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Submodular Maximization over Multiple Matroids via Generalized Exchange Properties

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Abstract

Submodular-function maximization is a central problem in combinatorial optimization, generalizing many important NP-hard problems including Max Cut in digraphs, graphs and hypergraphs, certain constraint satisfaction problems, maximum-entropy sampling, and maximum facility-location problems. Our main result is that for any $k \geq 2$ and any $\varepsilon > 0$, there is a natural local-search algorithm which has approximation guarantee of $1/(k+\varepsilon)$ for the problem of maximizing a monotone submodular function subject to k matroid constraints. This improves a $1/(k+1)$-approximation of Nemhauser, Wolsey and Fisher, obtained more than 30 years ago. Also, our analysis can be applied to the problem of maximizing a linear objective function and even a general non-monotone submodular function subject to k matroid constraints. We show that in these cases the approximation guarantees of our algorithms are $1/(k-1+\varepsilon)$ and $1/(k+1+1/k+\varepsilon)$, respectively.

Introduction

In this paper, we consider the problem of maximizing a non-negative submodular function f, defined on a (finite) ground set N, subject to matroid constraints. A function $f : 2^N \to \mathbb{R}$ is *submodular* if for all $S, T \subseteq N$, $f(S \cup T) + f(S \cap T) \leq f(S) + f(T)$. Throughout, we assume that our submodular function f is given by a *value oracle*; i.e., for a given set $S \subseteq N$, an algorithm can query an oracle to find the value $f(S)$. Furthermore, all submodular functions that we deal with are assumed to be non-negative. Without loss of generality, we take the ground set N to be $[n] := \{1, 2, \cdots, n\}$.

We assume some familiarity with matroids [16] and associated algorithmics [18]. Briefly, we denote a matroid $\mathcal{M}$ by an ordered pair $(N, \mathcal{I})$, where N is the ground set of $\mathcal{M}$ and $\mathcal{I}$ is the set of independent sets of $\mathcal{M}$. For a given matroid $\mathcal{M}$, the associated *matroid constraint* is $S \in \mathcal{I}(\mathcal{M})$. In our usage, we deal with k matroids $\mathcal{M}_i = (N, \mathcal{I}_i)$, $i = 1, \ldots, k$, on the common ground set N.

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It is no coincidence that we use N for the ground set of our submodular function f as well as for the ground set of our matroids $\mathcal{M}_i = (N, \mathcal{I}_i)$, $i = 1, \ldots, k$. Indeed, our optimization problem is

$$\max \left\{ f(S) \ : \ S \in \cap_{i=1}^{k} \mathcal{I}_i \right\}.$$

Where necessary, we make some use of other standard matroid notation: For a matroid $\mathcal{M} = (N, \mathcal{I})$, we denote its rank function by $r_{\mathcal{M}}$ and its dual by $\mathcal{M}^*$. A base of $\mathcal{M}$ is a $J \in \mathcal{I}$ having cardinality $r_{\mathcal{M}}(N)$. For a set $S \subset N$, we let $\mathcal{M} \backslash S$, $\mathcal{M}/S$, and $\mathcal{M}|S$ denote deletion of S, contraction of S, and restriction to S, respectively. We recall:

- $r_{\mathcal{M}}(S) := \max \{|J| \ : \ J \subseteq S \ , J \in \mathcal{I}\}$.
- $\mathcal{M}^*$ has ground set N and independent sets $\{X \in N \ : \ N \setminus X$ contains a base of $\mathcal{M}\}$
- $\mathcal{M} \backslash S$ ($\mathcal{M}$ *delete* S) has ground set $N \setminus S$ and independent sets $\{X \subseteq N \setminus S \ : \ X \in \mathcal{I}\}$
- Let J be a maximal subset of S that is independent in $\mathcal{M}$ Then $\mathcal{M}/S$ ($\mathcal{M}$ *contract* S) has ground set $N \setminus S$ and independent sets $\{X \subseteq N \setminus S \ : \ X \cup J \in \mathcal{I}\}$
- $\mathcal{M}|S$ ($\mathcal{M}$ *restricted to* S) is simply $M \backslash (N/ \setminus S)$ ($\mathcal{M}$ delete $N \setminus S$)

Previous Results. Optimizing submodular functions is a central topic in optimization [12]. While submodular minimization is a polynomially solvable problem [10, 19], the maximization variants are usually NP-hard because they include either Max Cut, some variant of facility location, or set packing problems.

There are essentially two algorithmic techniques that work for submodular maximization. The greedy algorithm was first applied to a wide range of submodular maximization problems in the late-70's and early-80's [3, 4, 5, 13, 14]. The most relevant result for our purposes is the proof that the greedy algorithm has approximation guarantee of $1/(k+1)$ for the problem of maximizing a monotone submodular function subject to k matroid constraints [14]. Until recently, this algorithm had the best established performance guarantee for the problem with general matroid constraints. Recently, Vondrák [20] designed the continuous greedy algorithm that achieves a $(1-1/e)$-approximation for the problem with $k=1$, i.e. subject to a single matroid constraint. This result is optimal in the oracle model even for the case of a uniform matroid constraint [15], and also optimal unless $P = NP$ for the special case of the maximum coverage problem [6].

The second algorithmic technique that works for submodular maximization is local search. Cornuéjols et al. [4] show that a local-search algorithm achieves a constant-factor approximation guarantee for the maximum uncapacitated facility-location problem which is a special case of submodular maximization. Note that in this case the greedy approach has a better performance guarantee. The maximum k-dimensional matching problem is a problem of maximizing a linear objective function subject to k special partition matroid constraints. The approximation guarantees for maximum k-dimensional matching are established with local-search algorithms and have performance guarantees $2/(k+\varepsilon)$ for linear function with $\{0,1\}$-coefficients, and $2/(k+1+\varepsilon)$ for a general linear function, even in the more general settings of set packing [9] and independent set problems in $(k+1)$-claw free graphs [1]. Unfortunately, general matroid constraints seem to complicate the matter; the best approximation guarantee for the problem of maximizing a linear function subject to k general-matroid constraints is $1/k$ [17].

The result of [14] can be improved in the case when all k constraints correspond to partition matroids. A simple local-search algorithm has approximation guarantee of $1/(k+\varepsilon)$ [11] for this variant of the problem. The analysis strongly uses the properties of partition matroids. It is based on the realtively simple augmenting-paths exchange properties of partition matroids that do not hold in general.

Local-search algorithms were also designed for non-monotone submodular maximization [7, 11]. The best approximation guarantee known for unconstrained submodular maximization is $2/5-\varepsilon$ [7]. For the problem with k general matroids the best known approximation is $1/(k+2+1/k+\varepsilon)$ [11].

Our Results and Techniques. In this paper we analyze a natural local-search algorithm: Given a feasible solution, i.e. a set S that is independent in each of the k matroids, our local-search algorithm tries to add at most p elements and delete at most kp elements from S. If there is such a local move that generates a feasible solution that improves the objective value, our algorithm repeats the local-search procedure with that new solution, until no improvement is possible. Our main result is that for $k \geq 2$, every locally-optimal feasible solution S satisfies the inequality

$$\left(k+\frac{1}{p}\right)\cdot f(S) \;\geq\; f(S\cup C)+\left(k-1+\frac{1}{p}\right)\cdot f(S\cap C),$$

for every feasible solution C.

We also provide an approximate variant of the local-search procedure that finds an approximate locally-optimal solution in polynomial time, while losing a factor of $1+\varepsilon$ in the left-hand side of the above inequality (Lemma 3.3) for arbitrary $\varepsilon > 0$. Therefore, we obtain a polynomial-time local-search algorithm with approximation guarantee $1/(k+\varepsilon)$ for the problem of maximizing a monotone increasing submodular function subject to k matroid constraints. This algorithm gives a $1/(k-1+\varepsilon)$-approximation in the case when the objective function is linear.

The main technical contributions of this paper are two new exchange properties for matroids. One is a generalization of the classical Rota Exchange Property (Lemma 2.8) and another is a generalization of the exchange property for partition matroids based on augmenting paths (Lemma 2.4). We believe that both properties and proofs are interesting in their own right.

In the case of a general non-monotone submodular objective functions, one round of local search is not enough, but applying the local search iteratively, as in [11], one can obtain an approximation algorithm with performance guarantee of $1/(k+1+1/(k-1)+\varepsilon)$.

In §1, we establish some useful properties concerning exact k-covers and submodular functions. In §2, we establish some exchange properties for matroids. In §3, we describe and analyze our local-search algorithm.

1 Some Useful Properties of Submodular Functions

Lemma 1.1. *Let f be a non-negative submodular function on N. Let $S' \subseteq S \subseteq N$, and let $\{T_l\}_{l=1}^t$ be a collection of subsets of $S \setminus S'$ such that each element of $S \setminus S'$ appears in exactly k of these subsets. Then*

$$\sum_{l=1}^{t} [f(S) - f(S \setminus T_l)] \;\leq\; k \;\big(f(S) - f(S')\big)\,.$$

Proof. Let $s = |S|$ and $c = |S'|$. Without loss of generality, we can assume that $S' = \{1, 2, \cdots, c\} = [c]$ and $S = \{1, 2, \cdots, s\} = [s]$ Then for any $T \subseteq S \setminus S'$, by submodularity: $f(S) - f(S \setminus T) \;\leq\; \sum_{p \in T} [f([p]) - f([p-1])]$. Using this we obtain

$$\begin{aligned}
\sum_{l=1}^{t} [f(S) - f(S \setminus T_l)] \;&\leq\; \sum_{l=1}^{t} \sum_{p \in T_l} [f([p]) - f([p-1])] \\
&=\; k \sum_{i=c+1}^{s} [f([i]) - f([i-1])] \\
&=\; k \;\big(f(S) - f(S')\big)\,.
\end{aligned}$$

The second equality follows from $S \setminus C = \{c+1, \cdots, s\}$ and the fact that each element of $S \setminus C$ appears in exactly k of the sets $\{T_l\}_{l=1}^t$. The last equality is due to a telescoping summation. □

Lemma 1.2. *Let f be a non-negative submodular function on N. Let $S \subseteq N$ and $C \subseteq N$, and let $\{T_l\}_{l=1}^t$ be a collection of subsets of $C \setminus S$ such that each element of $C \setminus S$ appears in exactly k of these subsets. Then*

$$\sum_{l=1}^{t} [f(S \cup T_l) - f(S)] \;\geq\; k \;\left(f(S \cup C) - f(S)\right).$$

Proof. Let $s = |S|$ and $c = |C \setminus S|$. Without loss of generality, we can assume that $S = \{1, 2, \cdots, s\}$ and that $C \setminus S = \{s+1, 2, \cdots, c\}$. Then for any $T_l \subseteq C \setminus S$, by submodularity: $f(S \cup T_l) - f(S) \;\geq\; \sum_{p \in T_l} [f(S \cup \{p\}) - f(S)]$. Using this we obtain

$$\begin{aligned}
\sum_{l=1}^{t} [f(S \cup T_l) - f(S)] \;&\geq\; \sum_{l=1}^{t} \sum_{p \in T_l} [f(S \cup \{p\}) - f(S)] \\
&=\; k \sum_{p \in C \setminus S} [f(S \cup \{p\}) - f(S)] \\
&\geq\; k\, [f(S \cup C) - f(S)]\,.
\end{aligned}$$

The last inequality follows from submodularity. □

2 New Exchange Properties of Matroids

2.1 Two matroids

An exchange digraph is a well-known construct for devising efficient algorithms for exact maximization of modular functions on a pair of matroids on a common ground set (for example, see [18]). We are interested in submodular maximization, k matroids and approximation algorithms; nevertheless, we are able to make use of such exchange digraphs, once we establish some new properties of them.

Let $\mathcal{M}_1 = (N, \mathcal{I}_1)$ and $\mathcal{M}_2 = (N, \mathcal{I}_2)$ be a pair of matroids on ground set N. For $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, we define a pair of digraphs $D_{\mathcal{M}_1}(I)$ and $D_{\mathcal{M}_2}(I)$ on node set N as follows:

- For each $i \in I, j \in N \setminus I$ such that $I - i + j \in \mathcal{I}_1$, we have an arc (i, j) of $D_{\mathcal{M}_1}(I)$;
- For each $i \in I, j \in N \setminus I$ such that $I - i + j \in \mathcal{I}_2$, we have an arc (j, i) of $D_{\mathcal{M}_2}(I)$.

The arcs in $D_{\mathcal{M}_1}(I)$ encode valid swaps in $\mathcal{M}_1$, and the arcs in $D_{\mathcal{M}_2}(I)$ encode valid swaps in $\mathcal{M}_2$.

In what follows, we assume that I is our current solution and J is the optimal solution. We also assume that $|I| = |J|$. If not, we extend I or J by dummy elements so that we maintain independence in both matroids (more details later). We use two known lemmas from matroid theory.

Lemma 2.1 ([18, Corollary 39.12a]). *If $|I| = |J|$ and $I, J \in \mathcal{I}_i$ ($i = 1$ or 2), then $D_{\mathcal{M}_i}(I)$ contains a perfect matching between $I \setminus J$ and $J \setminus I$.*

Lemma 2.2 ([18, Theorem 39.13]). *Let $|I| = |J|$, $I \in \mathcal{I}_i$, and assume that $D_{\mathcal{M}_i}(I)$ has a unique perfect matching between $I \setminus J$ and $J \setminus I$. Then $J \in \mathcal{I}_i$.*

Next, we define a digraph $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$ on node set N as the union of $D_{\mathcal{M}_1}(I)$ and $D_{\mathcal{M}_2}(I)$. A dicycle in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$ corresponds to a chain of feasible swaps. However, observe that it is not necessarily the case that the entire cycle gives a valid exchange in both matroids.

If $|I| = |J|$ and $I, J \in \mathcal{I}_1 \cap \mathcal{I}_2$, this means we have two perfect matchings on $I \Delta J$ which together form a collection of dicycles in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$. However, only the uniqueness of a perfect matching assures us that we can legally perform the exchange. This motivates the following definition.

Definition 2.3. We call a dicycle C in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$ *irreducible* if $C \cap D_{\mathcal{M}_1}(I)$ is the unique perfect matching in $D_{\mathcal{M}_1}(I)$ and $C \cap D_{\mathcal{M}_2}(I)$ is the unique perfect matching in $D_{\mathcal{M}_2}(I)$, covering the vertex set $V(C)$. Otherwise, we call C *reducible*.

The following, which is our main technical lemma, allows us to consider only irreducible cycles. The proof follows closely the ideas of matroid intersection (see [18, Lemma 41.5α]).

Lemma 2.4. *Let $\mathcal{M}_1 = (N, \mathcal{I}_1)$ and $\mathcal{M}_2 = (N, \mathcal{I}_2)$ be two matroids on ground set N. Suppose that $I, J \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $|I| = |J|$. Then there is $s \geq 0$ and a collection of irreducible dicycles $\{C_1, \ldots, C_m\}$ (possibly with repetition) in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$, using only elements of $I \Delta J$, so that each element of $I \Delta J$ appears in exactly 2^s of the dicycles.*

Proof. Consider $D_{\mathcal{M}_1,\mathcal{M}_2}(I) = D_{\mathcal{M}_1}(I) \cup D_{\mathcal{M}_2}(I)$. By Lemma 2.1, there exists a perfect matching between $I \setminus J$ and $J \setminus I$, both in $D_{\mathcal{M}_1}(I)$ and $D_{\mathcal{M}_2}(I)$. Let us denote these two perfect matchings by M_1, M_2. The union $M_1 \cup M_2$ forms a subgraph of out-degree 1 and in-degree 1 on $I \Delta J$. Therefore, it decomposes into a collection of dicycles $C_1, \ldots, C_m$. If they are all irreducible, we are done.

If C_i is not irreducible, it means that either $M_1' = C_i \cap D_{\mathcal{M}_1}(I)$ or $M_2' = C_i \cap D_{\mathcal{M}_2}(I)$ is not a unique perfect matching on $V(C_i)$. Let us assume, without loss of generality, that there is another perfect matching M_1'' in $D_{\mathcal{M}_1}(I)$. We consider the disjoint union $M_1' + M_1'' + M_2' + M_2'$, duplicating arcs where necessary. This is a subgraph of out-degree 2 and in-degree 2 on $V(C_i)$, which decomposes into dicycles $C_{i1}, \ldots, C_{it}$, covering each vertex of C_i exactly twice:

$$V(C_{i1}) + V(C_{i2}) + \ldots + V(C_{it}) = 2V(C_i).$$

Because $M_1' \neq M_1''$, we have a chord of C_i in M_1'', and we can choose the first dicycle so that it does not cover all of $V(C_i)$. So we can assume that we have $t \geq 3$ dicycles, and at most one of them covers all of $V(C_i)$. If there is such a dicycle among $C_{i1}, \ldots, C_{it}$, we remove it and duplicate the remaining dicycles. Either way, we end up with a collection of dicycles $C_{i1}, \ldots, C_{it'}$ such that each of them is shorter than C_i and together they cover each vertex of C_i exactly twice.

We repeat this procedure for each reducible dicycle C_i. For irreducible dicycles C_i, we just duplicate C_i to obtain $C_{i1} = C_{i2} = C_i$. This completes one stage of our procedure. After the completion of the first stage, we have a collection of dicycles $\{C_{ij}\}$ covering each vertex in $I \Delta J$ exactly twice.

As long as there exists a reducible dicycle in our current collection of dicycles, we perform another stage of our procedure. This means decomposing all reducible dicycles and duplicating all irreducible dicycles. In each stage, we double the number of dicycles covering each element of $I \Delta J$. To see that this cannot be repeated indefinitely, observe that every stage decreases the size of the longest reducible dicycle. All dicycles of length 2 are irreducible, and therefore the procedure terminates after a finite number of stages s. Then, all cycles are irreducible and together they cover each element of $I \Delta J$ exactly 2^s times. $\square$

We remark that of course the procedure in the proof of Lemma 2.4 is very inefficient, but it is not part of our algorithm — it is only used for this proof.

Next, we extend this Lemma 2.4 to sets I, J of different size, which forces us to deal with dipaths as well as dicycles.

Definition 2.5. We call a dipath or dicycle A *feasible* in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$, if

- $I \Delta V(A) \in \mathcal{I}_1 \cap \mathcal{I}_2$, and

- For any sub-dipath $A' \subset A$ such that each endpoint of A' is either an endpoint of A or an element of I, we also have $I \Delta V(A') \in \mathcal{I}_1 \cap \mathcal{I}_2$.

First, we establish that irreducible dicycles are feasible.

Lemma 2.6. *Any irreducible dicycle in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$ is also feasible in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$.*

Proof. An irreducible dicycle C consists of two matchings $M_1 \cup M_2$, which are the unique perfect matchings on $V(C)$, in $D_{\mathcal{M}_1}(I)$ and $D_{\mathcal{M}_2}(I)$ respectively. Therefore, we have $I \Delta V(C) \in \mathcal{I}_1 \cap \mathcal{I}_2$ by Lemma 2.2.

Consider any sub-dipath $A' \subset C$ whose endpoints are in I. (C has no endpoints, so the other case in Definition 2.5 does not apply.) This means that A' has even length. Suppose that $a_1 \in V(A')$ is the endpoint incident to an edge in $M_1 \cap A'$ and $a_2 \in V(A')$ is the other endpoint, incident to an edge in $M_2 \cap A'$. Note that any subset of M_1 or M_2 is again a unique perfect matching on its respective vertex set, because otherwise we could produce a different perfect matching on $V(C)$. We can view $I \Delta V(A')$ in two possible ways:

- $I \Delta V(A') = (I - a_1) \Delta (V(A') - a_1)$; because $V(A') - a_1$ has a unique perfect matching $M_2 \cap A'$ in $D_{\mathcal{M}_2}(I)$, this shows that $I \Delta V(A') \in \mathcal{I}_2$.
- $I \Delta V(A') = (I - a_2) \Delta (V(A') - a_2)$; because $V(A') - a_2$ has a unique perfect matching $M_1 \cap A'$ in $D_{\mathcal{M}_1}(I)$, this shows that $I \Delta V(A') \in \mathcal{I}_1$.

□

Finally, we establish the following property of possible exchanges between arbitrary solutions I, J (not necessarily of the same size).

Lemma 2.7. *Let $\mathcal{M}_1 = (N, \mathcal{I}_1)$ and $\mathcal{M}_2 = (N, \mathcal{I}_2)$ be two matroids and let $I, J \in \mathcal{I}_1 \cap \mathcal{I}_2$. Then there is $s \geq 0$ and a collection of dipaths/dicycles $\{A_1, \ldots, A_m\}$ (possibly with repetition), feasible in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$, using only elements of $I \Delta J$, so that each element of $I \Delta J$ appears in exactly 2^s dipaths/dicycles A_i.*

Proof. If $|I| = |J|$, we are done by Lemmas 2.4 and 2.6. If $|I| \neq |J|$, we extend the matroids by new "dummy elements" E, independent of everything else (in both matroids), and add them to I or J, to obtain sets of equal size $|\tilde{I}| = |\tilde{J}|$. We denote the extended matroids by $\tilde{\mathcal{M}_1} = (N \cup E, \tilde{\mathcal{I}_1}), \tilde{\mathcal{M}_2} = (N \cup E, \tilde{\mathcal{I}_2})$. We consider the graph $D_{\tilde{\mathcal{M}}_1,\tilde{\mathcal{M}}_2}(\tilde{I})$. Observe that the dummy elements do not affect independence among other elements, so the graphs $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$ and $D_{\tilde{\mathcal{M}}_1,\tilde{\mathcal{M}}_2}(\tilde{I})$ are identical on $I \cup J$.

Applying Lemma 2.4 to $\tilde{I}, \tilde{J}$, we obtain a collection of irreducible dicycles $\{C_1, \ldots, C_m\}$ on $\tilde{I} \Delta \tilde{J}$ such that each element appears in exactly 2^s dicycles. Let $A_i = C_i \setminus E$. Obviously, the sets $V(A_i)$ cover $I \Delta J$ exactly 2^s times. We claim that each A_i is either a feasible dicycle, a feasible dipath, or a collection of feasible dipaths (in the original digraph $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$).

First, assume that $C_i \cap E = \emptyset$. Then $A_i = C_i$ is an irreducible cycle in $D_{\mathcal{M}_1,\mathcal{M}_2}(I)$ (the dummy elements are irrelevant). By Lemma 2.6, we know that $A_i = C_i$ is a feasible dicycle.

Next, assume that $C_i \cap E \neq \emptyset$. C_i is still a feasible dicycle, but in the extended digraph $D_{\tilde{\mathcal{M}}_1,\tilde{\mathcal{M}}_2}(\tilde{I})$. We remove the dummy elements from C_i to obtain $A_i = C_i \setminus E$, a dipath or a collection of dipaths. Consider any sub-dipath A' of A_i, possibly $A' = A_i$, satisfying the assumptions of Definition 2.5. A_i does not contain any dummy elements. If both endpoints of A' are in I, it follows from the feasibility of C_i that $\tilde{I}\Delta V(A') \in \tilde{\mathcal{I}}_1 \cap \tilde{\mathcal{I}}_2$, and hence $I\Delta V(A') = (\tilde{I}\Delta V(A')) \setminus E \in \mathcal{I}_1 \cap \mathcal{I}_2$.

If an endpoint of A' is outside of I, then it must be an endpoint of A_i. This means that it has a dummy neighbor in $\tilde{I} \cap C_i$ that we deleted. (Note that this case can occur only if we added dummy elements to I, i.e. $|I| < |J|$.) In that case, extend the path to A'', by adding the dummy neighbor(s) at either end. We obtain a dipath from $\tilde{I}$ to $\tilde{I}$. By the feasibility of C_i, we have $\tilde{I}\Delta V(A'') \in \tilde{\mathcal{I}}_1 \cap \tilde{\mathcal{I}}_2$, and therefore $I\Delta V(A') = (\tilde{I}\Delta V(A'')) \setminus E \in \mathcal{I}_1 \cap \mathcal{I}_2$. □

2.2 A generalized Rota-exchange property

Next, we establish a very useful exchange property for a pair of bases of a single matroid.

Lemma 2.8. *Let $\mathcal{M} = (N, \mathcal{I})$ be a matroid and A, B bases in $\mathcal{M}$. Let $A_1, \ldots, A_m$ be subsets of A such that each element of A appears in exactly q of them. Then there are sets $B_1, \ldots, B_m \subseteq B$ such that each element of B appears in exactly q of them, and for each i*

$$A_i \cup (B \setminus B_i) \ \in \ \mathcal{I}.$$

Remark 2.9. A very special case of Lemma 2.8, namely when $m = 2$ and $q = 1$, attracted significant interest when it was conjectured by G.-C. Rota and proved in [2, 8, 21]; see [18, (39.58)].

Proof. (Lemma 2.8) We can assume for convenience that A and B are disjoint (otherwise we can make $\{B_i\}$ equal to $\{A_i\}$ on the intersection $A \cap B$ and continue with a matroid where $A \cap B$ is contracted).

For each i, we define a matroid $\mathcal{N}_i = (\mathcal{M}/A_i)|B$, where we contract A_i and restrict to B. In other words, $S \subseteq B$ is independent in $\mathcal{N}_i$ exactly when $A_i \cup S \in \mathcal{I}$. The rank function of $\mathcal{N}_i$ is

$$r_{\mathcal{N}_i}(S) \ = \ r_{\mathcal{M}}(A_i \cup S) - r_{\mathcal{M}}(A_i) \ = \ r_{\mathcal{M}/A_i}(S).$$

Let $\mathcal{N}_i^*$ be the dual matroid to $\mathcal{N}_i$. Recall that the ground set is now B. By definition, $T \subseteq B$ is a spanning set in $\mathcal{N}_i^*$ if and only if $B \setminus T$ is independent in $\mathcal{N}_i$, i.e. if $A_i \cup (B \setminus T) \in \mathcal{I}$. The bases of $\mathcal{N}_i^*$ are minimal such sets T; these are the candidate sets for B_i , which can be exchanged for A_i . The rank function of the dual matroid $\mathcal{N}_i^*$ is (by [18, (Theorem 39.3)])

$$\begin{aligned}
r^*_{\mathcal{N}_i}(T) &= |T| - r_{\mathcal{N}_i}(B) + r_{\mathcal{N}_i}(B \setminus T) \\
&= |T| - r_{\mathcal{M}}(A_i \cup B) + r_{\mathcal{M}}(A_i \cup (B \setminus T)) \\
&= |T| - |B| + r_{\mathcal{M}}(A_i \cup (B \setminus T)) \\
&= r_{\mathcal{M}/(B \setminus T)}(A_i) \ .
\end{aligned}$$

Observe that the rank of $\mathcal{N}_i^*$ is $r^*_{\mathcal{N}_i}(B) = |A_i|$.

Now, let us consider a new ground set $\hat{B} = B \times [q]$. We view the elements $\{(i,j) : j \in [q]\}$ as parallel copies of i. For a set $T \subseteq \hat{B}$, we define its "projection" to B as

$$\pi(T) \;=\; \{i \in B \;\mid\; \exists j \in [q] \text{ with } (i,j) \in T\}.$$

A natural extension of $\mathcal{N}_i^*$ to $\hat{B}$ is a matroid $\hat{\mathcal{N}}_i^*$ where a set T is independent if $\pi(T)$ is independent in $\mathcal{N}_i^*$. The rank function of $\hat{\mathcal{N}}_i^*$ is

$$r_{\hat{\mathcal{N}}_i^*}(T) \;=\; r_{\mathcal{N}_i^*}(\pi(T)) \;=\; r_{\mathcal{M}/(B\setminus\pi(T))}(A_i)\;. \tag{1}$$

The question now is whether $\hat{B}$ can be partitioned into $B'_1, \ldots, B'_m$ so that B'_i is a base in $\hat{\mathcal{N}}_i^*$. If this is true, then we are done, because each $B_i = \pi(B'_i)$ would be a base of $\mathcal{N}_i^*$ and each element of B would appear in q sets B_i.

To prove this, consider the union of our matroids, $\hat{\mathcal{N}}^* := \hat{\mathcal{N}}_1^* \vee \hat{\mathcal{N}}_2^* \vee \ldots \vee \hat{\mathcal{N}}_m^*$. By the matroid union theorem ([18, (Corollary 42.1a)]), its rank function is

$$r_{\hat{\mathcal{N}}^*}(\hat{B}) \;=\; \min_{T\subseteq \hat{B}} \left(|\hat{B}\setminus T| + \sum_{i=1}^{m} r_{\hat{\mathcal{N}}_i^*}(T) \right).$$

We claim that for any $T \subseteq \hat{B}$,

$$\sum_{i=1}^{m} r_{\hat{\mathcal{N}}_i^*}(T) \;=\; \sum_{i=1}^{m} r_{\mathcal{M}/(B\setminus\pi(T))}(A_i) \;\geq\; q \cdot r_{\mathcal{M}/(B\setminus\pi(T))}(A) \;=\; q|\pi(T)|\;. \tag{2}$$

The first equality follows from our rank formula (1), and the last equality holds because both A and B are bases of $\mathcal{M}$. To prove the inequality, without loss of generality, assume that $A = \{1,2,\ldots,n\} = [n]$, and let $g(S) := r_{\mathcal{M}/(B\setminus\pi(T))}(S)$, which is a monotone submodular function with $g(\emptyset) = 0$. We have

$$\begin{aligned}
\sum_{i=1}^{m} g(A_i) &= \sum_{i=1}^{m}\sum_{j\in A_i} [g(A_i\cap[j]) - g(A_i\cap[j-1])] \\
&\geq \sum_{i=1}^{m}\sum_{j\in A_i} [g([j]) - g([j-1])] \\
&= q\sum_{j\in A} [g([j]) - g([j-1])] \\
&= q\cdot f(A)\;,
\end{aligned}$$

using submodularity and the fact that each element of A appears in exactly q sets A_i . This proves (2), and we get $\sum_{i=1}^{m} r_{\hat{\mathcal{N}}_i^*}(T) \geq q|\pi(T)| \geq |T|$ for any $T \subseteq \hat{B}$. We conclude that the rank function of $\hat{\mathcal{N}}^*$ is

$$\hat{r}^*(\hat{B}) \;=\; \min_{T\subseteq\hat{B}} \left(|\hat{B}\setminus T| + \sum_{i=1}^{m} r_{\hat{\mathcal{N}}_i^*}(T) \right) \;=\; |\hat{B}|\;.$$

This means that $\hat{B}$ can be partitioned into sets $B'_1, \ldots, B'_m$ where B'_i is independent in $\hat{\mathcal{N}}_i^*$. However, the ranks of $\hat{B}$ in the $\hat{\mathcal{N}}_i^*$ sum up to $\sum_{i=1}^m r_{\hat{\mathcal{N}}_i^*}(\hat{B}) = \sum_{i=1}^m |A_i| = |\hat{B}|$, so the only way this can be possible is if each B'_i is a base of $\hat{\mathcal{N}}_i^*$. Then, each $B_i = \pi(B'_i)$ is a base of $\mathcal{N}_i^*$ and these are the sets demanded by the lemma. □

Finally, we present a version of Lemma 2.8 where the two sets are not necessarily bases.

Lemma 2.10. *Let $\mathcal{M} = (N, \mathcal{I})$ be a matroid and $I, J \in \mathcal{I}$. Let $I_1, \ldots, I_m$ be subsets of I such that each element of I appears in at most q of them. Then there are sets $J_1, \ldots, J_m \subseteq J$ such that each element of J appears in at most q of them, and for each i*

$$I_i \cup (J \setminus J_i) \ \in \ \mathcal{I}.$$

Proof. We reduce this statement to Lemma 2.8. Let A, B be bases such that $I \subseteq A$ and $J \subseteq B$. Also, we extend I_i arbitrarily to A_i, $I_i \subseteq A_i \subseteq A$, so that each element of A appears in exactly q of them. By Lemma 2.8, there are sets $B_i \subseteq B$ such that each element of B appears in exactly q of them, and $A_i \cup (B \setminus B_i) \in \mathcal{I}$ for each i. We define $J_i = J \cap B_i$. Then, each element of J appears in at most q sets J_i, and

$$I_i \cup (J \setminus J_i) \subseteq A_i \cup (B \setminus B_i) \ \in \ \mathcal{I}.$$

□

3 Local-Search Algorithm

As is usual, our local-search algorithm works in iterations. At each iteration, given a current feasible solution $S \in \cap_{j=1}^k \mathcal{I}_j$ our algorithm looks for an improved solution by looking at a polynomial number of options to change S. If the algorithm finds a better solution it moves to the next iteration, otherwise the algorithm stops. Specifically, given a current solution $S \in \cap_{j=1}^k \mathcal{I}_j$, the local moves that we consider are:

***p*-exchange operation**: If there is S' such that

- **(i)** $|S' \setminus S| \le p$, $|S \setminus S'| \le kp$, and
- **(ii)** $f(S') > f(S)$,

then $S \leftarrow S'$.

The p-exchange operation for $S' \subseteq S$ is called a *delete operation*. Our main result is the following lower bound on the value of the locally-optimal solution.

Lemma 3.1. *For every $k \ge 2$ and every $C \in \cap_{j=1}^k \mathcal{I}_j$, a locally-optimal solution S under p-exchanges, satisfies*

$$\left(k + \frac{1}{p}\right) \cdot f(S) \ \ge \ f(S \cup C) + \left(k - 1 + \frac{1}{p}\right) \cdot f(S \cap C).$$

Proof. Our proof is based on the new exchange properties of matroids: Lemmas 2.7 and 2.10. By applying Lemma 2.7 to the independent sets C and S in matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, we obtain a collection of dipaths/dicycles $\{A_1, \ldots, A_m\}$ (possibly with repetition), feasible in $D_{\mathcal{M}_1,\mathcal{M}_2}(S)$, using only elements of $C\Delta S$, so that each element of $C\Delta S$ appears in exactly 2^s paths/cycles A_i .

We would like to define the sets of vertices corresponding to the exchanges in our local-search algorithm, based on the sets of vertices in paths/cycles $\{A_1, \ldots, A_m\}$. The problem is that these paths/cycles can be much longer than the maximal cardinality of a set allowable in a p-exchange operation. To handle this, we index vertices of the set of $C \setminus S$ in each path/cycle A_i for $i = 1, \ldots, m$, in such a way that vertices along any path or cycle are numbered consecutively. The vertices of $S \setminus C$ remain unlabeled. Because one vertex appears in 2^s paths/cycles, it might get different labels corresponding to different appearances of that vertex. So one vertex could have up to 2^s different labels.

We also define $p+1$ copies of the index sets $\{A_1, \ldots, A_m\}$. For each copy $q = 0, \cdots, p$ of labeled $\{A_1, \ldots, A_m\}$, we throw away appearances of vertices from $C \setminus S$ that were labeled by q modulo $p+1$ from each A_i. By throwing away some appearances of the vertices, we are changing our set of paths in each copy of the original sets $\{A_1, \ldots, A_m\}$. Let $\{A_{q1}, \ldots, A_{qm_q}\}$ be the resulting collection of paths for $q = 0, \ldots, p$. Now each path A_{qi} contains at most p vertices from $C \setminus S$ and at most $p + 1$ vertices from $S \setminus C$.

Because our original collection of paths/cycles was feasible in $D_{\mathcal{M}_1,\mathcal{M}_2}(S)$ (see definition 2.5), each of the paths in the new collections correspond to feasible exchanges for matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, i.e. $S\Delta V(A_{qi}) \in \mathcal{I}_1 \cap \mathcal{I}_2$. Consider now the collection of paths $\{A_{qi} | q = 0, \ldots, p, i = 1, \ldots, m_q\}$. By construction, each element of the set $S \setminus C$ appears in exactly $(p+1)2^s$ paths, and each element of $C \setminus S$ appears in exactly $p2^s$ paths, because each vertex has $2^s(p + 1)$ appearances in total, and each appearance is thrown away exactly once. Let $L_{qi} = S \cap V(A_{qi})$ denote the set of vertices in the path A_{qi} belonging to the locally-optimal solution S, and let $W_{qi} = C \cap V(A_{qi})$ denote the set of vertices in the path A_{qi} belonging to the set C.

For each matroid $\mathcal{M}_i$ for $i = 3, \ldots, k$, independent sets $S \in \mathcal{I}_i$ and $C \in \mathcal{I}_i$, and collection of sets $\{W_{qi} \mid q = 0, \ldots, p; i = 1, \ldots, m_q\}$ (note that some of these sets might be empty), we apply Lemma 2.10. For convenience, we re-index the collection of sets $\{W_{qi} \mid q = 0, \ldots, p, i = 1, \ldots, m_q\}$. Let $W_1, \ldots, W_t$ be that collection, after re-indexing, for $t = \sum_{q=0}^{p} m_q$. By Lemma 2.10, for each $i = 3, \ldots, k$ there exist a collection of sets $X'_{1i}, \ldots, X'_{ti}$ such that $W_j \cup (S \setminus X'_{ji}) \in \mathcal{I}_i$. Moreover, each element of S appears in at most $p2^s$ of the sets from collection $X'_{1i}, \ldots, X'_{ti}$.

We consider the set of p-exchanges that correspond to adding the elements of the set W_j to the set S and removing the set of elements $\Lambda_j = L_j \cup (\cup_{i=3}^{k} X'_{ji})$ for $j = 1, \ldots, t$. Note that, $|\Lambda_j| \le (p + 1) + (k - 2)p = (k - 1)p + 1 \le kp$. By Lemmas 2.7 and 2.10, the sets

$$W_j \cup (S \setminus \Lambda_j)$$

are independent in each of the matroids $\mathcal{M}_1, \cdots, \mathcal{M}_k$. By the fact that S is a locally-optimal solution, we have

$$f(S) \;\geq\; f\left(\left(S \setminus \Lambda_j\right) \cup W_j\right), \qquad\qquad \forall j = 1, \ldots, t. \tag{3}$$

Using inequalities (3) together with submodularity for $j = 1, \ldots, t$, we have

$$f(S \cup W_j) - f(S) \;\leq\; f\left((S \setminus \Lambda_j) \cup W_j\right) - f\left(S \setminus \Lambda_j\right) \;\leq\; f(S) - f\left(S \setminus \Lambda_j\right). \tag{4}$$

Moreover, we know that each element of the set $C \setminus S$ appears in exactly $p2^s$ sets W_j , and each element $e \in S \setminus C$ appears in $n_e \leq (p+1)2^s + (k-2)p2^s$ sets Λ_j .

Consider the sum of t inequalities (4), and add $(p+1)2^s + (k-2)p2^s - n_e$ inequalities

$$f(S) \;\geq\; f(S \setminus \{e\}) \tag{5}$$

for each element $e \in S \setminus C$. These inequalities correspond to the delete operations. We obtain

$$\begin{aligned} \sum_{j=1}^{t} \left[f(S \cup W_j) - f(S)\right] \;&\leq\; \sum_{j=1}^{t} \left[f(S) - f\left(S \setminus \Lambda_j\right)\right] \;+ \\ \sum_{e \in S \setminus C} &\left((p+1)2^s + (k-2)p2^s - n_e\right) \left[f(S \setminus \{e\}) - f(S)\right]. \end{aligned} \tag{6}$$

Applying Lemma 1.1 to the right-hand side of the inequality (6) and Lemma 1.2 to the left-hand side of the inequality (6), we have

$$p2^s \left[f(S \cup C) - f(S)\right] \;\leq\; \left((p+1)2^s + (k-2)p2^s\right) \left[f(S) - f(S \cap C)\right],$$

which is equivalent to

$$\left(k + \frac{1}{p}\right) \cdot f(S) \;\geq\; f(S \cup C) + \left(k - 1 + \frac{1}{p}\right) \cdot f(S \cap C).$$

The result follows. □

A simple consequence of Lemma 3.1 implies bounds on the value of the locally-optimal solutions in the cases when the submodular function f has additional structure.

Corollary 3.2. *For $k \geq 2$, a locally-optimal solution S, and any $C \in \cap_{j=1}^{k} \mathcal{I}_j$, the following inequalities hold:*

1. *$f(S) \;\geq\; f(C)/\left(k + \frac{1}{p}\right)$ if function f is monotone,*

2. *$f(S) \;\geq\; f(C)/\left(k - 1 + \frac{1}{p}\right)$ if function f is linear.*

Input: Finite ground set $N := [n]$, value-oracle access to submodular function $f : 2^N \to \mathbb{R}$, and matroids $\mathcal{M} = (N, \mathcal{I}_i)$, for $i \in [k]$.

1. Set $v \leftarrow \arg\max\{f(u) \mid u \in N\}$ and $S \leftarrow \{v\}$.
2. While the following local operation is possible, update S accordingly:
   p-exchange operation. If there is S' such that
   (i) $|S' \setminus S| \le p$, $|S \setminus S'| \le kp$, and
   (ii) $f(S') \ge (1 + \frac{\epsilon}{n^4}) f(S)$,
   then $S \leftarrow S'$.

Output: S.

Figure 1: The approximate local-search procedure.

The local-search algorithm defined in the beginning of this section could run for an exponential amount of time until it reaches a locally-optimal solution. To ensure polynomial runtime, we follow the standard approach of an approximate local search under a suitable (small) parameter $\epsilon > 0$, as described in Figure 1. The following Lemma 3.3 is a simple extension of Lemma 3.1 for an approximate local optimum.

Lemma 3.3. *For an approximate locally-optimal solution S and any $C \in \cap_{j=1}^{k} \mathcal{I}_j$,*

$$(1+\epsilon)\left(k + \frac{1}{p}\right) \cdot f(S) \;\ge\; f(S \cup C) + \left(k - 1 + \frac{1}{p}\right) \cdot f(S \cap C),$$

where $\epsilon > 0$ is the parameter used in the procedure of Figure 1.

Proof. The proof of this lemma is almost identical to the proof of the Lemma 3.1 — the only difference is that left-hand sides of inequalities (3) and inequalities (5) are multiplied by $1 + \frac{\epsilon}{n^4}$. Therefore, after following the steps in the proof of Lemma 3.1, we obtain the inequality:

$$\left(k + \frac{1}{p} + \frac{\epsilon}{n^4} \cdot \frac{\lambda}{p 2^s}\right) \cdot f(S) \ge f(S \cup C) + \left(k - 1 + \frac{1}{p}\right) \cdot f(S \cap C),$$

where $\lambda = t + \sum_{e \in S \setminus C} [(p+1)2^s + (k-2)p2^s - n_e]$ is the total number of inequalities (3) and (5). because $t \le |C| p 2^s$ we obtain that $\lambda \le (n+k)p2^s$. Assuming that $n^4 >> n + k$, we obtain the result. ∎

Lemma 3.3 implies the following:

Theorem 3.4. *For any $k \geq 2$ and fixed constant $\delta > 0$, there exists a $\frac{1}{k+\delta}$-approximation algorithm for maximizing a non-negative non-decreasing submodular function subject to k matroid constraints. This bound improves to $\frac{1}{k-1+\delta}$ for linear functions.*

Remark 3.5. Combining the techniques from this paper and the iterative local-search technique from [11], we can improve the performance guarantees of the approximation algorithms for maximizing a general (non-monotone) submodular function subject to $k \geq 2$ matroid constraints from $k + 2 + \frac{1}{k} + \delta$ to $k + 1 + \frac{1}{k-1} + \delta$ for any $\delta > 0$.

References

[1] P. Berman, A $d/2$ approximation for maximum weight independent set in d-claw free graphs, Nordic J. Comput. 7 (2000), no. 3, 178–184.

[2] T.H. Brylawski. Some properties of basic families of subsets, *Discrete Mathematics* 6 (1973), 333–341.

[3] M. Conforti and G. Cornuéjols. Submodular set functions, matroids and the greedy algorithm: tight worst-case bounds and some generalizations of the Rado-Edmonds theorem, Discrete Appl. Math. 7 (1984), no. 3, 251–274.

[4] G. Cornuéjols, M. Fischer and G. Nemhauser. Location of bank accounts to optimize oat: An analytic study of exact and approximation algorithms, *Management Science*, 23 (1977), 789–810.

[5] G. Cornuéjols, M. Fischer and G. Nemhauser. On the uncapacitated location problem, *Annals of Discrete Math* 1 (1977), 163–178.

[6] U. Feige. A threshold of $\ln n$ for approximating set cover. *Journal of ACM* 45 (1998), 634–652.

[7] U. Feige, V. Mirrokni and J. Vondrák. Maximizing non-monotone submodular functions, *FOCS 2007.*

[8] C. Greene. A multiple exchange property for bases, *J Proc. Amer. Math. Soc.* 39 (1973), 45–50.

[9] C. Hurkens and A. Schrijver, On the size of systems of sets every t of which have an SDR, with an application to the worst-case ratio of heuristics for packing problems. SIAM J. Discrete Math. 2 (1989), no. 1, 68–72.

[10] S. Iwata, L. Fleischer and S. Fujishige. A combinatorial, strongly polynomial-time algorithm for minimizing submodular functions, *Journal of ACM* 48:4 (2001), 761–777.

[11] J. Lee, V. Mirrokni, V. Nagarajan and M. Sviridenko. Maximizing Non-Monotone Submodular Functions under Matroid and Knapsack Constraints, submitted for publication, preliminary version appeared in *Proceedings of STOC 2009.*

[12] L. Lovász. Submodular functions and convexity. In: A. Bachem et. al., eds, "Mathematical Programmming: The State of the Art, " 235–257.

[13] G.L. Nemhauser, L.A. Wolsey and M.L. Fisher. An analysis of approximations for maximizing submodular set functions I. *Mathematical Programming* 14 (1978), 265–294.

[14] G.L. Nemhauser, L.A. Wolsey and M.L. Fisher. An analysis of approximations for maximizing submodular set functions II. *Mathematical Programming Study* 8 (1978), 73–87.

[15] G. L. Nemhauser and L. A. Wolsey. Best algorithms for approximating the maximum of a submodular set function, *Math. Oper. Research*, 3(3):177–188, 1978.

[16] J.G. Oxley. "Matroid theory," Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992.

[17] J. Reichel and M. Skutella. Evolutionary algorithms and matroid optimization problems, in *Proceedings of the 9th Genetic and Evolutionary Computation Conference (GECCO'07)*, 947–954, 2007.

[18] A. Schrijver. "Combinatorial Optimization: Polyhedra and Efficiency". Springer, 2003.

[19] A. Schrijver. A combinatorial algorithm minimizing submodular functions in strongly polynomial time, *Journal of Combinatorial Theory, Series B* 80 (2000), 346–355.

[20] J. Vondrák. Optimal approximation for the submodular welfare problem in the value oracle model. In *STOC, 2008.*

[21] D.R. Woodall. An exchange theorem for bases of matroids, *J. Combinatorial Theory Ser. B* 16 (1974), 227–228.