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Mathematics

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Carry-Free Multiplication

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Abstract. Using the meaning of a number we show three novel and simpler ways to do ordinary or carry type multiplication. It is assumed that one knows the multiplication table by heart. Let $p = ab$ where a and b are positive integers. Write $a = a_1 + a_2$ where a_1 contain a's even powers of ten and a_2 contains a's odd powers of ten. Form $p_1 = a_1 b$, $p_2 = a_2 b$ via ordinary multiplication. Then $p = p_1 + p_2$. The subproducts of p_1, p_2 will not have any carries and can be written down by rote. Alternatively, each row of p_1 and p_2 can be paired as two rows and then p can be computed directly. This second way is another way to do box multiplication. Furthermore, if one now adds each pair of rows together producing a single row again one would obtain carry type multiplication done another way. We also expand our approach to handle doing these new algorithms in base 100 and base 1000.

1 Introduction

This paper will describe Carry Free Multiplication (CFM). This work was inspired by reading the book Math Magic [1] and during the preparation of this paper the book Secrets of Mental Math [2]. The work presented here is believed to be novel. The book [1] gives five rules for doing elementary arithmetic:

1. Know what a number means.
2. Math is a universal language. Usually do math left to right as this is the way we read.
3. Memorize.
4. Practice.
5. Be creative.

Rule (3) would at least have one know by heart the addition and multiplication tables; see Figure 1. One way to memorize the tables is to use counting. They say "reading is fundamental" for learning a language. For math the saying is "counting is fundamental". For example, to learn the c's times table, just count by c's. Here c is one of the digits 0 to 9. As a specific example, counting by three's gives: 0, 3, 6, 9, 12, 15, 18, 21, 24, 27. Finally, note that we do not display the upper parts of each table as these values are obtained by reflecting the tables along their diagonals. For addition the diagonal contains double the number while for multiplication it contains the square of the number.

```
       Addition Table                      Multiplication Table
+ | 0  1  2  3  4  5  6  7  8  9 | x | 0 1  2  3  4  5  6  7  8  9
--|------------------------------|---|---------------------------
0 | 0                            | 0 | 0
1 | 1  2                         | 1 | 0 1
2 | 2  3  4                      | 2 | 0 2  4
3 | 3  4  5  6                   | 3 | 0 3  6  9
4 | 4  5  6  7  8                | 4 | 0 4  8 12 16
5 | 5  6  7  8  9 10             | 5 | 0 5 10 15 20 25
6 | 6  7  8  9 10 11 12          | 6 | 0 6 12 18 24 30 36
7 | 7  8  9 10 11 12 13 14       | 7 | 0 7 14 21 28 35 42 49
8 | 8  9 10 11 12 13 14 15 16    | 8 | 0 8 16 24 32 40 48 56 64
9 | 9 10 11 12 13 14 15 16 17 18 | 9 | 0 9 18 27 36 45 54 63 72 81
```

Fig. 1. Addition and Multiplication Tables. Entries above the main diagonal are obtained via relflection indicating that $c + d = d + c$ and $c \times d = d \times c$ for single digits c, d.

Here is my answer for Rule (1). The number 473 is short hand notation for 400 + 70 + 3. Now what about the single digits 1 to 9 and 0? We postpone this discussion to Section 2. For Rule (2): Doing operations left to right gives a continually refined approximation to the answer of the operation. For example, in a restuarant it usually suffices to add together the rounded dollar amounts of the bill in order to check its accuracy. Please do not regard Rule (4) as drudgery! Rule (5) hopefully will convince you that there are many way to do a problem and hence can alleviate drudgery. In fact, this paper is describing three new ways to do ordinary multiplication. Let me give you one example of creativity. What is 151×34? It is $150 \times 34 + 34$. But $150 \times 34 = 300 \times 17 = 5100$. So, the answer is 5134.

In Section 2 we briefly describe the meaning of zero to nine and + and ×. Section 3 describes CFM and in Section 4 we consider base 100 and base 1000 CFM. We close the paper in Section 5.

2 The Meaning of Zero to Nine and +, ×

Zero, or 0, means nothing. We can see this fact by observing the first columns of the addition and multiplication table. Note that zero is not a postive integer. However, 0 is indespensible for describing the positive integers in the positional power of ten notation that is universally used throughout the world for denoting the integers. Plus, or + means combine together. It signifies the operation of addition. Thus, $2 = 1 + 1$ from the addition table means two ones. Similary, 3 means $2 + 1$ and since 2 means two ones, 3 means three ones. It follows that a means a ones where a is any positive integer. Finally we come to ten which we write as 10. 10 is one more than 9 as we see from the addition table. There are ten single digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. All other positive integers are made up of these ten digits and we call our number system decimal or base 10.

Now 1 is the fundamental or the atomic unit. Positive integers have many applications. Money, measurement of length, area, volume, weigths, time, etc. are a few. For young people one needs to be concrete. Let us use money. In the US the penny is a good choice. Then 10 is the dime, 100 the dollar bill, etc.

Times, or $\times$ means repeatedly add together. It signifies the operation of multiplication. To be more specific: Multiply $a \times b$ means add a to itself b times or add b to itself a times. Here a, b are any positive integers. As an example, let $a = 3$ and $b = 2$. From the multiplication table, we see $3 \times 2 = 6$. And from the addition table, if we double 3 we get $3 + 3 = 6$ which are two rows of three ones.

3 Carry Free Multiplication

A prerequisite for learning to do CFM quickly is to know the addition and multiply tables by heart. This is part of rule (3) above. We will illustrate how to do CFM by an example. Hopefully, it will be clear that this example defines CFM in general. To do so, it will be very useful to have quad ruled paper. The reason is that the number $a = 57934$ is shorthand notation for $50000 + 7000 + 900 + 30 + 4$. This is an answer to rule (1) above. Let $b = 826$ which is short hand notation for $800 + 20 + 6$. We want to find their product $p = a \times b$. We need to form $5 \times 3 = 15$ products from the multiplication table and sum them together using the addition table. So, why is quad ruled essential? Some young people are "messy writers". Having square boxes to place the entries of the multiplication table in makes all children "non-messy" in doing their arithmetic. And, perhaps even more importantly, it forces them to think about which boxes they should put the multiplication table entries in.

Ordinary multiplication requires, in this case, multiplying a by 6, by 20, and by 800. In each of these three cases one can use both the addition and the multiplication tables in concert by doing one digit carry type multiplications by the three digits of b. It should be noted, and this is important for understanding, that multiplying by 20 and by 800 is the same as multiplying a by 2 and appending a single zero and by multiplying by 8 and appending two zeroes. It follows that by learning 100 addition facts and 100 multiply facts that one is actually learning an infinite number of facts. Again note that 100 facts is really 55 facts as $a + b = b + a$ and $a \times b = b \times a$. Now when one multiplies a by 6 one uses five multiply facts from row six and column six of the multiply table above: $50000 \times 6 = 300000$, $7000 \times 6 = 42000$, $900 \times 6 = 5400$, $30 \times 6 = 180$, and $4 \times 6 = 24$. Similarly, for 20 one uses five multiply facts from row two and column two with an appropriate number of zeroes appended. And for 800 one uses five multiply facts from row eight and column eight again with an appropriate number of zeroes appended.

It might appear that CFM is *not* possible to do in a consise straightforward way as Carry Multiplication (CM) can be done. However, it is. To see why we look at the multiplication table and see that $c \times d$ is never greater than $9 \times 9 = 81$. Here c and d are each one of the single digits 0 to 9. It follows that $c \times d < 100$. To effect no carry multiplication or CFM we write a as the sum of two numbers:

$a = a_1 + a_2$. a_1 contains the a's even powers of ten and a_2 contains a's odd powers of ten. In our example, $a_1 = 50904$ and $a_2 = 7030$. By rule (1) it should be clear that $a = a_1 + a_2$. Let $p_1 = a_1 \times b$ and $p_2 = a_2 \times b$. Then $p = p_1 + p_2$ as $a \times b = p_1 + p_2$ for all positive integers.

Note that by multiplying a_1 or a_2 by any single digit c *cannot* produce a carry because of the inserted zeros. A proof rests on the fact, stated above, that $c \times d < 100$ when c and d are single digits. Also needed here is the multiplication fact that $c \times 0 = 0$ and the addition fact that $c + 0 = c$. Let us now do these two products along with computing the product p by CM:

```
p1 = a1 x b  +   p2 = a2 x b = p = a x b

   50904     +       7030     =    57934
    x826             x826           x826
--------          -------        -------
  305424     +      42180     =   347604
 1018080     +     140600     =  115868
40723200     +    5624000     = 463472
--------          -------       --------
42046704     +    5806780     = 47853484
 5806780
--------
47853484
```

Fig. 2. Doing CFM as Two Multiplication Problems and Adding Their Results Together.

In the above computations of p_1 and p_2 note the first partial products are the sums $300000 + 5400 + 24$ and $42000 + 180$ from 5 partial products associated with multiplying a by six. One can see the three of the five math facts 30, 54, 24 concatenated together in the first partial product of p_1 and the remaining two of the five math facts 42, 18 concatenated together with a zero appended in the first partial product of p_2. Similarly, one can see ten more concatenated math table facts associated with multiplying a by both 20 and 800 in rows two and three of Figure 2.

There are ten carries in the multiply part of CM problem $p = a \times b$ out of a maximum of twelve. In the addition part of the CM problem there are three carries out a maximum of seven. In contrast, in the CFM problem, there are nine rote copies for $p1$ and six rote copies for $p2$. No carry occurs in either problem; that is why we call this multipication algorithm CFM. There are three addition problems in CFM. The first requires three carries and the second requires one carry. The final simple addition of the even and odd product terms requires two carries out of a maximum of seven.

3.1 CFM as a Single Multiplication

We had $p = p_1 + p_2$. Clearly, these problems can be combined into a single problem. Let c be a single digit. We now want to form $a \times c$ in a carry free manner. We illustrate with our example $a = 57934$ and $c = 6$. Instead of forming a single product row we form two product rows just like we did above. However, we form these two rows together in the order, even, odd, even, odd, even. We do as a child might do it. Say, $4 \times 6 = 24$ and place 24 in the first row at the far right. Say, $30 \times 6 = 180$ and place 180 in the second row at the far right. Say $900 \times 6 = 5400$ and concatenate 54 to the left of 24 in the first row. Note that we are really doing the addition problem $5400 + 24 = 5424$. For our algorithm this addition becomes concatenation. Next say $7000 \times 6 = 42000$ and concatenate 42 to the right of 180 in row two. Again note that we are really doing the addition problem $42000 + 180 = 42180$. Finally, say $50000 \times 6 = 300000$ and concatenate 30 to the left of 5424 in the first row.

The above procedure is done exactly the same for all the digits of $b = 826$. Thus we will get $2 \times 3 = 6$ rows for our example where, in general, the 3 is the number of digits in the multipler b; see the left half of Figure 3.

```
p = a x b    p = a x b

   57934        57934
    x826         x826
--------     --------
  305424 +
   42180   =   347604
 1018080 +
  140600   =  115868
40723200 +
 5624000   = 463472
--------     --------
47853484     47853484
```

Fig. 3. CFM as a Single Multiplication Problem. This is also Box Multiplication without using the Boxes.

3.2 Another Way To Do Ordinary Multiplication

In our second way to do CFM, see Section 3.1 and Figure 3, we had two rows for each digit c of the multiplier b. For each pair of rows obtained we could stop and add these two rows together. It should be obvious that this gives a new procedure for doing ordinary multiplication or CM. This is indicated in both parts of Figure 3. In our new procedure the multiplication part of $a \times c$ is carry free. The carries only occur when the pairs of CFM rows are added together.

4 Base 100 and 1000 CFM

In subsections 4.1 and 4.2 we describe base 100 CFM and in subsection 4.3 we describe base 1000 CFM. In Section 4.1 we show two ways to multiply two digit numbers together in your head. This is a requirement for doing base 100 CFM. This requirement could be removed if students wants to do scratch work while learning how to multiply two digit numbers together in their heads. In subsection 4.2 we use this ability to describe base 100 CFM. Finally, in subsection 4.3 we describe base 1000 CFM by using base 100 CFM.

4.1 Base 100 CFM

To do Base 100 CFM one needs to be able to multiply two digit numbers in their heads and arrive at a four digit answer. I just read the book [2] which convinces me this is possible. A prerequisite for it is to be able to do two and three digit addition in a very simple one digit at a time manner. This brings us to an application of Rule (2). Consider a two digit addition: $45 + 56 = 101$. Say $45 + 50 + 6 = 95 + 6$ and then say $95 + 6 = 101$. The idea is to do a series of very simple adds at each step. To not forget as one proceeds one can say "in their minds" each simple problem. Consider three digit addition: An example is $437+683 = 1120$. Say $437+600+83 = 1037+83$. Then say $1037+80+3 = 1117+3$ and finally say $1117 + 3 = 1120$. This approach is used in [2].

Now consider how we can use the above to do two digit by two digit multiplications. Consider $ab \times cd$ where $ab = 79$ and $cd = 26$, or 79×26. This means $(70 + 9) \times (20 + 6)$. We show two new ways to do this:

Method 1 for Two Digit by Two Digit Multiply in your Head We start with the two outer terms $a \times c$ and $b \times d$. For our example, we need to add $70 \times 20 + 9 \times 6$ However, by knowing the multiplication table by heart this is $1400 + 54$. Now this addition is trivial as it is merely a concatenation of two multiplication table entries to form an initial four digit product or starting answer. So, we have 1454 as our partial answer and we are half way home in a very trivial way. The next thing to note is that the two cross terms are multiples of ten. Hence, the units digit, 4 in our example, will not change. This means our answer can be obtained by adding two table entries to a single three digit number, 145 in our example. We will not do this; however, we should be aware of this general fact for all 10 thousand two digit by two digit multiplications. We now take the $a \times d$ cross term which is 70×6 or 420 from the table and we want to add it to 1454. Say, $1454 + 420$=$1454 + 400 + 20 = 1854 + 20$. Next say, $1854 + 20 = 1874$. We now have 1874 in our mind and we want add $b \times c$ the second and last cross term to it. This is 9×20 which from the table is 180. Hence, our answer is $1874 + 180 = 1874 + 100 + 80 = 1974 + 80$. Now say $1974 + 80 = 2054$. This is our final answer.

Method 2 for Two Digit by Two Digit Multiply in your Head We start with the two cross terms 70×6 and 9×20. Say $420 + 180 = 520 + 80$ and then say $520 + 80 = 600$. Now say, $1400 + 600 = 2000$. Finally, say $2000 + 54 = 2050 + 4$ and say $2050 + 4 = 2054$. What we have done is to use Rules (3) and (2) and silently talk our way to the answer. Note the cross terms ad and bc always are a multiple of 10 and so we can forget about the appended zero. The leading term ac is a multiply of 100 and hence we can drop one zero when we add it to the sum of the cross terms. The addition of the leading term + sum of the cross terms is the the sum of a three digit number ending in zero and usually a two digit number. Occasionly, the sum of the cross terms is a three digit number; if so, the leading digit must be a one. Finally, the last term bd is easily added to this sum of the three other terms which is a multiple of ten. To see this just add the tens part of bd to the three term sum and then append the ones part of bd to the resulting sum.

4.2 Conclusion of Base 100 CFM

Below, we carry out our first method, see Section 3, using base 100 arithmetic.

```
1st part      2nd part

   50034          7900
    x826          x826
--------      --------
 1300884        205400
40027200       6320000
--------      --------
41328084       6525400
 6525400
--------
47853484
```

First, we compute 34×26 in our heads by Method 2 of Section 4.1. Say $180 + 080 = 260$ and the add $600 + 260 = 860$. Finally, say $860 + 24 = 884$ and enter 0884 to start the first row. Next compute 05×26. Because $c = 0$ there is only one cross term and no leading term. We get 100 which gets added to 30 to give 130 as our answer. Also, we can use Rule (5) here and note that $5 \times 26{=}10 \times 13 = 130$. In either case append 130 to 0884. Next we do row two of the first problem. We compute 34×8 in our heads. Let us use rule (5) and see that $34 \times 8 = 68 \times 4 = 136 \times 2 = 272$. Or, see that $34 \times 8 = 17 \times 16 = 16^2 + 16 = 256 + 16 = 272$. Thus place 027200 to start the second row. Next append $05 \times 08 = 40$ to row two to get the result 40027200. Now turn to the second problem. We have done above the computation 79×26. Place 205400 to start the first row. For our example, this also ends the first row. Now we do 79×8 using Rule (5). The answer is $80 \times 8 - 8 = 632$. So place 6320000 to start and finish the second row. What now remains to be done are three addition problems which are straightforward.

4.3 Base 1000 CFM

We briefly sketch the Base 1000 CFM via our common example. $a = 57934$ is $57000+934 = 057\times10^3+934$. This is the base 1000 representation of a. The base 1000 representation of $b = 826$ is b itself. Hence in base 1000 $a \times b$ is $57000 \times b + 934 \times b$. Now we do these two computations as base 100 computations. We use method two of Section 4.1. We have already done $34\times26 = 0884$ and $9\times26 = 234$ and $34 \times 8 = 272$. These are the first three rows of the problem 934×826. The last row is trivial. It is multiplication table fact $900 \times 800 = 720000$. For the second problem, we compute 826×57000 instead of 57000×826. We have to do mental computations 26×57 and 8×57. For the first say $140 + 300 = 440$. Then say $1000 + 440 = 1440$. Finally, say $1440 + 42 = 1482$. Row one of problem two becomes 1482000. Lastly, $8 \times 57 = 4 \times 114 = 2 \times 228 = 456$. Row two of problem two becomes 45600000. Now add the two problem answers together to get the final answer.

```
Problem 1     Problem 2

      934           826
     x826        x57000
--------      --------
      884       1482000
    23400      45600000
    27200      --------
   720000      47082000
--------        +771484
   771484      --------
               47853484
```

5 Summary and Conclusion

Our first way of doing CFM presents ordinary multiplication or CM as doing two simpler CFM problems. With or without quad ruled paper the multiply part of CFM consist of rote memory recall of multiplication table facts and writing these table sub-products in their correct power of 10 positions. Having quad ruled paper make this process foolproof for "messy writers". For the addition part of the CFM each of two subproblens will usually have fewer carries than the CM problem. The final adding of p_1 and p_2 together is a straightforward two addend sum.

Our second way of doing CFM is a merging together of our first way. Instead of having three addition problems we now have one larger addition problem consisting of double the number of rows. We mention that this second way is another way of doing box multiplication which many elementary school students prefer over doing CM.

In our second way, since we have double the numbers of rows to add we can add pairs of rows together as the multiplication proceeds. If we do this we get another way to do ordinary multiplication. This new way then has its multiplication part as carry free. It also breaks CM into three distinct phases and hence has pedigodical value in the learning of CM.

In conclusion, we think it can be argued that CFM as described here in three different ways teaches better the actual meaning of doing ordinary multiplication.

In Section 4 we have described how to do CFM in base 100 and 1000.

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