n} \mu_k = (1 - b_1)\frac{b_2 - b_1}{1 - b_1} + b_1 = b_2.$$

For $k = 3, \ldots, n$

$$\begin{aligned} \alpha_k &= \sum_{l=3}^{k-1} \mu_l (b_k - b_{k-1}) + \mu_k (b_k - b_1) \\ &= (b_k - b_{k-1})(1 - \frac{b_2 - b_1}{b_{k-1} - b_1}) + (b_2 - b_1)(\frac{1}{b_{k-1} - b_1} - \frac{1}{b_k - b_1})(b_k - b_1) \\ &= (b_k - b_{k-1}) - \frac{(b_k - b_{k-1})(b_2 - b_1)}{b_{k-1} - b_1} + \frac{(b_2 - b_1)(b_k - b_1)}{b_{k-1} - b_1} - (b_2 - b_1) \\ &= b_k - b_{k-1}. \end{aligned}$$

Finally,

$$\beta = b_n \sum_{k=3}^{n} \mu_k + b_1(\mu_1 + \mu_2) = b_n(1 - \frac{b_2 - b_1}{b_n - b_1}) + b_1 \frac{b_2 - b_1}{b_n - b_1} = b_n - (b_2 - b_1)$$

and therefore equation (9) is the same as

$$s + b_2 z_1 + \sum_{k=3}^{n} (b_k - b_{k-1}) z_k - \sum_{k=1}^{n} \mu_k v_k = b_n - (b_2 - b_1). \tag{10}$$

Further, $\sum_{i=1}^{n+1} \mu_k M_k$ equals

$$b_2(z_1 - z_2) + v_{n+1} - \sum_{k=1}^{n} \mu_k v_k = -(b_2 - b_1).$$

Therefore, if we define $\lambda_k = \mu_k / b_2$, and drop terms with negative coefficients for v_k variables in $\sum_{i=1}^{n+1} \lambda_k M_k$ we get

$$(z_1 - z_2) + v_{n+1}/b_2 \geq -(b_2 - b_1)/b_2.$$

If we let γ stand for the right-hand-side of the inequality above, then $\hat{\gamma} = b_1/b_2$. Applying the basic mixed-integer inequality, we get $(b_1/b_2)(z_1 - z_2) + v_{n+1}/b_2 \geq 0$ or

$$b_1(z_1 - z_2) + v_{n+1} \geq 0$$

as an MIR inequality for S. Substituting out v_{n+1} in the previous inequality using (M_{n+1}), we get $mix1_{\{1,\ldots,n\}}$.

Notice that Theorem 2.1 also implies that inequality $mix1_{I'}$ has MIR rank at most $|I'|$ for $proj_{[I']}(S)$. Remember that $proj_{[I']}(S) = \{s \in \mathbb{R}_+, z \in \mathbb{Z}^{|I'|} \ : \ s + z_k \geq b_k, \ k \in I'\}$. As all inequalities defining $proj_{[I']}(S)$ are present in the definition of S, we can make the following observation.

Corollary 2.2 *The $|I'|$-term mixing inequality $mix1_{I'}$ has MIR rank at most $|I'|$.*

We next show that the upper bound on rank can be tight for two-term mixing inequalities of type I. Let

$$S_2 = \left\{ s \in \mathbb{R}_+, z \in \mathbb{Z}^2 \ : \ s + z_1 \geq b_1; \ s + z_2 \geq b_2 \ \right\}$$

and remember that the 2-term mixing inequality for S_2 is $s + b_1 z_1 + (b_2 - b_1) z_2 \geq b_2$.

Theorem 2.3 *If $0 < b_1 < b_2 < 1/2$, then the 2-term mixing inequality $mix1_{\{1,2\}}$ for S_2 has MIR rank 2.*

Proof. We will construct a point (s^*, z^*) which satisfies all MIR cuts, but violates $mix1_{\{1,2\}}$. Choose $\delta > 0$ such that $b_2 + 2\delta < 1/2$, and set $z_2^* = 1 - \delta$ and $z_1^* = 1 - 2\delta$. For any $s^* \geq 0$, $(s^*, z^*) \in S_2^{LP}$, the LP relaxation of S_2. We will now choose s^* such that the MIR inequalities $s + b_1 z_1 \geq b_1$ and $s + b_2 z_2 \geq b_2$ are satisfied by (s^*, z^*), but it violates $mix1_{\{1,2\}}$. Now

$$b_2 - (b_2 - b_1) z_2^* - b_1 z_1^* = b_2(1 - z_1^*) + b_1(z_2^* - z_1^*) = b_2\delta + b_1\delta,$$

and $b_2\delta + b_1\delta$ is greater than

$$b_1(1 - z_1^*) = b_1(2\delta) \text{ and } b_2(1 - z_2^*) = b_2\delta.$$

We choose s^* to be any number less than $b_2\delta + b_1\delta$ and larger than $\max\{b_1(2\delta), b_2\delta\}$. Then (s^*, z^*) is violated by $mix1_{\{1,2\}}$, and satisfies the MIR inequalities above.

Assume that some other MIR inequality is violated by (s^*, z^*), and assume that this inequality is derived using the multipliers $\lambda = (\lambda_1, \lambda_2)$. Define $v_1^* = s^* + z_1^* - b_1$ and $v_2^* = s^* + z_2^* - b_2$. Then, as $b_2 + 2\delta < 1/2$ implies that $b_2 + 1/2 < 1 - 2\delta = z_1^*$, it follows that $v_1^*, v_2^* > 1/2$. Recall that $\lambda \in \mathbb{Z}^2$, and $\lambda^+ v^* < 1$. As the MIR inequality defined by λ is the same as the MIR inequality defined by $-\lambda$, we can assume that the multiplier with maximum magnitude is positive. The only nonzero λ values satisfying the conditions above and yielding distinct inequalities are $(1, 0), (0, 1)$, and $(-1, 1)$. We constructed (s^*, z^*) so that it satisfied the MIR inequalities obtained with the multiplier vectors $(1, 0)$ and $(0, 1)$. The vector $(-1, 1)$ simply yields inequality $mix2_{\{1,2\}}$ (as proved in Lemma 2.4) which is trivially satisfied by (s^*, z^*). ∎

2.2 MIR rank of type II mixing inequalities

In this section we study the MIR rank of type II mixing inequalities and show that the rank of the $|I'|$-term mixing inequality (5) is at most $|I'| - 1$. As in Section 2.2, it is sufficient to analyze the MIR rank of inequality (2). We start with studying the set S_2 defined earlier.

Lemma 2.4 *Inequality $mix2_{\{1,2\}}$ is an MIR inequality for S_2.*

Proof. We first convert the inequalities defining S_2 to equations by adding nonnegative slacks v_1, v_2 as follows:

$$s + z_1 - v_1 = b_1, \quad s + z_2 - v_2 = b_2.$$

Subtracting the first equation from the second, and dropping the term $-v_2$, we get $z_2 - z_1 + v_1 \geq b_2 - b_1$ as a valid inequality for S_2. Applying (6), we obtain $(b_2 - b_1)(z_2 - z_1) + v_1 \geq b_2 - b_1$ as an MIR inequality for S_2; substituting out v_1, we obtain $mix2_{\{1,2\}}$. ∎

Note that the derivation above does not use the nonnegativity of s and therefore $mix2_{\{1,2\}}$ is valid for the relaxed mixing set

$$S_{2R} = \Big\{ (s, z) \in \mathbb{R} \times \mathbb{Z}^2 \ : \ s + z_1 \geq b_1; \ s + z_2 \geq b_2 \ \Big\}.$$

We will now prove a result on the rank of type II mixing inequalities. The proof will be similar to the proof that $mix1_{\{1,\dots,n\}}$ is an MIR inequality for $S^{[n-1]}$ given after Theorem 2.1.

Theorem 2.5 *Inequality $mix2_{\{1,\dots,n\}}$ is valid for $S^{[n-1]}$ and therefore it has MIR rank at most $n - 1$.*

Proof. We showed that this result is true for $n = 2$. Assume it is true for $n - 1$; in other words, assume all $n-1$ term mixing inequalities of type II are valid for $S^{[n-2]}$. Consider inequality (M_1) (that is $s + z_1 - v_1 = b_1$) together with the following (type II) mixing inequalities (expressed as equations via slacks)

$$s + (b_1 + 1 - b_n)z_1 + (b_k - b_1)z_k + \sum_{i=k+1}^{n} (b_i - b_{i-1})z_i - v_k = b_n \qquad (M_k')$$

for $k = 3, \ldots, n$ and the type II mixing inequality

$$s + (b_2 + 1 - b_n)z_2 + \sum_{i=3}^{n}(b_i - b_{i-1})z_i - v_{n+1} = b_n \qquad (M'_{n+1})$$

that are valid for $S^{[n-2]}$.

We next define multipliers λ' to obtain a base inequality which yields inequality (2) as an MIR inequality. Let

$$\mu'_k = \begin{cases} (b_2 - b_1)(\dfrac{1}{b_n - b_1}) & \text{if } k = 1, \\ (b_2 - b_1)(\dfrac{1}{b_{k-1} - b_1} - \dfrac{1}{b_k - b_1}) & \text{for } k = 3, \ldots, n \\ -1 & \text{if } k = n + 1, \end{cases}$$

and note that $\mu'_1 + \sum_{k=3}^{n} \mu'_k = 1$ and $\sum_{k=3}^{p} \mu'_k = 1 - (b_2 - b_1)/(b_p - b_1)$ for $p = 3, \ldots, n$.

Now consider $\mu'_1(M_1) + \sum_{k=3}^{n} \mu'_k(M'_k)$ and denote it by

$$\alpha_0 s + \sum_{k=1}^{n} \alpha_k z_k - \sum_{k=1}^{n} \mu_k v_k = \beta. \qquad (11)$$

As before, $\beta = b_n - (b_2 - b_1)$, $\alpha_0 = 1$, $\alpha_2 = 0$, and $\alpha_k = b_k - b_{k-1}$ for $k = 3, \ldots, n$. In addition,

$$\begin{aligned} \alpha_1 &= \mu'_1 + (b_1 + 1 - b_n)\sum_{k=3}^{n} \mu'_k = \mu'_1(b_n - b_1) + (b_1 + 1 - b_n)\sum_{k=1}^{n} \mu'_k \\ &= \frac{b_2 - b_1}{b_n - b_1}(b_n - b_1) + (b_1 + 1 - b_n) = b_2 + 1 - b_n. \end{aligned}$$

Therefore equation 11 is the same as

$$s + (b_2 + 1 - b_n)z_1 + \sum_{k=3}^{n}(b_k - b_{k-1})z_k - \sum_{k=1}^{n} \mu'_k v_k = b_n - (b_2 - b_1), \qquad (12)$$

and $\sum_{k=1}^{n+1} \mu'_k(M'_k)$ equals

$$(b_2 + 1 - b_n)(z_1 - z_2) + v_{n+1} - \sum_{k=1}^{n} \mu'_k v_k = -(b_2 - b_1).$$

Finally, setting $\lambda' = \mu'/(b_2 + 1 - b_n)$, and dropping the variables v_k with negative coefficients in $\sum_{k=1}^{n+1} \lambda'_k(M'_k)$, we get

$$(z_1 - z_2) + v_{n+1}/(b_2 + 1 - b_n) \geq -(b_2 - b_1)/(b_2 + 1 - b_n) \qquad (13)$$

as a valid inequality for $S^{[n-2]}$. Let γ stand for the right-hand-side of the above inequality. Then $\hat{\gamma} = (b_1 + 1 - b_n)/(b_2 + 1 - b_n)$. Applying the basic mixed-integer inequality we get $(b_1 + 1 - b_n)(z_1 - z_2) + v_{n+1} \geq 0$ as an MIR inequality for $S^{[n-2]}$. Substituting out v_{n+1} from the above expression, we obtain $mix2_{\{1,\ldots,n\}}$. ∎

Using Theorem 2.5, we make the following observation.

Corollary 2.6 *The $|I'|$-term mixing inequality $mix2_{I'}$ has MIR rank at most $|I'| - 1$.*

3 Mixing inequalities for general mixed-integer sets

In this section we define mixing inequalities for a general mixed-integer set $P_I = P \cap Z(I)$ where

$$P = \{x \in \mathbb{R}^n \,:\, Ax \geq b\,\}, \qquad Z(I) = \{x \in \mathbb{R}^n \,:\, x_i \in \mathbb{Z}\ \forall i \in I\},$$

I is a subset of $N = \{1, \ldots, n\}$ and A has m rows indexed by $M = \{1, \ldots, m\}$. We assume that if there are non-negativity constraints on some variables, they are included in $Ax \geq b$.

Let $\Lambda \in \mathbb{R}^{t\times m}$ and let λ^k denote the kth row of Λ for $k \in K = \{1, \ldots, t\}$. We call Λ *a good mixing matrix for* P_I if it satisfies the following properties for all $k \in K$:

1. $0 < \beta_1 < \beta_2 < \ldots < \beta_t$ where $\beta_k = \lambda^k b - (\lceil \lambda^k b \rceil - 1)$,
2. $(\lambda^k A)_i \in \mathbb{Z}$ for all $i \in I$,
3. $(\lambda^k A)_i = 0$ for all $i \notin I$.

If Λ is a good mixing matrix for P_I, then the inequality

$$\sum_{j\in M} (\max_{k\in K}\{-\lambda^k_j\})^+ (Ax - b)_j + \sum_{k\in K} (\beta_k - \beta_{k-1})(\lambda^k Ax - \lceil \lambda^k b \rceil + 1) \geq \beta_t \tag{14}$$

where $\beta_0 = 0$ is called a *mixing inequality of type I generated by* Λ. We use $(\cdot)^+$ to denote $\max\{0, \cdot\}$ and $(Ax - b)_j$ to denote the jth row of $(Ax - b)$. Similarly, we define a *mixing inequality of type II generated by* Λ to be

$$\begin{aligned}\sum_{j\in M} (\max_{k\in K}\{-\lambda^k_j\})^+ (Ax - b)_j \quad &+ \quad (\beta_1 + 1 - \beta_t)(\lambda^1 Ax - \lceil \lambda^1 b \rceil + 1) \\ &+ \sum_{k\in K\setminus\{1\}} (\beta_k - \beta_{k-1})(\lambda^k Ax - \lceil \lambda^k b \rceil + 1) \geq \beta_t. \end{aligned}\tag{15}$$

To see that mixing inequalities (14) and (15) are valid for P_I, let $v = Ax - b$ and note that $v \geq 0$ for all $x \in P$. For any matrix $\Lambda \in \mathbb{R}^{t\times m}$, clearly the equality system $(\Lambda A)x - \Lambda v = \Lambda b$ is satisfied by all $x \in P$. Dropping v_j variables with negative coefficients from these equations, one obtains the following valid inequalities

$$\lambda^k Ax + \sum_{j\in M} (-\lambda^k_j)^+ v_j \geq \lambda^k b \tag{16}$$

for all $k \in K$. Further relaxing inequality (16) we obtain

$$\lambda^k Ax + \sum_{j\in M} \max_{k'\in K}\{(-\lambda^{k'}_j)^+\} v_j \geq \lambda^k b \tag{17}$$

as valid inequalities for P. Now letting $s = \sum_{j\in M} \max_{k'\in K}\{(-\lambda^{k'}_j)^+\} v_j$, and $z_k = \lambda^k Ax - (\lceil \lambda^k b \rceil - 1)$ inequality (17) becomes $s + z_k \geq \beta_k$. As Λ is a good mixing matrix for P_I, for any given $x \in P_I$, we have the corresponding $s \geq 0$ and $z_k \in \mathbb{Z}$ for all $k \in K$. Furthermore, as $\beta_k \in (0, 1]$ and is strictly increasing, any point $x \in P_I$ can be mapped to a point in $S = \{s \in$

$\mathbb{R}_+, z \in \mathbb{Z}^n \ : \ s + z_k \geq \beta_k \ \forall k \in K\}$. Writing the mixing inequalities (1) and (2) for S and replacing the surrogate variables with the original ones, one obtains inequalities (14) and (15).

We next present bounds on the maximum violation of mixing inequalities and then formulate the separation problem for mixing inequalities as an optimization problem.

3.1 Bounding the violation of mixing inequalities

Consider the mixing set $S = \{s \in \mathbb{R}_+, z \in \mathbb{Z}^t \ : \ s + z_k \geq \beta_k \ \forall k \in K\}$ and let S_{LP} denote its continuous relaxation. For a given $(\bar{s}, \bar{z}) \in S_{LP}$, let the violation of the mixing inequality (1) be defined as, $\Delta^1(\bar{s}, \bar{z}) = \beta_t - \bar{s} - \sum_{k=1}^{t} \delta_k \bar{z}_k$ where $\delta_1 = \beta_1$ and $\delta_k = \beta_k - \beta_{k-1}$ for $k = 2, \ldots, t$. Similarly, let the violation of the mixing inequality (2) be defined as, $\Delta^2(\bar{s}, \bar{z}) = \beta_t - \bar{s} - \sum_{k=1}^{t} \epsilon_k \bar{z}_k$ where $\epsilon_1 = \beta_1 + 1 - \beta_t$ and $\epsilon_k = \beta_k - \beta_{k-1}$ for $k = 2, \ldots, t$.

Lemma 3.1 *Let $(\bar{s}, \bar{z}) \in S_{LP}$, then $\Delta^1(\bar{s}, \bar{z}) \leq \frac{1}{2}(1 - \frac{1}{t+1})$ and $\Delta^2(\bar{s}, \bar{z}) \leq \frac{1}{2}(1 - \frac{1}{t})$.*

Proof. As $(\bar{s}, \bar{z})$ satisfies $\bar{s} + \bar{z}_k - \beta_k \geq 0$ for all $k \in K$, we have

$$\sum_{k=1}^{t} \delta_k \bar{s} + \sum_{k=1}^{t} \delta_k \bar{z}_k - \sum_{k=1}^{t} \delta_k \beta_k \geq 0 \tag{18}$$

Define $\delta_{t+1} = 1 - \beta_t$ so that $(1 - \beta_k) = \sum_{j=k+1}^{t+1} \delta_j$ for $k \geq 1$. Adding (18) to $\Delta^1(\bar{s}, \bar{z})$, we obtain

$$\Delta^1(\bar{s}, \bar{z}) \ \leq \ \beta_t - (1 - \beta_t)\bar{s} - \sum_{k=1}^{t} \delta_k \beta_k \quad \leq \quad \sum_{k=1}^{t} \delta_k - \sum_{k=1}^{t} \delta_k \beta_k \ = \ \sum_{k=1}^{t} \delta_k (1 - \beta_k) \ = \ \sum_{k=1}^{t} \sum_{j=k+1}^{t+1} \delta_k \delta_j.$$

Note that $\sum_{k=1}^{t+1} \delta_k = 1$. We can now rewrite the last term in this expression using the following observation

$$(\sum_{k=1}^{t+1} \delta_k)^2 \ = \ \sum_{k=1}^{t+1} \delta_k^2 + 2 \sum_{k=1}^{t} \sum_{j=k+1}^{t+1} \delta_k \delta_j \implies \sum_{k=1}^{t} \sum_{j=k+1}^{t+1} \delta_k \delta_j \ = \ \frac{1}{2}(1 - \sum_{k=1}^{t+1} \delta_k^2). \tag{19}$$

For $q \geq 1$, define $w(q) = \min\{\sum_{k=1}^{q} (\delta_k)^2 \ : \ \sum_{k=1}^{q} \delta_k = 1\}$ and note that $w(q) = 1/q$. Therefore

$$\Delta^1(\bar{s}, \bar{z}) \ \leq \ \frac{1}{2}(1 - \sum_{k=1}^{t+1} \delta_k^2) \ \leq \ \frac{1}{2}(1 - w(t+1)) \ = \ \frac{1}{2}(1 - \frac{1}{t+1}).$$

Similarly for $\Delta^2(\bar{s}, \bar{z})$ note that $\bar{s} + \sum_{k=1}^{t} \epsilon_k \bar{z}_k - \sum_{k=1}^{t} \epsilon_k \beta_k \geq 0$ as $(\bar{s}, \bar{z})$ satisfies $\bar{s} + \bar{z}_k - \beta_k \geq 0$ for all $k \in K$ and $\sum_{k=1}^{t} \epsilon_k = 1$. Adding this expression to $\Delta^2(\bar{s}, \bar{z})$, we obtain

$$\Delta^2(\bar{s}, \bar{z}) \ \leq \ \beta_t - \sum_{k=1}^{t} \epsilon_k \beta_k \ = \ \sum_{k=1}^{t} \epsilon_k (\beta_t - \beta_k) \ = \ \sum_{k=1}^{t-1} \epsilon_k (\beta_t - \beta_k) \ = \ \sum_{k=1}^{t-1} \sum_{j=k+1}^{t} \epsilon_k \epsilon_l.$$

Combining the fact that $\sum_{k=1}^{t} \epsilon_k = 1$ with the observation (19) above,

$$\Delta^2(\bar{s}, \bar{z}) \;\leq\; \frac{1}{2}(1 - \sum_{k=1}^{t} \epsilon_k^2) \;\leq\; \frac{1}{2}(1 - w(t)) \;=\; \frac{1}{2}(1 - \frac{1}{t}).$$

$\blacksquare$

Using the fact that the validity of mixing inequalities (14) and (15) for the general mixed integer set P_I was shown by mapping points in P_I to points in the mixing set S, we have the following observation. We define the *violation* of an inequality to be the right-hand-side minus the left-hand-side.

Corollary 3.2 *For a given a point $\hat{x} \in P$ the violation of any t-term mixing inequality (14) is at most $\frac{1}{2}(1 - \frac{1}{t+1})$. Similarly, the violation of any t-term mixing inequality (15) is at most $\frac{1}{2}(1 - \frac{1}{t})$.*

Notice that for $t = 1$ this observation implies that the maximum violation of a type I mixing inequality (14) is 1/4. This is same as the bound shown in [5] for the maximum violation of an MIR inequality. In addition, when $t = 1$, the maximum violation of a type II mixing inequality (14) is zero, as the inequality is implied by $Ax \geq b$.

3.2 Separating violated mixing inequalities

For a given a point $\hat{x} \in P$, a most violated mixing inequality (14) generated by a good mixing matrix that has up to t rows can be obtained by solving the following quadratic mixed-integer program which we call *Mix-Sep-I*:

$$\text{Maximize} \quad \beta_t - \sum_{j \in M} \delta_j \hat{v}_j \;-\; \beta_1 z_1 \;-\; \sum_{k \in K \setminus \{1\}} (\beta_k - \;\beta_{k-1}) z_k$$

Subject to

$$\begin{aligned}
\alpha^k \;&= (\lambda^k A) && k \in K, \\
\alpha_i^k \;&=\; 0 && \forall k \in K,\; i \in N \setminus I, \\
z_k \;&=\; (\lambda^k A)\hat{x} - \theta_k && \forall k \in K, \\
\beta_k \;&=\; (\lambda^k b) - \theta_k && \forall k \in K, \\
\beta_k \;&\geq\; \beta_{k-1} && \forall k \in K, \\
\delta \;&\geq\; -\lambda^k && \forall k \in K, \\
1 \;&\geq\; \beta_t, &&
\end{aligned}$$

$$\lambda^k \in \mathbb{R}^m,\; \alpha^k \in \mathbb{Z}^n \; k \in K;\; \theta \in \mathbb{Z}^t,\; z \in \mathbb{R}^t, \beta \in \mathbb{R}_+^t,\; \delta \in \mathbb{R}_+^m.$$

where $\hat{v} \in \mathbb{R}^m$ denotes $A\hat{x} - b$. In this formulation, variable z_k stands for $(\lambda^k A\hat{x} - \lceil \lambda^k b \rceil + 1)$ and θ_k stands for $\lceil \lambda^k b \rceil - 1$ (if $\lambda^k b$ is integral, then θ_k can take on the value $\lambda^k b$ or $\lambda^k b - 1$).

The objective function measures the violation of the mixing inequality (14), defined to be the right-hand-side of the inequality minus the left-hand-side.

Lemma 3.3 *For a given point $\hat{x} \in P$, an optimal solution of Mix-Sep-I corresponds to a most violated mixing inequality of type I that can be generated by a good mixing matrix with t or fewer rows.*

Proof. Given a good mixing matrix $\Lambda' \in \mathbb{R}^{t' \times m}$ where $t' \leq t$, it is easy to construct a feasible solution to Mix-Sep-I where the objective value is the same as the violation of the mixing inequality generated by Λ'. This can simply be done by first appending $t - t'$ copies of the the last row of Λ' to obtain the matrix $\Lambda \in \mathbb{R}^{t \times m}$. Letting $\lambda^k = k$th row of Λ and $\alpha^k = \lambda^k A$ for $k \in K$, $\theta = \Lambda b - 1$ and $\delta_j = (\max_{k \in K}\{-\lambda_j^k\})^+$ for $j \in M$ gives the desired solution to Mix-Sep-I.

On the other hand, given an optimal solution to Mix-Sep-I, let $\Lambda \in \mathbb{R}^{t \times m}$ be the matrix with kth row equal to the value of λ^k in the solution and note that Λ is not necessarily a good mixing matrix as Mix-Sep-I does not guarantee that $\beta_k > \beta_{k-1}$ for $k \in K \setminus \{1\}$. Furthermore, the β_k values produced by Mix-Sep-I are guaranteed to be equal to $\lambda^k b - (\lceil \lambda^k b \rceil - 1)$ only when $\lambda^k b \notin \mathbb{Z}$. If $\lambda^k b \in \mathbb{Z}$ for some $k \in K$, it is possible that $\theta_k = \lambda^k b$ and $\beta_k = 0$ in the optimal solution. Let Λ' be obtained from Λ by deleting rows λ^k such that $\beta_k = 0$ or $\beta_k = \beta_{k-1}$ in the optimal solution to Mix-Sep-I. Notice that Λ' is a good mixing matrix with at most t rows. Furthermore, the violation of the mixing inequality (14) generated by Λ' equals the optimal value of Mix-Sep-I. ∎

Similarly, we define *Mix-Sep-II* to be the quadratic mixed-integer program obtained from Mix-Sep-I by changing its objective function to

$$\beta_t - \sum_{j \in M} \delta_j \hat{v}_j \; - \; (\beta_1 + 1 - \beta_t) z_1 \; - \sum_{k \in K \setminus \{1\}} (\beta_k - \beta_{k-1}) z_k.$$

Lemma 3.4 *For a given point $\hat{x} \in P$, an optimal solution of Mix-Sep-II corresponds to a most violated mixing inequality of type II that can be generated by a good mixing matrix with t or fewer rows.*

Proof. As in the proof of Lemma 3.3, for a given good mixing matrix Λ' with at most t rows, it is easy to construct a feasible solution to Mix-Sep-II with an objective value equal to the violation of the mixing inequality (15) generated by Λ'.

Further, for a given optimal solution of Mix-Sep-II let $\Lambda' \in \mathbb{R}^{l \times m}$ be obtained by collecting rows λ^k such that $\beta_k > \beta_{k-1}$ in the optimal solution. Let z^* denote the objective value of this solution. If $\beta_1 > 0$ in the optimal solution, Λ' is a good mixing matrix and gives a mixing inequality (15) with violation at least z^*. On the other hand, if $\beta_1 = 0$, Λ' is not a good mixing matrix, however, it is possible to obtain a new matrix $\bar{\Lambda}$ by moving the first row of Λ' to the end. The matrix $\bar{\Lambda}$ gives $\bar{\lambda}^1 b - (\lceil \bar{\lambda}^1 b \rceil - 1) > 0$ and $\bar{\lambda}^l b - (\lceil \bar{\lambda}^l b \rceil - 1) = 1$. If $\bar{\lambda}^{l-1} b - (\lceil \bar{\lambda}^{l-1} b \rceil - 1)$

is also 1, then we further delete the last row of $\bar{\Lambda}$ to obtain a good mixing matrix where the violation of the associated mixing inequality (15) equals z^*. ∎

4 Mixing closure of mixed-integer sets

We define the *mixing closure* of P with respect to I to be the set of points in P that satisfy all mixing inequalities (14) and (15) that can be generated by good mixing matrices. Let $clo(P_I)$ denote the mixing closure of P with respect to I. Our main result in this section is that $clo(P_I)$ can be described using a bounded number of mixing inequalities each of which has a bounded number of terms. In other words, it is sufficient to consider a bounded number of good mixing matrices, each having a bounded number of rows. As we only consider rational data, without loss of generality, we assume that (after scaling, if necessary) $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$ in the definition of P. Before presenting our main result, we first study a special case where all variables are integral.

4.1 Mixing closure of pure integer sets

It is significantly easier to analyze $clo(P_I)$ when there are no continuous variables in the definition of P_I; that is, when $I = N$. Let $\Lambda \in \mathbb{R}^{t\times m}$ be a good mixing matrix for P_N and consider the mixing inequality of type I generated by Λ

$$\sum_{j\in M} \delta_j v_j + \sum_{k\in K} (\beta_k - \beta_{k-1})(\lambda^k Ax - \lceil \lambda^k b \rceil + 1) \geq \beta_t \tag{20}$$

where $v = Ax - b$ and $\delta_j = (\max_{k\in K}\{-\lambda_j^k\})^+$. We next observe that it is sufficient to consider good mixing matrices with small entries.

Lemma 4.1 *Let $\bar{x} \in P$ and $\Lambda \in \mathbb{R}^{t\times m}$ be a good mixing matrix for P_N. If $\bar{x}$ violates a type I mixing inequality inequality (20) generated by Λ then there exists a good mixing matrix $\Lambda' \in \mathbb{R}^{t\times m}$ for P_N such that $\mathbf{1} > \Lambda' > -\mathbf{1}$ and $\bar{x}$ also violates the mixing inequality generated by Λ'.*

Proof. Assume that $\delta_j > 0$ for some $j \in M$. In other words $0 > \min_{k\in K}\{\lambda_j^k\}$. Consider Λ' obtained by replacing λ_j^k with $\lambda_j^k + \lfloor \delta_j \rfloor$ for all $k \in K$. Clearly Λ' is a good mixing matrix. The left-hand-side of the mixing inequality generated by Λ' is

$$\sum_{j\in M} \delta_j \bar{v}_j + \sum_{k\in K} (\beta_k - \beta_{k-1})(\lambda^k A\bar{x} - \lceil \lambda^k b \rceil + 1) - \lfloor \delta_j \rfloor \bar{v}_j + \sum_{k\in K} (\beta_k - \beta_{k-1}) \lfloor \delta_j \rfloor (a_j \bar{x} - b_j)$$

where $\bar{v}_j = a_j\bar{x} - b_j \geq 0$ is the surplus variable associated with the jth row. Note that the right-hand-side of the inequality is the same as the right-hand-side of inequality (20) as A and b are integral. Using $1 \geq \beta_t = \sum_{k=1}^t (\beta_k - \beta_{k-1})$ and $\bar{v}_j, \lfloor \delta_j \rfloor \geq 0$ we have

$$-\lfloor \delta_j \rfloor \bar{v}_j + \sum_{k\in K} (\beta_k - \beta_{k-1}) \lfloor \delta_j \rfloor (a_j\bar{x} - b_j) = -\lfloor \delta_j \rfloor \bar{v}_j + \beta_t \lfloor \delta_j \rfloor \bar{v}_j \leq -\lfloor \delta_j \rfloor \bar{v}_j + \lfloor \delta_j \rfloor \bar{v}_j = 0.$$

Therefore, the mixing inequality generated by Λ' is violated at least as much as the original inequality (20). Without loss of generality, we can therefore assume that $\Lambda' > -\mathbf{1}$.

Now assume $\lambda_j^k \geq 1$ for some $j \in M$ and $k \in K$ and consider Λ' obtained by replacing λ_j^k with $\lambda_j^k - \lfloor \lambda_j^k \rfloor$. Λ' is a good mixing matrix and the left-hand-side of the mixing inequality generated by Λ' is

$$\sum_{j \in M} \delta_j \bar{v}_j + \sum_{k \in K} (\beta_k - \beta_{k-1})(\lambda^k A\bar{x} - \lceil \lambda^k b \rceil + 1) - (\beta_k - \beta_{k-1}) \lfloor \lambda_j^k \rfloor (a_j \bar{x} - b_j)$$

and the right-hand-side is the same as inequality (20) as the data is integral. Clearly the new inequality is violated at least as much as the original inequality (20) as $(\beta_k - \beta_{k-1}) \lfloor \lambda_j^k \rfloor \geq 0$ and $a_j \bar{x} \geq b_j$. Therefore all $\lambda_j^k \geq 1$, can be replaced with $\lambda_j^k - \lfloor \lambda_j^k \rfloor$ to obtain a good mixing matrix $\Lambda' < \mathbf{1}$. ∎

Based on this observation, we next show that there are a finite number of good mixing matrices for P_N and therefore the elementary closure of mixing inequalities of type I is polyhedral. Let $\Delta \in \mathbb{Z}_+$ denote the absolute value of the largest entry in $[A, b]$ and let $t^* = (2m\Delta)^{(n+1)}$.

Lemma 4.2 *If $\bar{x} \in P$ violates a type I mixing inequality inequality (20) then it violates one with at most t^* terms.*

Proof. By definition $[A, b] \in [-\Delta, \Delta]^{m \times (n+1)}$. Using Lemma 4.1, and without loss of generality, we can therefore assume that if $\bar{x} \in P$ violates a mixing inequality with t terms, then it violates one generated by a good mixing matrix that satisfies $(\Lambda A, \lfloor \Lambda b \rfloor) \in (-m\Delta, m\Delta)^{t \times (n+1)}$. Therefore, in Mix-Sep-I it suffices to consider only $\kappa = (2m\Delta)^{t \times (n+1)}$ possible choices for variables (α, θ).

In addition, note that for any $\bar{x} \in P$, it suffices to consider mixing inequalities with at most t^* terms as the term $q_k \stackrel{\text{def}}{=} \lambda^k A\bar{x} - \lceil \lambda^k b \rceil + 1$ in inequality (20) can be assumed to be strictly increasing and there are only t^* possible choices for $(\lambda^k A, \lceil \lambda^k b \rceil - 1)$. Given any violated mixing inequality, if $q_k \geq q_{k+1}$ then one can throw away the term q_k for $k > 1$ and replace the coefficient of q_{k+1} with $\beta_{k+1} - \beta_{k-1}$ to obtain a mixing inequality with fewer terms and at least as much violation. ∎

Let $clo^1(P_I)$ denote the set of points in P that satisfy all mixing inequalities of type I that can be generated by good mixing matrices. Define $clo^2(P_I)$ similarly using mixing inequalities of type II. We next observe that $clo^1(P_I)$ is polyhedral.

Corollary 4.3 *$clo^1(P_N)$ is a polyhedron.*

Proof. Using Lemma 4.2, it suffices to consider at most $\kappa^* = (2m\Delta)^{t^* \times (n+1)}$ possible choices for (α, θ) in Mix-Sep-I to obtain a violated mixing inequality of type I. Notice that after fixing (α, θ), the value of the z variables are implied and therefore, for each fixed value of (α, θ), the most violated inequality can be obtained by solving a linear program obtained from Mix-Sep-I by fixing α, θ and z variables. As it is sufficient to consider the basic feasible solutions when solving a linear program, and as there a finite number of such basic feasible solutions, say w^*, one only needs to consider $w^*\kappa^*$ inequalities to obtain a violated one. ■

Note that for each good mixing matrix Λ it is possible to write a type I mixing inequality and a type II mixing inequality. In other words, Mix-Sep-I and Mix-Sep-II have identical feasible regions and only differ in their objective functions. Using this basic observation, it is possible to adopt Lemmas 4.1 and 4.2 to mixing inequalities of type II and show that $clo^2(P_N)$ and therefore $clo(P_N) = clo^1(P_N) \cap clo^2(P_N)$ is a polyhedron. As our results in the next section subsume this result, we do not present it and avoid repetition.

4.2 Mixing closure of mixed-integer sets

In this section we show that $clo(P_I)$ is a polyhedron. Unlike the pure integer case ($I = N$), we are not able to show that $clo(P_I)$ is given by good mixing matrices with small entries. We instead argue that it suffices to consider good mixing matrices with "bounded fractionality", i.e., matrices whose entries are integer multiples of some rational number that depends on A and b. We also argue that fractionality of the coefficients β_i ($i = 1, \ldots, t$) in a non-redundant mixing inequality is also bounded and therefore it suffices to consider mixing inequalities with a bounded number of terms. Using these observations, we then show that $clo(P_I)$ is a polyhedron. This result is motivated by a similar result for non-redundant MIR cuts in [5], but the proof is substantially more complicated.

Remember that, without loss of generality, A and b are assumed to be integral. Let $g(A)$ stand for the maximum subdeterminant of A, and let $f(A)$ stand for the product of distinct subdeterminants of A. Clearly, $f(A)$ is a divisor of $g(A)!$.

One can obtain trivial upper bounds for $g(A)$ and $f(A)$ as follows. For a square $t \times t$ matrix B with columns $b_1, \ldots, b_t$, $det(B) \leq \Pi_{i=1}^t ||b_i|| \leq (\sqrt{t} \max_{i,j} |B_{ij}|)^t$. For positive integers k, q, define $h(k, q) = (\sqrt{k}q)^k$. Then $g(A) \leq h(\min\{m, n\}, \max_{i,j} |A_{ij}|)$, and

$$f(A) \text{ is a divisor of } h(\min\{m, n\}, \max_{i,j} |A_{ij}|)!.$$

Let $\Omega = h(m, (g(A) + 1)!)!$ and note that Ω only depends on the matrix A.

Theorem 4.4 *Let $\bar{x} \in P$ and assume it violates a type I mixing inequality (14). Then $\bar{x}$ violates a type I mixing inequality with at most $\Omega f(A)^2$ terms such that each β_i is an integral multiple of $1/\Omega f(A)^2$.*

Proof. Consider the collection of violated type I mixing inequalities for $\bar{x}$ and from among them let $\mathcal{I}$ be one that has fewest number of terms. Let $\bar{\Lambda} \in \mathbb{R}^{t \times m}$ be a good mixing matrix that generates $\mathcal{I}$, with rows denoted as $\bar{\lambda}_i$, for $i = 1, \ldots, t$. Let the violation of $\mathcal{I}$ be $\Delta > 0$. Consider the family of type I mixing inequalities generated by Λ where $\Lambda A = \bar{\Lambda} A$ and $\lceil \Lambda b \rceil = \lceil \bar{\Lambda} b \rceil$. Let $\bar{z} = \bar{\Lambda} A \bar{x} - \lceil \bar{\Lambda} b \rceil + 1$ and $\bar{v} = A\bar{x} - b$. Then every such inequality is a solution of the linear program *Mix-Sep-LP-I*, defined as

$$\text{Maximize} \qquad \beta_t - \sum_{i=1}^{m} \delta_i \bar{v}_i - \sum_{i=1}^{t} (\beta_i - \beta_{i-1}) \bar{z}_i \tag{21}$$

Subject to

$$\lambda^i A = \bar{\lambda}^i A \qquad (i = 1, \ldots, t), \tag{22}$$

$$\lambda^i b - \beta_i = \lceil \bar{\lambda}^i b \rceil - 1 \qquad (i = 1, \ldots, t), \tag{23}$$

$$\lambda_i + \delta \geq 0 \qquad (i = 1, \ldots, t), \tag{24}$$

$$\beta_i - \beta_{i-1} \geq 0 \qquad (i = 2, \ldots, t), \tag{25}$$

$$\delta \geq 0, \quad 1 \geq \beta_i \geq 0 \qquad (i = 1, \ldots, t). \tag{26}$$

Every optimal solution of Mix-Sep-LP-I gives a type I mixing inequality violated by at least Δ.

Let $\beta = (\beta_1, \ldots, \beta_t)$, and $\mathcal{X} = (\lambda^1, \ldots, \lambda^t, \delta, \beta)$. Define

$$\mathcal{A} = \begin{bmatrix} [A\, b\, I_m] & & & & & & \\ & \cdots & & & & & \\ & & [A\, b\, I_m] & & & & \\ [0_1\, 0_2\, I_m] & \cdots & [0_1\, 0_2\, I_m] & I_m & & & \\ [0_3\ -e_1\, 0_4] & \cdots & [0_3\ -e_t\, 0_4] & & C & I_t & -I_t \end{bmatrix}.$$

Here $0_1, 0_2, 0_3$ and 0_4 are matrices with components equal to 0 and dimensions $m \times n$, $m \times 1$, $t \times n$ and $t \times m$ respectively. I_m and I_t are $m \times m$ and $t \times t$ identity matrices, respectively. Here e_i stands for the unit vector in $\mathbb{R}^t$ with a one in the ith row. C is a $t \times (t-1)$ matrix where the ith column equals $-e_i + e_{i+1}$ for $i = 1, \ldots, t-1$. The left-hand-side of Mix-Sep-LP-I can be written as $\mathcal{X}\mathcal{A}$, and the nonzero entries of the right-hand-side are equal to a component of $\bar{\Lambda} A$ or $\lceil \bar{\Lambda} b \rceil - 1$ or are equal to 1. $\mathcal{A}$ has $p = tm + m + t$ rows and has full row rank. Further, $\mathcal{I}$ defines a feasible solution of Mix-Sep-LP-I with objective value Δ.

As $\mathcal{A}$ has full row rank, the lineality space associated with Mix-Sep-LP-I has dimension 0. In other words, the feasible solutions of Mix-Sep-LP-I define a pointed polyhedron (having vertices). Therefore any basic feasible solution of Mix-Sep-LP-I is defined by p linearly independent constraints, and corresponds to a $p \times p$ submatrix of $\mathcal{A}$, say $\mathcal{B}$ (which we refer to as a *basis matrix*). For each column in $\mathcal{B}$, the corresponding constraint is satisfied as an equation by the basic feasible solution, and we will say that that constraint is *present* in $\mathcal{B}$.

We will prove that a basic optimal solution of Mix-Sep-LP-I defines a mixing inequality with the properties stated in the theorem. To prove this, we will show that all components of the inverse of an optimal basis are integral multiples of $1/\Omega f(A)^2$. This will imply that in a basic optimal solution, the components of β and Λ are integral multiples of $1/\Omega f(A)^2$, as the right-hand-side of Mix-Sep-LP-I is integral.

Consider a basic optimal solution $\mathcal{X}' = (\Lambda', \beta', \delta')$ of Mix-Sep-LP-I with associated basis matrix

$\mathcal{B}$. It defines a mixing inequality, say $\mathcal{I}'$, with violation at least Δ. Further, it satisfies $0 < \beta'_1 < \beta'_2 < \ldots < \beta'_t \leq 1$, otherwise there exists a mixing inequality having fewer than t terms and violation $\geq \Delta$, a contradiction to the minimality of $\mathcal{I}$. Therefore, out of the last $3t-1$ columns of $\mathcal{A}$, only the last one (corresponding to $\beta_t \leq 1$) can be present in $\mathcal{B}$. If any of the other $3t-2$ columns is present in $\mathcal{B}$, then one of the following constraints is satisfied by β' as an equation: $\beta_{i+1} - \beta_i \geq 0$ for $i = 1, \ldots t-1$, or $\beta_i \geq 0$ for $i = 1, \ldots, t$ or $\beta_i \leq 1$ (or $-\beta_i \geq -1$) for $i = 1 \ldots t-1$.

This implies that the columns corresponding to the constraints $\lambda^i b - \beta_i = \lceil \lambda^i b \rceil - 1$ for $i = 1, \ldots, t-1$ must be present in $\mathcal{B}$: if any column (say the ith one) is absent, then the $(tm+m+i)$th row of $\mathcal{B}$ does not have any nonzero entries and $\mathcal{B}$ does not have full row rank, a contradiction. Finally, at least one of the constraints $\beta_t \leq 1$ and $\lambda^t b - \beta_t = \lceil \lambda^t b \rceil - 1$ is present in $\mathcal{B}$. We now assume that the constraints involving β present in $\mathcal{B}$ are permuted to the end of $\mathcal{B}$.

Case 1: If only one of the constraints $\beta_t \leq 1$ and $\lambda^t b - \beta_t = \lceil \lambda^t b \rceil - 1$ is present in $\mathcal{B}$, it has the form

$$\mathcal{B} = \begin{bmatrix} M & B \\ 0_5 & -I_t \end{bmatrix} \Rightarrow \mathcal{B}^{-1} = \begin{bmatrix} M^{-1} & M^{-1}B \\ 0_5 & -I_t \end{bmatrix}.$$

Here 0_5 is a $t \times (tm+m)$ matrix with zero entries, M is a nonsingular square matrix with $tm+m$ rows and B is a matrix with nonzero components drawn from the vector b (and thus has only integral entries).

Case 2: If both the constraints $\beta_t \leq 1$ and $\lambda^t b - \beta_t = \lceil \lambda^t b \rceil - 1$ are present in $\mathcal{B}$, it has the form

$$\mathcal{B} = \begin{bmatrix} M' & 0_6 & B' \\ 0_7 & -e_t & -I_t \end{bmatrix},$$

where M' has $tm+m$ rows but $tm+m-1$ columns, 0_6 and 0_7 are matrices of appropriate dimension with zero entries, the $(tm+m)$th column corresponds to $\beta_t \leq 1$, and subsequent columns correspond to $\lambda^i b - \beta_i = \lceil \lambda^i b \rceil - 1$ for $i = 1, \ldots, t$. Let $\mathcal{B}'$ be obtained from $\mathcal{B}$ by subtracting the pth column from the $(tm+m)$th column. Then $\mathcal{B}' = \mathcal{B} \times T$, where T is a $p \times p$ matrix with $T_{ij} = 1$ if $i = j$, or $T_{ij} = -1$ if $(i,j) = (p, tm+m)$, and 0 otherwise. Then $\mathcal{B}^{-1} = T(\mathcal{B}')^{-1}$. Notice that $\mathcal{B}'$ has the same block upper triangular structure as $\mathcal{B}$ in Case 1. We will therefore focus on analyzing the components of the inverse (especially their denominators) of a basis matrix having the form in Case 1.

Let the columns of M corresponding to the constraints involving λ^i be M_i for $i = 1, \ldots, t$. Let N_i be the submatrix of M_i obtained by choosing the m rows corresponding to the variables λ^i; clearly N_i is a submatrix of $[A\, I_m]$ for $i = 1, \ldots, t-1$, and a submatrix of $[A\, b\, I_m]$ for $i = t$ (the vector b is present in N_t only in the matrix $\mathcal{B}'$ in Case 2). Further, the only nonzeros in the m rows of M corresponding to λ^i are contained in N_i; as these rows are linearly independent, N_i has rank m, and at least m columns. Also, the columns of $[A\, b]$ present in N_i are linearly independent, as the columns in M_i are linearly independent, for $i = 1, \ldots, t$ (see the depiction of M_i below). We can combine the above facts to conclude that N_i contains a nonsingular $m \times m$ submatrix A_i containing all columns from $[A\, b]$ present in N_i. Then the columns in N_i but not in A_i (denote these by $N_i \setminus A_i$) are unit vectors, and correspond to constraints $\lambda^i_j + \delta_j \geq 0$ for different j. We depict M_i below, and its various submatrices which we refer to in this proof. Assume M_i has $m+l$ columns for some $l \geq 0$, and A_i has k columns of A for some $k \leq m$. Let a_i be the ith column of A, for $i = 1, \dots, n$, and let $i_1, \dots, i_k$ be distinct integers in $[1, n]$, and let $i_{k+1}, \dots, i_{m+l}$ be distinct integers in $[1, m]$. Then M_i has the form below:

$$M_i \rightarrow \left[\begin{array}{c} 0_{m(i-1)\times(m+l)} \\ \left[\overset{A_i}{\boxed{a_{i_1}\dots a_{i_k}\; e_{i_{k+1}}\dots e_{i_m}}}\;\overset{N_i\setminus A_i}{\boxed{e_{i_{m+1}}\dots e_{i_{m+l}}}}\right] N_i \\ 0_{m(t-i)\times(m+l)} \\ \underbrace{\boxed{0_{m\times k}\; e_{i_{k+1}}\dots e_{i_m}}}_{B_i}\; e_{i_{m+1}}\dots e_{i_{m+l}} \end{array}\right];$$

where e_j stands for a unit vector in $\mathbb{R}^m$ with a one in the jth position and zeros elsewhere.

Let $M(A_i)$ stand for the columns of M_i which intersect A_i and $M(N_i \backslash A_i)$ stand for the remaining columns in M_i. Let the columns $M(N_i \setminus A_i)$ be arranged at the end of M, for $i = 1, \dots, t$. Then M is a non-singular block arrow matrix having the following form:

$$M = \begin{bmatrix} A_1 & \dots & 0 & C_1 \\ & \dots & & \\ 0 & \dots & A_t & C_t \\ B_1 & \dots & B_t & D \end{bmatrix}.$$

Here M has $(t+1)^2$ blocks of $m \times m$ matrixes, where the diagonal blocks are $A_1, \dots, A_t, D$ (we will describe D in a moment). The blocks in the last row (other than D) are $B_1, \dots, B_t$. Each block B_i is a square submatrix of $M(A_i)$ with m rows corresponding to the variables $\delta_i (i = 1, \dots, m)$; some of its columns are distinct unit vectors, and the remaining columns have only zero entries (see the depiction of M_i above). Each block C_i is an $m \times m$ matrix and consists of the columns of $N_i \setminus A_i$ along with columns with zero entries. As discussed above, the nonzero columns of C_i are distinct unit vectors. Further, the columns of C_i and C_j for $i \neq j$ have nonzeros in non-overlapping columns. We can conclude that at most m of the blocks C_i are nonzero. Finally, each column of D is a unit vector; either it corresponds to a constraint $\delta_j \geq 0$ or $\lambda_j^i + \delta_j \geq 0$ for some $j \in [1, m]$.

The inverse of M is not hard to compute. We start off with the LU decomposition of M:

$$LU = M \implies L = \begin{bmatrix} I_m & \dots & 0 & 0 \\ & \dots & & \\ 0 & \dots & I_m & 0 \\ B_1 A_1^{-1} & \dots & B_t A_t^{-1} & I_m \end{bmatrix}, U = \begin{bmatrix} A_1 & \dots & 0 & C_1 \\ & \dots & & \\ 0 & \dots & A_t & C_t \\ 0 & \dots & 0 & \bar{D} \end{bmatrix},$$

where $\bar{D} = D - \sum_{i=1}^{t} B_i A_i^{-1} C_i$. The products $B_i A_i^{-1} C_i \neq 0$ for only m distinct values of i; without loss of generality, we assume that $\bar{D} = D - \sum_{i=1}^{m} B_i A_i^{-1} C_i$. As M is non-singular, so is $\bar{D}$. Further, as the unit vectors in C_i and C_j for $i \neq j$ are in non-overlapping columns, and as the nonzero rows of B_i are unit vectors, the nonzero entries in $\bar{D} - D$ are simply components of

A_i^{-1}, and are thus ratios of subdeterminants of $[A\, b]$. This implies that every entry of $\bar{D} - D$ is an integral multiple of $1/f(A)$. As the components of D are either zero or one, every component of $f(A)\bar{D}$ is integral and is bounded in magnitude by $f(A) + f(A)g(A) = f(A)(1 + g(A))$ which is a divisor of $(g(A) + 1)!$. Therefore every component of $(f(A)\bar{D})^{-1}$ is an integral multiple of $1/f(f(A)\bar{D})$ which is an integral multiple of $1/\Omega = 1/h(m, (g(A) + 1)!)!$. Therefore every component of $\bar{D}^{-1} = f(A)^m(f(A)\bar{D})^{-1}$ is an integral multiple of $1/\Omega$.

Finally,

$$L^{-1} = \begin{bmatrix} I_m & \dots & 0 & 0 \\ & \dots & & \\ 0 & \dots & I_m & 0 \\ -B_1A_1^{-1} & \dots & -B_tA_t^{-1} & I_m \end{bmatrix}, U^{-1} = \begin{bmatrix} A_1^{-1} & \dots & 0 & -A_1^{-1}C_1\bar{D}^{-1} \\ & \dots & & \\ 0 & \dots & A_t^{-1} & -A_t^{-1}C_t\bar{D}^{-1} \\ 0 & \dots & 0 & \bar{D}^{-1} \end{bmatrix}$$

Clearly every component of L^{-1} is an integral multiple of $1/f(A)$. Every component of U^{-1} is an integral multiple of $1/\Omega f(A)$. Therefore every component of $M^{-1} = U^{-1}L^{-1}$ is an integral multiple of $1/\Omega f(A)^2$. As the matrix B' in $\mathcal{B}$ has integral entries, and because of the relationship between $\mathcal{B}^{-1}$ and M^{-1}, every entry in $\mathcal{B}^{-1}$ is an integral multiple of $1/\Omega f(A)^2$ (this is also true in Case 2 above).

As the right-hand-side of Mix-Sep-LP-I has integral components, it follows that the components of $\mathcal{X}' = (\Lambda', \beta', \delta')$ are all integral multiples of $1/\Omega f(A)^2$. Further, as the associated mixing inequality $\mathcal{I}'$ has distinct values of β_is contained in the interval $(0, 1]$, it follows that $\mathcal{I}'$ has at most $\Omega f(A)^2$ terms. ∎

Corollary 4.5 *Let $\bar{x} \in P$ and assume it violates a type II mixing inequality (15). Then $\bar{x}$ violates a type II mixing inequality with at most $\Omega f(A)^2$ terms such that each β_i is an integral multiple of* $1/\Omega f(A)^2$.

Proof. As in the proof of Theorem 4.4, consider the collection of violated type II mixing inequalities for $\bar{x}$ and from among them let $\mathcal{J}$ be one that has fewest number of terms. Let the violation of $\mathcal{J}$ be $\Delta > 0$ and assume that it is generated by the mixing matrix $\bar{\Lambda} \in \mathbb{R}^{t \times m}$. Let $\bar{z} = \bar{\Lambda}A\bar{x} - \lceil \bar{\Lambda}b \rceil + 1$ and $\bar{v} = A\bar{x} - b$. Let Mix-Sep-LP-II be the linear program defined by optimizing

$$\min \sum_{i=1}^{m} \delta_i \bar{v}_i + (\beta_1 + 1 - \beta_t)\bar{z}_1 + \sum_{i=2}^{t} (\beta_i - \beta_{i-1})\bar{z}_i - \beta_t$$

subject to the constraints of Mix-Sep-LP-I, i.e., to (22) - (26). As $\mathcal{J}$ corresponds to a solution of Mix-Sep-LP-II, an optimal solution of Mix-Sep-LP-II defines a violated mixing inequality of type II with violation at least Δ.

Consider a basic optimal solution of Mix-Sep-LP-II, with associated mixing inequality $\mathcal{J}'$. If it satisfies $\beta_i = \beta_{i+1}$ for $1 \leq i \leq t - 1$, then there exists another violated mixing inequality of type II with fewer terms than $\mathcal{J}$, a contradiction. In addition, if $\beta_0 = 0$ and $\beta_t = 1$, then, as discussed in the proof of Lemma 3.4, there exists a good mixing matrix, and a corresponding type II mixing inequality with violation Δ and $\beta_0 > 0$ and $\beta_t = \beta_{t-1} = 1$, again a contradiction.

Therefore, we can assume that any basic optimal solution satisfies $\beta_1 < \beta_2 < \ldots < \beta_t$ and at most one of $\beta_0 = 0$ and $\beta_t = 1$ holds. In the proof of Theorem 4.4 we showed that any such basic feasible solution of Mix-Sep-LP-I has the property that $\beta_1, \ldots, \beta_t$ are integral multiples of $1/\Omega f(A)^2$. As the basic solution which yields $\mathcal{J}'$ is a basic feasible solution of Mix-Sep-LP-I satisfying the above condition on the β_is, we can conclude that the β_i values in $\mathcal{J}'$ are integral multiples of $1/\Omega f(A)^2$. ■

To prove that $clo(P_I)$ is a polyhedron, we will use a proof technique similar to the one in [4] used for showing that the MIR closure of P with respect to I is a polyhedron.

Theorem 4.6 *The mixing closure of P with respect to I is a polyhedron.*

Proof. Let $q = \Omega f(A)^2$. Define

$$C = \{\beta \in \mathbb{R}^q : 0 \leq \beta_1 \leq \ldots \leq \beta_q \leq 1, \beta_i \text{ is an integral multiple of } 1/q, \text{ for } i = 1, \ldots, q\}.$$

C is clearly a finite set. For some vector $\bar{\beta} \in C$, define Mix-Sep-I($\bar{\beta}$) to be the integer program obtained by fixing the values of β_i in Mix-Sep-I to $\bar{\beta}_i$; notice that the objective function of Mix-Sep-I($\bar{\beta}$) is a linear function of the variables. The convex hull of solutions of this integer program (call it the *integer hull*) has finitely many vertices. Define Mix-Sep-II($\bar{\beta}$) in a similar manner.

Given a point $\bar{x} \in P \setminus clo(P_I)$, Theorem 4.4 implies that there exists a violated mixing inequality which defines a solution of Mix-Sep-I($\bar{\beta}$) or Mix-Sep-II($\bar{\beta}$) for some $\bar{\beta} \in C$. Therefore, there exists a violated mixing inequality associated with a vertex of the integer hull of Mix-Sep-I($\bar{\beta}$); note that Mix-Sep-II($\bar{\beta}$) has the same integer hull. This implies that $clo(P_I)$ is the set of points satisfying the mixing inequalities associated with the vertices of the integer hull of Mix-Sep-I($\bar{\beta}$) for all $\bar{\beta} \in C$. Therefore $clo(P_I)$ is a polyhedron. ■

5 Lengths of MIR proofs for mixing inequalities

Let $cx \geq d$ be a valid inequality for P_I. An *MIR cutting-plane proof* (or MIR proof) of $cx \geq d$ from P with respect to I is a sequence of inequalities $a_i x \geq d_i$ $(i = 1, \ldots, L)$ such that the last inequality in the sequence is $cx \geq d$, and for $i = 1, \ldots, L$, the inequality $a_i x \geq d_i$ is an MIR inequality derived from the previous inequalities in the sequence and the inequalities in $Ax \geq b$. The *length* of this proof is said to be L. Cutting plane proofs for Gomory-Chvátal cuts or lift-and-project cuts are defined similarly where each inequality in the sequence is required to be a Gomory-Chvátal or lift-and-project cut, respectively, obtained using the previous inequalities in the sequence and $Ax \geq b$, see [3, 9]. Cutting-plane proofs were introduced by Chvátal in [1].

Pudlák in [9] showed that that there are valid inequalities for a particular mixed-integer set P_I (arising from a graph problem) that cannot have a polynomial-length Gomory-Chvátal cutting-plane proof. Later Dash [3] showed that the same inequalities cannot have a polynomial-length

MIR cutting-plane proof either. In other words, for these particular inequalities, any MIR cutting-plane proof has exponential length.

In this section, we show that the same negative result holds for mixing inequalities. We define a *mixing cutting-plane proof* the same way as above where each inequality in the cutting plane proof is now derived from previous inequalities via mixing as in (14) and (15). We first show that mixing inequalities (1) and (2) have an MIR proof of length $O(n^2)$ from S_{LP}. An immediate consequence of this result is that mixing inequalities (14) and (15) for P_I with t terms have $O(t^2)$ length MIR proofs from $Ax \geq b$. These observations, when combined with results in [3], imply that mixing proofs have exponential encoding size for Pudlák's inequality system.

Theorem 5.1 *The inequalities $mix1_{\{1,\ldots,n\}}$ and $mix2_{\{1,\ldots,n\}}$ have MIR proofs of length $O(n^2)$ from the set S.*

Proof. For $1 \leq i < j \leq n$, let $ineq(i,j)$ denote the mixing inequality $mix1_{\{i,j,j+1\ldots,n\}}$. In Section 2.1 we showed that $mix1_{\{1,\ldots,n\}}$ can be derived as an MIR inequality using inequalities $s + z_1 \geq b_1$, $s + b_1 z_1 \geq b_1$, $ineq(2,3)$ and $ineq(1,k)$ for $k = 3,\ldots,n$. It is easy to see that this also implies that any mixing inequality $ineq(i,j)$ can be derived as an MIR inequality using inequalities $s+z_i \geq b_i$, $s+b_i z_i \geq b_i$ together with $ineq(j,j+1)$ and $ineq(i,k)$ for $k = j+1,\ldots,n$.

Note that for any $1 \leq i < j \leq n$, inequality $ineq(i,j)$ has $n-j+3$ terms and it is derived using mixing inequalities with fewer terms. Based on this observation, it is possible to produce a short MIR cutting-plane proof as follows: First generate all simple MIR inequalities $s + b_i z_i \geq b_i$ for $i = 1,\ldots,n$. Next generate all mixing inequalities $ineq(i,j)$ with 3 terms using base inequalities $s + z_i \geq b_i$ and simple MIR inequalities $s + b_i z_i \geq b_i$. Finally, for all $k = 4,\ldots,n$ generate all k-term mixing inequalities $ineq(i,j)$ using the base inequities, simple MIR inequalities and mixing inequalities $ineq(i,j)$ with $k-1$ or fewer terms. Notice that all mixing inequalities $ineq(i,j)$ with $k-1$ or fewer terms are generated before any mixing inequality $ineq(i,j)$ with k or more terms.

Clearly, this procedure produces n simple MIR inequalities and $(n^2-n)/2$ mixing inequalities and therefore the MIR proof of $mix1_{\{1,\ldots,n\}}$ has length at most $O(n^2)$.

An MIR proof of $mix2_{\{1,\ldots,n\}}$ with length $O(n^2)$ is derived in a similar manner by defining $ineq'(i,j)$ to denote the mixing inequality $mix2_{\{i,j,j+1\ldots,n\}}$. ∎

Corollary 5.2 *Mixing proofs have exponential worst-case encoding size.*

Proof. Given a mixing proof of $cx \geq d$ from $Ax \geq b$ of length L, assume the ith mixing inequality in the proof has t_i terms. It follows from Theorem 5.1 that there is an MIR proof of $cx \geq d$ from $Ax \geq b$ with length $O(\sum_{i=1}^{L} t_i^2)$. Letting $cx \geq d$ and $Ax \geq b$ stand for the appropriate inequality systems in Pudlák's exponentiality result for Gomory-Chvátal cutting-plane proofs, the results in Dash [3] imply that $\sum_{i=1}^{L} t_i^2$ is exponential in the number of variables and constraints in $Ax \geq b$. ∎

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