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Strengthening Lattice-free Cuts Using Non-negativity

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Strengthening lattice-free cuts using non-negativity

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Abstract

In recent years there has been growing interest in generating valid inequalities for mixed-integer programs using sets with 2 or more constraints. In particular, Andersen et.al (2007) and Borozan and Cornuéjols (2007) study sets defined by equations that contain exactly one integer variable per row. The integer variables are not restricted in sign. Cutting planes based on this approach have already been used by Espinoza [12] for general mixed-integer problems and there is also ongoing computational research in this area.

In this paper, we restrict the model studied in the earlier papers and require the integer variables to be non-negative. We extend the results in Andersen et.al (2007) and Borozan and Cornuéjols (2007) to our case and show that cuts generated by their approach can be strengthened by using the non-negativity of the integer variables. In particular, it is possible to obtain cuts which have negative coefficients for some variables.

Keywords: Mixed integer programming, valid inequalities, lattice free polyhedra.

1 Introduction

Given a mixed-integer program (MIP) and a basic feasible solution to its linear programming (LP) relaxation, one can define a relaxation of the feasible solution set

$$X = \Big\{(x, s) \in \mathbb{Z}^m \times \mathbb{R}^n_+ \;:\; x_i - \sum_{j=1}^{n} a_{ij} s_j \;=\; f_i \;\text{ for } i \in \{1, \ldots, m\}\Big\}$$

which is obtained by starting with the associated simplex tableau and (i) deleting rows associated with basic continuous variables, (ii) relaxing integrality of the non-basic variables and (iii) relaxing the non-negativity of basic variables. Notice that variables x can be projected out by requiring s to satisfy $f_i + \sum_{j=1}^{n} a_{ij} s_j \in \mathbb{Z}$ for all i. This set can also be viewed as a continuous relaxation of the corner polyhedra of Gomory [13].

In a recent paper Andersen at. al [2] study the set X when $m = 2$ and show that all valid inequalities for X can be represented by maximal lattice-free bodies in $\mathbb{R}^2$. Later Borozan and Cornuéjols [5] extend this and show that minimal valid inequalities for the semi-infinite relaxation of X are in one-to-one correspondence with maximal lattice point free bodies in $\mathbb{R}^m$ that contain b

(provided that b is not on the boundary). In addition, Cornuéjols and Margot [6] extend the results in [2] and study conditions under which valid inequalities for the set X (when $m = 2$) become facet defining. More recently, Andersen at. al [3] extend their earlier work by considering upper bounds on some of the continuous variables. There has also been some initial computational work by Espinoza [12] as well as ongoing computational work by other groups [1, 8] that use the results in [2, 5] to produce cutting planes for MIPs.

In this paper, we study the set

$$X_+ = \Big\{(x,s) \in \mathbb{Z}^m_+ \times \mathbb{R}^n_+ \ : \ x_i - \sum_{j=1}^{n} a_{ij}s_j \ = \ f_i \ \text{ for } i \in \{1,\dots,m\}\Big\}$$

which contains non-negative points in X. As $X^+ \subseteq X$, it gives a tighter relaxation of MIPs for which integer basic variables are required to be non-negative.

Our main result in this paper is to show inequalities derived in [5] (using maximal lattice point free sets) can be strengthened using the fact that x variables are required to be non-negative in X_+. This strengthening, for example, leads to minimal valid inequalities of the form $\alpha x \geq 1$ where α has negative components. We next present an example to emphasize the difference between X and X_+.

Example 1.1 *Let $r_1, r_1, r_2, r_3, r_4, r_5, f \in \mathbb{R}^2$ be defined as follows:*

$$r_1 = \begin{pmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{pmatrix} \quad r_2 = \begin{pmatrix} -\frac{1}{4} \\ -\frac{5}{4} \end{pmatrix} \quad r_3 = \begin{pmatrix} \frac{7}{4} \\ -\frac{5}{4} \end{pmatrix} \quad r_4 = \begin{pmatrix} \frac{5}{4} \\ -\frac{5}{4} \end{pmatrix} \quad r_5 = \begin{pmatrix} \frac{3}{4} \\ -\frac{5}{4} \end{pmatrix} \quad \textit{and} \quad f = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$$

and consider the following set,

$$X = \Big\{(x,s) \in \mathbb{Z}^2_+ \times \mathbb{R}^5_+ \ : \ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \sum_{j=1}^{5} r_j s_j = f\Big\}$$

defined by 2 rows. Using a the results in [6, 2], it is possible to show that the following inequality:

$$s_1 + s_2 + s_3 + s_4 + s_5 \geq 1$$

is valid and facet-defining for X. However, using the non-negativity of the x variables in $X_+ = X \cap \mathbb{R}^+_7$, it is possible to show that the following stronger inequality:

$$s_1 + s_2 + s_3 - s_5 \geq 1$$

is valid (and facet defining) for X_+. We will come back to this example in Section 2.

The rest of the paper is organized as follows: In Section 2, we define the semi-infinite extension of X^+ where we essentially study the set X^+ when it has infinitely many variables, one for each rational coefficient vector. For this extension, we characterize the basic properties of minimal valid functions, relate them to convex sets that do not contain non-negative integer points and show that certain polyhedral sets lead to minimal valid functions. In Section 3, we focus on the semi-infinite extension of X^+ when it is defined by two rows and give a complete characterization of minimal valid functions and how they are related to convex sets that do not contain non-negative integer points. In Section 4, we show how to strengthen valid inequalities for X based on maximal lattice point free sets to obtain valid inequalities for X^+.

2 The semi-infinite extension of X^+

In this section, we study the semi-infinite extension of X^+ and show basic properties of minimal valid functions for it. We define the set

$$P_f^+ \;=\; \Big\{(x,s) \in \mathbb{Z}_+^m \times \mathbb{J} \;:\; x_i - \sum_{r\in\mathbb{Q}^m} r s_r \;=\; f_i \;\text{ for } i = 1,\dots,m\Big\}$$

where $\mathbb{J} = \{s \in \mathbb{R}^{\mathbb{Q}^m} \;:\; s$ has finite support$\}$, to be the semi-infinite extension of X^+. Our main observation is that most of the fundamental results known to hold for semi-infinite extension of X (called P_f and studied in [5]) can be extended to the semi-infinite relaxation of X^+. We are, however, not able to show that there is a one-to-one correspondence between minimal valid functions and maximal positive lattice point free sets, which are analogous to maximal lattice-free convex sets. More precisely, we show that given a maximal positive lattice point free set, one can construct a minimal valid function but we are not able to show that any minimal valid function can be constructed using a maximal positive lattice point free set.

2.1 Valid functions for P_f^+

We say that a function $\psi : \mathbb{Q}^m \to \mathbb{R}$ is a *valid function* for P_f^+ if

$$\sum_{r\in\mathbb{Q}^m} \psi(r) s_r \geq 1$$

for all $(x,s) \in P_f^+$. As discussed in [5] (also see [2]) all valid functions violated by the point $(x,s) = (f,0)$ can be written in this form. Note that the x variables do not appear in this expressions as they are substituted out using the equations defining P_f^+. Also note that we are restricting ourselves to finite functions ψ, since in the context of generating cutting planes for mixed-integer programming, functions that can assume the value $\pm\infty$ are not useful in practice. We say that ψ is a *minimal* valid function if it is a valid function and there is no other valid function ψ' such that (i) $\psi(r) \geq \psi'(r)$ for all $r \in \mathbb{Q}^m$, and (ii) $\psi(r) > \psi'(r)$ for some $r \in \mathbb{Q}^m$.

For the sake of completeness, we first define the following: A function $f : \mathbb{Q}^m \to \mathbb{R}$ is called

(i) *convex,* if $\alpha f(x') + (1-\alpha) f(x'') \geq f(\alpha x' + (1-\alpha)x'')$ for all $x', x'' \in \mathbb{Q}^m$ and $\alpha \in [0,1] \cap \mathbb{Q}$.

(ii) *positively homogeneous,* if $f(\alpha x') = \alpha f(x')$ for all $x' \in \mathbb{Q}^m$ and $\alpha \in \mathbb{Q}_+$.

(iii) *subadditive,* if $f(x') + f(x'') \geq f(x' + x'')$ for all $x', x'' \in \mathbb{Q}^m$.

Lemma 2.1 *If ψ is a minimal valid function for P_f^+ then ψ is (i) subadditive, (ii) positively homogeneous, and (iii) convex.*

Proof. The proof essentially summarizes and adopts proofs of Lemmas 2.2, 2.3, 2.4 and 2.5 in [5]. (i) Assume that ψ is not subadditive, then $\psi(r') + \psi(r'') < \psi(r' + r'')$ for $r', r'' \in \mathbb{Q}^m$. Define $\phi : \mathbb{Q}^m \to \mathbb{R}$ to be the same as ψ except let $\phi(r' + r'') = \psi(r') + \psi(r'')$. We will show that ϕ is valid, and therefore ψ can not be minimal, a contradiction. If ϕ is not valid there exists a point $(x', s') \in P_f^+$ such that $\sum_{r\in\mathbb{Q}^m} \phi(r)s'_r < 1$. But in this case ψ can not be valid either as $\sum_{r\in\mathbb{Q}^m} \phi(r)s'_r = \sum_{r\in\mathbb{Q}^m} \psi(r)s''_r < 1$ where $(x', s'') \in P_f^+$ and s'' is obtained from s' by reducing its $(r' + r'')$th component to zero and increasing the r'th and r''th components by $s'_{(r'+r'')}$. Therefore, ϕ is indeed valid, and ψ is not minimal.
(ii) As ψ is subadditive, we have that $\psi(r) + \psi(0) \geq \psi(r) \Rightarrow \psi(0) \geq 0$. Let $(\bar{x}, \bar{s})$ be a feasible solution to P_f^+. Since s has finite support and ψ is finite, we know that $\sum \psi(r)s_r < +\infty$. Note that the point $(\bar{x}, \tilde{s})$ defined by $\tilde{s}_r := \bar{s}_r$ for $r \neq 0$ and $\tilde{s}_0 = 0$ is also feasible for P_f^+. Hence, $0 + \sum_{r\neq 0} \psi(r)s_r \geq 1$. Therefore ψ is still a valid function if we change $\psi(0) = 0$, so minimality of ψ implies $\psi(0) = 0$.

Therefore, if $\alpha = 0$ then $\psi(\alpha r) = \alpha\psi(r)$ for all $r \in \mathbb{Q}^m$. Assume that $\psi(\alpha r') \neq \alpha\psi(r')$ for some $\alpha > 0$ and $r', \alpha r' \in \mathbb{Q}^m$. Let $\beta = \min\{\psi(\alpha r')/\alpha, \psi(r')\}$ and define $\phi : \mathbb{Q}^m \to \mathbb{R}$ be same as ψ except let $\phi(\alpha r') = \alpha\beta$ and $\phi(r') = \beta$. As in the first part of the proof, it is straight forward to reach a contradiction by observing that ϕ is valid function P_f^+ provided that ψ is valid.
(iii) Notice that ψ positively homogeneous and therefore for all $\alpha \in [0, 1]$ and $r', r'' \in \mathbb{Q}^m$

$$\alpha\psi(r') + (1 - \alpha)\psi(r'') = \psi(\alpha r') + \psi((1 - \alpha)r'') \geq \psi(\alpha r' + (1 - \alpha)r'')$$

where the last inequality follows from subadditivity. ∎

In [5], Borozan and Cornuéjols show that all valid functions for P_f are (i) subadditive, (ii) positively homogeneous, (iii) convex and (iv) non-negative. We note that the last property does not hold for all valid functions for P_f^+. In Section 2.3, we describe a family of valid functions for P_f^+ that assume negative values for some $r \in \mathbb{Q}^m$.

Lemma 2.2 *If ψ is positively homogeneous and subadditive, then it is valid for P_f^+ if and only if $\psi(x - f) \geq 1$ for all $x \in \mathbb{Z}_+^m$.*

Proof. The only if part is straight forward: if $\psi(\bar{x} - f) < 1$ for some $\bar{x} \in \mathbb{Z}_+^m$, define $\bar{s} \in \mathbb{J}$ to have all zero components except $\bar{s}_{(\bar{x}-f)} = 1$. We therefore have $(\bar{x}, \bar{s}) \in P_f^+$ and yet $\sum_{r\in\mathbb{Q}^m} \psi(r)\bar{s}_r < 1$, a contradiction.

For the if part, note that for all $(\bar{x}, \bar{s}) \in P_f^+$ we have $\bar{x} \in \mathbb{Z}_+^m$ and $\sum_{r\in\mathbb{Q}^m} r\bar{s}_r = \bar{x} - f$. First using homogeneity and then using subadditivity, we have $\sum_{r\in\mathbb{Q}^m} \psi(r)\bar{s}_r = \sum_{r\in\mathbb{Q}^m} \psi(\bar{r}s_r) \geq \psi(\sum_{r\in\mathbb{Q}^m} r\bar{s}_r)$ Implying $\sum_{r\in\mathbb{Q}^m} \psi(r)\bar{s}_r \geq 1$ and therefore ψ is a valid function for P_f^+. ∎

2.2 Positive lattice point free sets and minimal valid functions for P_f^+

We call a set $S \subset \mathbb{R}^m$ *positive lattice point free* if $int(S) \cap \mathbb{Z}_+^m = \emptyset$, where $int(S)$ denotes the interior of the set S (a point is in the interior if it is possible to construct a ball around it that is contained in the set). For a given function $\psi : \mathbb{Q}^m \to \mathbb{R}$ we define a closed set in $\mathbb{R}^m$ associated with the function as follows:

$$S(\psi, f) = cl\Big(\big\{x \in \mathbb{Q}^m : \psi(x - f) \leq 1\big\}\Big).$$

Using this definition, notice that Lemma 2.2 can be re-stated as follows:

Remark 2.3 *If ψ is positively homogeneous and subadditive, then it is valid for P_f^+ if and only if $S(\psi, f)$ is a positive lattice point free set.*

Moreover, remember that the proof of Lemma 2.1 shows that if the function ψ is positively homogeneous and subadditive, then it is a convex function. This, in turn, implies that $S(\psi, f)$ is a convex set. As all minimal valid functions for P_f^+ are positively homogeneous and subadditive, we also observe that $S(\psi, f)$ is convex for all minimal valid functions ψ.

For a set B, let $RC(B)$ denote the recession cone of B and $RC^o(B) = RC(B) \setminus int(RC(B))$ denote the boundary of the recession cone of B. We first make a basic observation regarding minimal valid functions.

Lemma 2.4 *Let $f \in \mathbb{Q}^m$ and $\psi : \mathbb{Q}^m \to \mathbb{R}$ be a positively homogeneous and subadditive function. Then $f \in int(S(\psi, f))$. Moreover, for every $r \in \mathbb{Q}^m$, the function ψ satisfies the following:*

(i) $\psi(r) \leq 0$, if $r \in RC(S(\psi, f))$,

(ii) $\psi(r) = 0$, if $r \in RC^o(S(\psi, f))$, and,

(iii) $\psi(r) = 1/\max\{\lambda \in \mathbb{R}_+ \ : \ f + \lambda r \in S(\psi, f)\}$, if $r \notin RC(S(\psi, f))$.

Proof. To simplify notation, let $S = S(\psi, f)$, $RC = RC(S(\psi, f))$ and $RC^o = RC^o(S(\psi, f))$. We start with showing that $\psi(r) < \infty$ for all $r \in \mathbb{Q}^m$ implies that $f \in int(S)$. Let e_d be the unit vector with a 1 in the d-th component and zero everywhere else. Since $\psi(e_d) < \infty$, we have that $1 = \frac{1}{\psi(e_d)}\psi(e_d) = \psi\left(\frac{1}{\psi(e_d)}e_d\right) = \psi\left(f + \frac{1}{\psi(e_d)}e_d - f\right)$ and hence $f + \frac{1}{\psi(e_d)}e_d \in S$. Since the same argument is valid for all e_d and $-e_d$ for all $d = 1, \ldots, m$, we have that there exists $\epsilon > 0$ such that $f \pm \epsilon e_d \in S$ for all $d = 1, \ldots, m$ and hence $f \in int(S)$. We next prove (i), (ii) and (iii).

(i) Consider $r \in RC$. As $f \in S$, we have $f + \lambda r \in S$ for all $\lambda \in \mathbb{Q}_+$ implying $\psi(f + \lambda r - f) \leq 1$. Hence $\psi(\lambda r) = \lambda\psi(r) \leq 1$. Since λ can be arbitrarily large, we have $\psi(r) \not> 0$, or equivalently, $\psi(r) \leq 0$.

(ii) We first show that $\psi(r) > 0$ when $r \notin RC$. For the sake of contradiction assume that $\psi(r) \leq 0$. Then for any $x \in S \cap \mathbb{Q}^m$ and any $\lambda \in \mathbb{Q}_+$ we have that $\psi(x + \lambda r - f) \leq \psi(x - f) + \lambda\psi(r) \leq \psi(x - f) \leq 1$, therefore $x + \lambda r \in S$. Since S is convex, $x + \lambda r \in S$ for all $\lambda \in \mathbb{R}_+$ and hence $r \in RC$.

Now consider $r \in RC^o$ and note as $r \in RC$ we have $\psi(r) \leq 0$. Suppose, for the sake of contradiction, that $\psi(r) = -\beta$ for some $\beta > 0$. Since $r \in RC^o$, there exists a nonzero vector $v \notin RC$ such that $r + \delta v \notin RC$ for all $\delta > 0$. Now choose a δ' such that $0 < \delta' < \beta/\psi(v)$ and remember that $v \notin RC$ implies $0 < \psi(v) < +\infty$. As $r + \delta' v \notin RC$ we have $f + \lambda(r + \delta' v) \notin S$ for some sufficiently large $\lambda > 0$. In other words, $\psi(\lambda(r + \delta' v)) > 1$. As ψ is subadditive and positively homogeneous, we also have

$$\lambda\psi(r) + \lambda\delta'\psi(v) \geq \psi(\lambda(r + \delta' v)) > 1 \Rightarrow \psi(r) \geq 1/\lambda - \delta'\psi(v) > 1/\lambda - \beta \geq -\beta,$$

which is a contradiction and therefore $\psi(r) = 0$.

(iii) Finally, we consider $r \notin RC$. Notice that we have already shown in part (ii) that $\psi(r) > 0$ and therefore,

$$1 = \frac{1}{\psi(r)}\psi(r) = \psi(\frac{1}{\psi(r)}r) = \psi(f + \frac{1}{\psi(r)}r - f)$$

implying $f + r/\psi(r) \in S$ and hence

$$1/\psi(r) \leq \bar{\lambda} = \max\{\lambda \in R_+ \ : \ f + \lambda r \in S\}.$$

Now if $\bar{\lambda} > 1/\psi(r)$, we have that $\psi(f + \bar{\lambda} r - f) = \psi(\bar{\lambda} r) = \bar{\lambda}\psi(r) > 1$, a contradiction. Therefore, $\bar{\lambda} = 1/\psi(r)$. ∎

Notice that the first part of the proof of Lemma 2.4 can be easily extended to show that, even if we allow ψ to take on the value ∞, $\psi < \infty$ if and only if $f \in int(S(\psi, f))$. As we assume ψ to be finite, we only consider maximal positive lattice point free sets that contain f in their interior. We remark that Zambelli [22] showed that all cutting-planes for X can be generated using maximal lattice-free convex sets that contain f in the interior.

2.3 A minimal valid function for P_f^+

We next present a family of minimal valid functions that are derived using polyhedral maximal positive lattice point free sets. Throughout this section we assume that B is a polyhedral set that satisfies the following properties: (i) it is full-dimensional, (ii) it contains f in its interior, (iii) it does not contain any non-negative integer points in its interior. Therefore, B can be represented as

$$B = \{x \in \mathbb{R}^m : a_i^T x \leq b_i, \forall i = 1, \ldots, k\},$$

where all inequalities are facet defining, $a_i^T f < b_i$ for $i \in I = \{1, \ldots, k\}$, and $int(B) \cap \mathbb{Z}_+ = \emptyset$. We now define the function $\psi_B : \mathbb{Q}^m \to \mathbb{R}$ as follows:

$$\psi_B(r) = \max_{i \in I} \left\{r^T \hat{a}_i\right\}$$

where $\hat{a}_i = a_i/(b_i - a_i^T f)$. Note that ψ_B is a positively homogeneous function. In this section we show that ψ_B is valid for P_f^+. In addition, we show that if B is maximal, then ψ_B is minimal.

Given a vector $r \in \mathbb{Q}^m$, clearly $r \in RC(B)$ if and only if $a_i^T r \leq 0$ for all $i \in I$, and $r \in RC(B) \setminus RC^o(B)$ if and only if $a_i^T r < 0$ for all $i \in I$. Now consider a vector $r \notin RC(B)$ and notice that in this case the function value $\psi_B(r)$ is identical to that of the function defined by Borozan and Cornuéjols in [5]. The function used in [5], which we call $\phi_B : \mathbb{Q}^m \to \mathbb{R}$, is defined as follows:

$$\phi_B = \begin{cases} 0 & \text{if } r \in RC(B) \\ 1/\max\{\lambda \in R_+ \ : \ f + \lambda r \in B\} & \text{if } r \notin RC(B). \end{cases}$$

Notice that the condition $f + \lambda r \in B$ above can also be written as

$$a_i^T f + a_i^T r\lambda \le b_i \text{ for all } i \in I \iff 1/\lambda \ge r^T a_i/(b_i - a_i^T f) \text{ for all } i \in I$$

and therefore, when $r \notin RC(B)$ both functions are indeed the same and assume the value $1/\lambda'$ where $\lambda' > 0$ is the scalar for which the point $f + \lambda' r$ is on the boundary of B. Furthermore, if $r \in RC^o(B)$, both functions assume the value 0 and therefore are again equal.

For $r \in int(RC(B))$, however, then the function values are different as $\phi_B(r) = 0 > \psi_B(r)$. To see that $\psi_B(r) < 0$, notice that $a_i r < 0$ for all $i \in I$ and therefore $\max_{i\in I}\left\{a_i^T r/(b_i - a_i^T f)\right\} < 0$. More precisely, we can give the following geometric description of ψ_B when $r \in int(RC(B))$.

Lemma 2.5 *Let $r \in int(RC(B))$. Then $\psi_B(r) = -1/\lambda'$ where $\lambda' > 0$ is the largest scalar for which the condition $a_i^T(f - \lambda' r) \le b_i$ for at least one $i \in I$.*

Proof. Let $\hat{\lambda} = -1/\psi_B(r)$ and let $l \in \arg\max\limits_{i\in I}\left\{a_i^T r/(b_i - a_i^T f)\right\}$. Therefore

$$a_l^T(f - \hat{\lambda} r) = a_l^T f + \frac{1}{\psi_B(r)} a_i^T r = a_l^T f + \frac{b_l - a_l^T f}{a_l^T r} a_l^T r = b_l$$

and therefore we have $a_i^T(f - \hat{\lambda} r) \le b_i$ for at least one $i \in I$. Now let $\lambda > \hat{\lambda}$ and as $r \in int(RC(B))$ we have $a_i^T r < 0$ for all $i \in I$ and

$$a_i^T(f - \lambda r) > a_i^T f - \lambda a_i^T r = a_i^T f + \frac{1}{\psi_B(r)} a_i^T r \ge a_i^T f + \frac{b_i - a_i^T f}{a_i^T r} a_i^T r = b_i.$$

Therefore, if $\lambda > \hat{\lambda}$, the condition $a_i^T(f - \lambda r) \le b_i$ is not satisfied by any $i \in I$, implying that $\hat{\lambda}$ indeed is the largest scalar for which $a_i^T(f - \lambda' r) \le b_i$ for at least one $i \in I$. ∎

In the context of [5], the set B used in defining the function ϕ_B is required to be lattice point free and therefore $int(RC(B)) = \emptyset$ as such sets can not have a full-dimensional recession cones. Consequently, the functions ϕ and ψ coincide for the sets considered in [5]. In our context, however, the sets B has to be positive lattice point free and therefore it can have $int(RC(B)) \neq \emptyset$. We next show that ψ_B is valid for P_f^+ if B is positive lattice point free.

Lemma 2.6 *If B is positive lattice point free, then the function ψ_B is valid for P_f^+.*

Proof. Clearly ψ_B is positively homogenous. We next show that it is also subadditive: Let $r^1, r^2 \in \mathbb{Q}^m$ and let $\psi_B(r^1 + r^2) = \hat{a}_l(r^1 + r^2)$ for some $l \in I$. Then

$$\psi_B(r^1) + \psi_B(r^2) = \max_{i\in I}\{\hat{a}_i^T r^1\} + \max_{i\in I}\{\hat{a}_i^T r^2\} \ge \hat{a}_l^T r^1 + \hat{a}_l^T r^2 = \hat{a}_l^T(r^1 + r^2) = \psi_B(r^1 + r^2).$$

Therefore, ψ_B is subadditive and by Lemma 2.2 and Remark 2.3, it is valid if and only if $S(\psi_B, f)$ is positive lattice point free. Let $r \in S(\psi_B, f)$ and note that for all $i \in I$ we have

$$1 \ge \psi(r - f) \ge \hat{a}_i(r - f) = (a_i^T r - a_i^T f)/(b_i - a_i^T f) \;\Rightarrow\; b_i - a_i^T f \ge a_i^T r - a_i^T f) \;\Rightarrow\; b_i \ge a_i^T x$$

and therefore $r \in B$. As r is arbitrary, we have $S(\psi_B, f) \subseteq B$ and therefore $S(\psi_B, f)$ is positive lattice point free and the proof is complete. ■

Note that it is possible to extend the last argument in the proof to show that the following remark is true.

Remark 2.7 $S(\psi_B, f) = B$.

To prove it, let $r \in B \cap \mathbb{Q}^m$ and therefore $a_i^T r \le b_i$ for all $i \in \{1, \dots, k\}$. Let $\psi(r - f) = \hat{a}_t^T(r - f)$ for some $t \in \{1, \dots, k\}$. Then,

$$\psi(r - f) = (a_t^T r - a_t^T f)/(b_t - a_t^T f) \le (b_t - a_t^T f)/(b_t - a_i^T f) = 1$$

and therefore $r \in S(\psi_B, f)$. Since $B \cap \mathbb{Q}^m \subseteq S(\psi_B, f)$ then $B = cl(B \cap \mathbb{Q}^m) \subseteq S(\psi_B, f) \Rightarrow S(\psi_B, f) = B$.

We next show that maximality of B is sufficient to obtain a minimal valid function.

Lemma 2.8 *If B is maximal positive lattice point free, then ψ_B is a minimal valid function for P_f^+.*

Proof. Suppose not and let ψ be a minimal valid function for P_f^+ such that $\psi \le \psi_B$ and $\psi(\bar{r}) < \psi_B(\bar{r})$ for some $\bar{r} \in \mathbb{Q}^m$. We next consider two cases.

<u>*Case 1:*</u> $\bar{r} \notin RC(B)$.

For simplicity, let $S = S(\psi, f)$. By Lemma 2.4, we have $\psi(\bar{r}), \psi_B(\bar{r}) > 0$ and by positive homogeneity of ψ_B and ψ, we have that there exist $\mu > \lambda > 0$ such that $\psi_B(\lambda\bar{r}) = \psi(\mu\bar{r}) = 1$. Let $\bar{x} = f + \lambda\bar{r}$ and let $\bar{\bar{x}} = f + \frac{\mu+\lambda}{2}\bar{r}$. Then $\psi_B(\bar{\bar{x}} - f) > 1$, which implies that $\bar{\bar{x}} \notin B$. But $\psi(\bar{\bar{x}} - f) < 1$, which implies $\bar{\bar{x}}$ is in the interior of $cl(S)$. It follows that B is strictly contained in $cl(S)$. By Remark 2.3, $cl(S)$ is a positive lattice point free set. This contradicts the assumption that B is a maximal positive lattice point free set. Therefore $\psi(r) = \psi_B(r)$ for all $\bar{r} \notin RC(B)$.

<u>*Case 2:*</u> $\bar{r} \in RC(B)$.

First note that for all $i \in I$ there exists a vector $v_i \in \mathbb{Q}^m \setminus RC(B)$ such that $\psi_B(v_i) = \hat{a}_i v_i \ge \hat{a}_t v_i$ for all $t \in I$. To show that v_i exists, we use the fact that $a_i x \le b_i$ is facet defining for B and therefore there exists a point x^i such that $a_i x^i = b_i$ and $a_t x^i \le b_t$ for all $t \ne i$. Then $v_i = x^i - f$ satisfies the desired properties as

$$\hat{a}_i v_i = \frac{a_i x^i - a_i f}{b_i - a_i f} = 1 \ \text{ and } \ \hat{a}_t v_i = \frac{a_t x^i - a_t f}{b_t - a_t f} \le \frac{b_t - a_t f}{b_t - a_t f} = 1$$

for all $t \in I$. The fact that $v_i \notin RC(B)$ follows from the fact that $a_i v^i = \hat{a}_i v^i (b_i - a_i f) > 0$.

If $\bar{r} \in bd(RC(B))$, we have that $a_t \bar{r} \le 0$ for all $t \in I$, with $a_i \bar{r} = 0$ for some $i \in I$. In this case, $\psi_B(\bar{r}) = \hat{a}_i \bar{r} = 0$. Note that $a_i(v^i + \bar{r}) = a_i v^i > 0$ and hence $(v^i + \bar{r}) \notin RC(B)$. Moreover, note that

$$\hat{a}_i(v^i + \bar{r}) = \frac{a_i v^i}{b_i - a_i f} = 1 \ \text{ and } \ \hat{a}_t(v^i + \bar{r}) = \frac{a_t v^i + a_t \bar{r}}{b_i - a_i f} \le \frac{a_t v^i}{b_i - a_i f} \le 1$$

for all $t \in I$ and therefore $\psi_B(v^i + \bar{r}) = \hat{a}_i(v^i + \bar{r})$. Since ψ is minimal, it is subadditive and hence $\psi(v^i) + \psi(\bar{r}) \geq \psi(v^i + \bar{r})$. But then, $\psi(\bar{r}) \geq \psi(v^i + \bar{r}) - \psi(v^i) = \psi_B(v^i + \bar{r}) - \psi_B(v^i) = 0 = \psi_B(\bar{r}) > \psi(\bar{r})$, a contradiction. Hence $\psi(\bar{r}) = \psi_B(\bar{r})$ for all $\bar{r} \notin int(RC(B))$.

If $\bar{r} \in int(RC(B))$, let i be such that $\psi_B(\bar{r}) = \hat{a}_i\bar{r}$. Since $\bar{r} \in int(RC(B))$, we have $a_i\bar{r} < 0$. By the choice of v_i, we have that $a_i v_i = \hat{a}_i v_i(b_i - a_i f) > 0$. Let $\alpha = |a_i\bar{r}|/a_i v_i$ and note that $a_i(\bar{r} + \alpha v_i) = 0$ implying that $(\bar{r} + \alpha v_i) \notin int(RC(B))$ and hence $\psi(\bar{r} + \alpha v_i) = \psi_B(\bar{r} + \alpha v_i) \geq 0$. As ψ is valid and therefore subadditive, we have

$$\psi(\bar{r}) + \psi(\alpha v_i) = \psi(\bar{r}) + \psi_B(\alpha v_i) = \psi(\bar{r}) + \alpha\hat{a}_i v_i \geq \psi(\bar{r} + \alpha v_i).$$

As $\psi(\bar{r} + \alpha v_i) = \psi_B(\bar{r} + \alpha v_i) \geq 0$, we have

$$\psi(\bar{r}) + \alpha\hat{a}_i v_i \geq 0 \implies \psi(\bar{r}) \geq -\alpha\hat{a}_i v_i = \hat{a}_i\bar{r} = \psi_B(\bar{r}) > \psi(\bar{r}),$$

again a contradiction. ∎

We next revisit the example presented in Section 1 to illustrate how (maximal) positive lattice point free sets lead to valid (and facet defining) inequalities for X^+.

Example 1.1 (continued) *Remember the set*

$$X = \left\{(x,s) \in \mathbb{Z}^2_+ \times \mathbb{R}^5_+ \;:\; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \sum_{i=1}^{5} r_i s_i = f\right\} \tag{1}$$

where f and r_i are defined in Section 1. As shown in Figure 1(a), the triangle T defined by the corner points p_1, p_2, p_3 is a maximal lattice point free set in $\mathbb{R}^2$. Notice that $p_i = f + r_i$ for $i = 1, \ldots, 5$ and consequently $\phi_T(r_i) = 1$ for all i and, by [6], the inequality $s_1 + s_2 + s_3 + s_4 + s_5 \geq 1$ is valid and facet-defining for X.

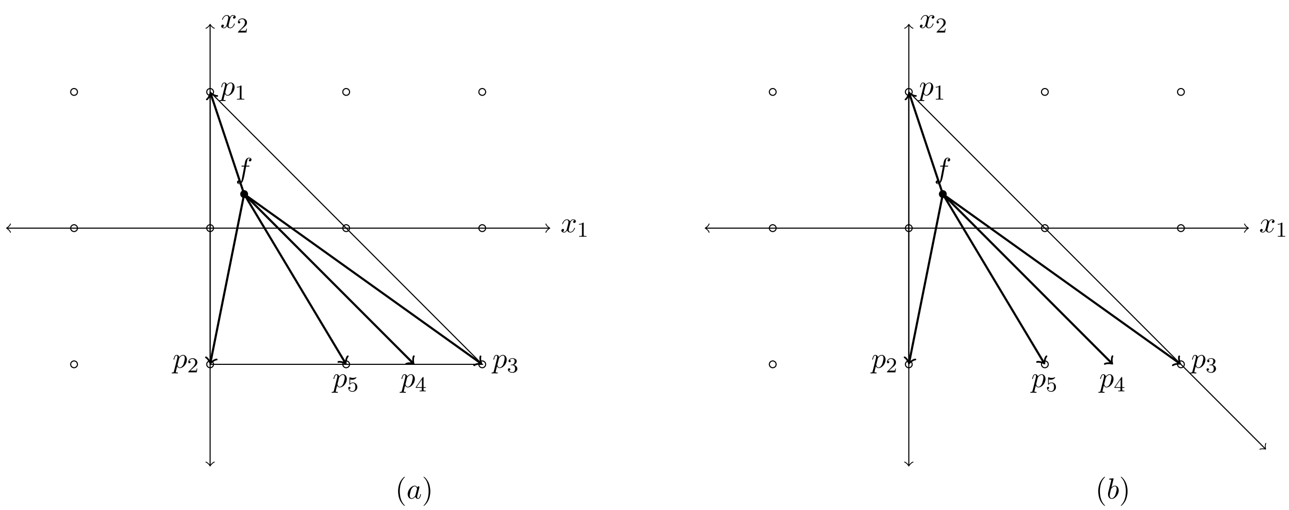

Figure 1: (a) A maximal lattice point free set and
. (b) a maximal positive lattice point free set in $\mathbb{R}^2$, both containing $f > 0$.

In comparison, notice that the translated cone C (shown in Figure 1(b)) defined by the rays $\overrightarrow{p_1p_2}$ and $\overrightarrow{p_1p_3}$ is a maximal positive lattice point free set. This set can be written as

$$C = \{x \in \mathbb{R}^2 : -x_1 \leq 0, x_1 + x_2 \leq 1\}$$

and notice that $p_4 \in RC^o(C)$ and $p_5 \in RC(C) \setminus RC^o(C)$. The set C leads to the minimal valid function

$$\psi_C(r) \quad = \quad \max\left\{\frac{r^T \cdot [-1,0]^T}{0 - [-1,0] \cdot f}\ ,\ \frac{r^T \cdot [1,1]^T}{1 - [1,1] \cdot f}\right\} \quad = \quad \max\left\{r^T \cdot \begin{bmatrix} -4 \\ 0 \end{bmatrix},\ r^T \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\}$$

which gives the following stronger valid inequality for $X^+ = X \cap \mathbb{R}^+_7$,

$$s_1 + s_2 + s_3 - s_5 \geq 1.$$

Furthermore, this inequality is facet defining as the dimension of X^+ is 5, and the following 5 affinely independent points are in X^+ and satisfy the inequality as equality: $p_1 = [0, 1; 1, 0, 0, 0, 0]$, $p_2 = [0, 0; 1/2, 1/2, 0, 0, 0]$, $p_3 = [1, 0; 1/2, 0, 1/2, 0, 0]$, $p_4 = [1, 0; 1, 0, 0, 4/5, 0]$, $p_5 = [1, 0; 3, 0, 0, 0, 2]$.

We end this section by noting that for $r \in RC(B)$, $\psi_B(r)$ is only determined by the inequalities that define facets of $RC(B)$. This property will be used later in Section 4 to strengthen inequalities for P^+_f .

Lemma 2.9 *Let $r \in RC(B)$. Then $\psi_B(r) = a^l r/(b^l - a^l f)$ for some $l \in I$ such that $a^l x \leq 0$ defines a facet-defining of $RC(B)$.*

Proof. Since $r \in RC(B)$, we have by Lemma 2.4 that $\psi(r) \leq 0$. If $r \notin int(RC(B))$, then $a^l r = 0$ for some facet $a^l x \leq 0$ of $RC(B)$ and $\psi_B(r) = \max_{i \in I} \left\{a^i r/(b^i - a^i f)\right\} = 0$ and the result follows. Thus, we may assume $r \in int(RC(B))$. Let $I^c \subseteq I$ be such that $a^i x \leq 0$ defines a facet of $RC(B)$ if and only if $i \in I^c$. If $\psi_B(r) > a^i r/(b^i - a^i f)$ for all $i \in I^c$, then let $j \notin I^c$ be such that $\psi_B(r) = a^j r/(b^j - a^j f)$. By Lemma 2.4, $\psi_B(r) < 0$ and hence $b^j = a^j(f + \frac{1}{\psi_B(r)} r)$ and $b^i < a^i(f + \frac{1}{\psi_B(r)} r)$ for all $i \in I^c$. However, for all $j \notin I^c$, we have that there exist $\mu_i \geq 0, \forall i \in I^c$ such that $a^j = \sum_{i \in I^c} \mu_i a^i$. Therefore $b^j = a^j(f + \frac{1}{\psi_B(r)} r) = \sum_{i \in I^c} \mu_i a^i(f + \frac{1}{\psi_B(r)} r) \geq \sum_{i \in I^c} \mu_i b_i$. But this contradicts the fact that $a^j x \leq b^j$ defines a facet of B. Therefore, there exists $l \in I^c$ such that $a^T_l(f - \lambda r) = b_l$. ∎

3 Special case: $m = 2$

We now consider $m = 2$ and show that any minimal valid function $\psi : \mathbb{Q}^2 \to \mathbb{R}$ is defined by the maximal positive lattice point free set $B = S(\psi, f)$. We note that in $\mathbb{R}^2$ any cone is polyhedral and therefore the recession cone of any set is described by at most three inequalities.

Let $B \subseteq \mathbb{R}^2$ be a full-dimensional closed convex set that is positive lattice point free and has f in its interior. (B is not necessarily polyhedral.) Let $RC(B) = \{x \in \mathbb{R}^2 : c_i x \leq 0, i \in J\}$, where

$|J| \leq 3$ and $c_i x \leq 0$ defines a facet of $RC(B)$ for all $i \in J$. Let $d_i = \sup\{c_i x : x \in B\}$ for $i \in J$ and note that $B \subseteq C = \{x \in \mathbb{R}^2 : c_i x \leq d_i, i \in J\}$. We now let $\hat{c}_i = c_i/(d_i - c_i^T f)$ and note that, as f is in the interior of B, we have $c_i f < d_i$. We now extend the definition of the function ψ_B in two dimensions as follows:

$$\psi_B(r) = \begin{cases} \max\limits_{i \in J} \{r^T \hat{c}_i\} & \text{if } r \in int(RC(B)) \\ 1/\max\{\lambda \in R_+ \ : \ f + \lambda r \in B\} & \text{if } r \notin RC(B) \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to show that if B is polyhedral, the above definition coincides with the one in Section 2.3. To see that $\psi_B(r)$ is subadditive and positively homogeneous, just notice that $B = \{x : c_i x \leq d_i, i \in I\}$, where I is a (possibly infinite) index set. Then

$$\psi_B(r) = \sup_{i \in I \cup J} \{\hat{c}_i r\}$$

and the result follows. We next show that essentially all minimal valid functions have the form above.

Lemma 3.1 *Let $\psi : \mathbb{Q}^2 \to \mathbb{R}$ be a minimal valid function such that the positive lattice point free set $B = S(\psi, f)$ contains f in its interior. Then $\psi = \psi_B$.*

Proof. By Lemma 2.4 we have $\psi(r) = \psi_B(r)$ for all $r \notin int(RC(B))$. We therefore consider $r \in int(RC(B))$. Let $\epsilon \in (0, 1/2)$. We first construct vectors $v_i \notin RC(B)$, for $i \in J$, that satisfy the following properties: (i) $\hat{c}_i v_i \geq \hat{c}_t v_i$ for all $t \in J$ and (ii) $\psi_B(v_i) \leq \hat{c}_i v_i + \epsilon$. Remember that $d_i = \sup\{c_i x : x \in B\}$ and as $\epsilon(d_i - c_i f) > 0$ there exists $x^i \in B$ such that $c_i x^i \geq d_i - \epsilon(d_i - c_i f)$. As $x_i \in B$, we also have $c_t x^i \leq d_t$ for all $t \in J$. Furthermore, as $\{x \in \mathbb{R}^2 : c_i x \leq 0, i \in J\}$ is a minimal polyhedral representation of $RC(B)$, it is possible to pick points $r^i \in RC(B)$ for all $i \in I$ such that (i) $c_i r^i = 0$ and (ii) $c_t r^i < 0$ for all $t \neq i$. Now let $\lambda > -\epsilon(d_t - c_t f)/c_t r^i$ for all $t \neq i$, and note that $v_i = x^i + \lambda r^i - f$ satisfies the first desired property as

$$\hat{c}_i v_i = \frac{c_i x^i - c_i f}{d_i - c_i f} \geq \frac{d_i - c_i f}{d_i - c_i f} - \epsilon = 1 - \epsilon = \frac{d_t - c_t f}{d_t - c_t f} - \epsilon \geq \frac{c_t x^i + \lambda c_t r^i - c_t f}{d_t - c_t f} = \hat{c}_t v_i.$$

Also note that as $c_i v^i = \hat{c}_i v^i (d_i - c_i f) > 0$, it follows that $v_i \notin RC(B)$. Moreover, as $v^i + f = x^i + \lambda r^i$ where $x^i \in B$ and $r^i \in RC(B)$, we have $f + v^i \in B$, implying $\max\{\lambda \in \mathbb{R}_+ : f + \lambda v^i \in B\} \geq 1$ and therefore $\psi_B(v_i) \leq 1 \leq \hat{c}_i v_i + \epsilon$.

Let i be such that $\psi_B(r) = \hat{c}_i r$. Since $r \in int(RC(B))$, we have $c_i r < 0$ and remember that $c_i v_i = \hat{c}_i v_i (d_i - c_i f) > 0$. Let $\alpha = |c_i r|/c_i v_i$ and note that $c_i(r + \alpha v_i) = 0$ implying that $(r + \alpha v_i) \notin int(RC(B))$ and hence $\psi(r + \alpha v_i) = \psi_B(r + \alpha v_i) \geq 0$. Also remember that $v_i \notin RC(B)$ and therefore $\psi(v_i) = \psi_B(v_i)$. As ψ is a minimal valid function, it is subadditive, and therefore have

$$\psi(r) + \alpha \hat{c}_i v_i \geq \psi(r) + \alpha \psi_B(v_i) - \alpha\epsilon = \psi(r) + \psi(\alpha v_i) - \alpha\epsilon \geq \psi(r + \alpha v_i) - \alpha\epsilon \geq -\alpha\epsilon$$

implying

$$\psi(r) \geq -\alpha \hat{c}_i v_i - \alpha\epsilon = \hat{c}_i r - \alpha\epsilon = \psi_B(r) - \alpha\epsilon.$$

Notice that since $\hat{c}_i v_i \geq 1 - \epsilon > 1/2$, we have that $\alpha = |c_i r|/c_i v_i = |\hat{c}_i r|/\hat{c}_i v_i \leq 2|\hat{c}_i r|$. Hence $\psi(r) \geq \psi_B(r) - 2|\hat{c}_i r|\epsilon$. Since this is valid for any $\epsilon > 0$, it follows that $\psi(r) \geq \psi_B(r)$. ∎

3.1 Maximal positive lattice point free sets in $\mathbb{R}^2$

We now characterize all maximal positive lattice point free sets and show that they are polyhedral. In particular, the main result of this section is the following theorem, which is similar to theorems by Bell [4], Doignon [11], Lovász [17], and Scarf [19] for maximal lattice-free convex sets and will be used in Section 3.2 to characterize $S(\psi, f)$ for minimal valid functions ψ.

Theorem 3.2 *A maximal positive lattice point free set in $\mathbb{R}^2$ is either a full-dimensional polyhedron with at most 4 facets or an irrational hyperplane.*

The rest of this section is devoted to prove of Theorem 3.2. We first study full-dimensional maximal positive lattice point free set s and show that we can restrict ourselves to the case where such sets contains a positive point in the interior.

Lemma 3.3 *Let $K \subseteq \mathbb{R}^m$ be a full-dimensional maximal positive lattice point free set. If there does not exist a point $f > 0$ in $int(K)$, then K is a half-space.*

Proof. Notice first that if K contains a point $f' > 0$, then K contains a point $f > 0$ in its interior. Indeed pick $y \in int(K)$ and since $f_\lambda = \lambda y + (1-\lambda)f' \in int(K)$ for all $\lambda \in (0, 1)$, we can pick λ arbitrarily close to 1 such that $f_\lambda > 0$.

Therefore, there exists a hyperplane $ax \leq b$ such separating K from $cl(\{x \in \mathbb{R}^m : x > 0\}) = \{x \in \mathbb{R}^m : x \geq 0\}$, that is $ax \leq b$ for all $x \in K$ and $ax \geq b$ for all $x \geq 0$. But then $\{x \in \mathbb{R}^m : ax \leq b\} \supseteq K$ and does not contain any nonnegative integer points in its interior, hence by maximality of K, $K = \{x \in \mathbb{R}^m : ax \leq b\}$. ∎

We now show that, regardless of the value of m, maximal positive lattice point free sets are polyhedral under certain conditions on their recession cones.

Lemma 3.4 *Let $K \subseteq \mathbb{R}^m$ be a maximal positive lattice point free set. If $RC(K) \cap \mathbb{R}^m_+ = \{0\}$, then K is polyhedral.*

Proof. First note that $K \neq \emptyset$ as it is maximal. Moreover, $K \cap \mathbb{R}^m_+$ cannot be empty, otherwise convex hull of K and the origin contains K and is a positive lattice point free set, a contradiction. Since $K \cap \mathbb{R}^m_+ \neq \emptyset$, we have that the condition $RC(K) \cap \mathbb{R}^m_+ = \{0\}$ is equivalent to $K \cap \mathbb{R}^m_+$ is bounded.

As $K \cap \mathbb{R}^m_+$ is bounded, there exists numbers $u_i \in R_+$ for all $i \in I = \{1, \dots, n\}$ such that $x_i \leq u_i$ for all $x \in K \cap \mathbb{R}^m_+$. For $i \in I$, define the sets $C_i = \{x \in \mathbb{R}^m \,:\, x \geq 0,\ x_i \geq u_i + 1\}$. Note that if K is a positive lattice point free set, so is its closure and therefore by maximality, K has to be closed.

Therefore K and all C_i are non-empty, convex and closed sets. Furthermore, for all $i \in I$ the sets K and C_i are pairwise disjoint and have no common directions of recession.

Therefore, for each $i \in I$ there exits a hyperplane $\alpha^i x = \beta^i$ that strongly separates K and C_i (see, for example, [18] Separation Theorems). In other words, there exists $\alpha^i \in \mathbb{R}^m$ and $\beta^i \in \mathbb{R}$ such that for all $x' \in K$ and $x'' \in C_i$ we have $\alpha^i x' < \beta^i$ and $\alpha^i x'' > \beta^i$. Notice that for all $i, j \in I$ the unit direction e_j is a direction of recession for C_i and therefore $\alpha^i \geq 0$ for all $i \in I$.

As $K \cap \mathbb{R}^m_+$ is not be empty, there exists some $\bar{x} \in K \cap \mathbb{R}^m_+$. Combining this with $\alpha^i \geq 0$ and $\alpha^i \bar{x} < \beta^i$, we therefore have $\beta^i > 0$ for all $i \in I$. Finally, let $\tilde{x}^i$ be a vector of all zeroes except the i'th component which is equal to $u_i + 1$. Note that $\tilde{x}^i \in C_i$ and as $\alpha^i \tilde{x}^i > \beta^i > 0$, we have $\alpha^i_i > 0$.

Now, let $\bar{\alpha} = \sum_{i \in I} \alpha^i$ and $\bar{\beta} = \sum_{i \in I} \beta^i$ and note that $\bar{\alpha} x < \bar{\beta}$ for all $x \in K$. Therefore, $K \cap \mathbb{R}^m_+ \subseteq X = \{x \in \mathbb{R}^m : x \geq 0,\ \bar{\alpha} x \leq \bar{\beta}\}$. Let $X^L = X \cap Z^m_+$ be the collection of lattice points in X and note that X^L contains a finite number of points as $\bar{\alpha} > 0$ and $\bar{\beta} > 0$. As K does not contain positive lattice points in its interior, for each $p \in X^L$, there exists a closed half-space defined by $\alpha^p x \leq \beta^p$ that contains K and has p on its boundary. Therefore the following polyhedral set

$$P = \{x \in \mathbb{R}^m \ : \ \bar{\alpha} x \leq \bar{\beta},\ \alpha^p x \leq \beta^p \text{ for all } p \in X^L\}$$

contains K and does not contain any positive lattice points. As K is assumed to be maximal, $K = P$ and the proof is complete. ∎

Notice that if B is a full-dimensional maximal positive lattice point free set with $dim(RC(B)) = 0$, then $RC(B) \cap \mathbb{R}^m_+ = \{0\}$ and hence Lemma 3.4 implies that B is polyhedral. Lemmas 3.5 and 3.6 complete the proof that B is polyhedral by considering other possible dimensions $RC(B)$.

Lemma 3.5 *Let $S \subseteq \mathbb{R}^2$ be a positive lattice point free set such that there is a point $f > 0$ in its interior. If $dim(RC(S)) = 2$ then $RC(S) \cap \mathbb{R}^2_+ = \{0\}$.*

Proof. Suppose there is a vector $v \in RC(S)$ such that $v \geq 0$ and $v \neq 0$. Since $v \neq 0$, we may assume, by symmetry, that $v_1 = 1$. Since $RC(S)$ is full-dimensional, there exists a nonzero vector u such that $u_1 = 0$ and such that $v + \epsilon u \in RC(S)$ for some ϵ small enough. Now, for any $\alpha > 0$ we have that $w = f + \alpha v \in S$ and $z = f + \alpha(v + \epsilon)u \in S$. But then choose $\alpha > 1/|\epsilon u_2|$ such that $f_1 + \alpha \in \mathbb{Z}_+$. Then $|w_2 - z_2| = |\alpha \epsilon u_2| > 1$. Since $w_1 = z_1 = f_1 + \alpha \in \mathbb{Z}_+$, then we have a nonnegative integer point in the interior of the line segment between w and z and hence a nonnegative integer point in the interior of S, which is a contradiction. ∎

Lemma 3.6 *Let $S \subseteq \mathbb{R}^2$ be a maximal positive lattice point free set that contains a point $f > 0$ in its interior. If $dim(RC(S)) = 1$, then S is a polyhedron.*

Proof. If for all $r \in RC(S)$ we have that $r \not\geq 0$, then by Lemma 3.4 the result follows. Thus, we may assume that there exists $r \in RC(S)$ such that $r \geq 0$. In addition, we can assume that there exists a point $\bar{y} \in \mathbb{Z}^2$ in the interior of S such that $\bar{y} \not\geq 0$, since otherwise, S is maximal lattice-free and hence, by [17], it polyhedral. We will next show that if all these assumptions are made, then S has a nonnegative lattice point in its interior, which is a contradiction.

Case 1: r has one zero component.

Without loss of generality, assume that $r_1 = 0$. In addition, after scaling, we can assume that $r_2 = 1$. In this case, if $\bar{y}_1 \geq 0$, then $\bar{y} + |\bar{y}_2|r$ is a nonnegative integer point in the interior of S, which is a contradiction. Therefore, we may assume that $\bar{y}_1 < 0$. But since $f > 0$ is a point in the interior of S, then there exists a point w in the interior of S such that $w_1 = 0$. But then there exists $\lambda > 0$ such that $w + \lambda r$ is a nonnegative integer point in the interior of S.

Case 2: $r > 0$.

If r is rational, then we may assume that r is integer and thus, there exists $\lambda \in \mathbb{Z}_+$ such that $\bar{y} + \lambda r$ is a nonnegative integer point in the interior of S. Thus, we may assume that r is not rational. Without loss of generality, let $r_1 = 1$.

Now consider the line $-r_2 x_1 + x_2 = b$ generated by $\bar{y} + \lambda r$ for $\lambda \in \mathbb{R}$. Note that r_2 and $b = -r_2\bar{y}_1 + \bar{y}_2$ are irrational numbers. From the approximation of r_2 by continued fractions (see for instance [20]), it follows that there exists a sequence (p_n, q_n) such that $p_n \in \mathbb{Z}_+$ and $q_n \in \mathbb{Z}_+$ and $\lim\limits_{n\to\infty} p_n = \lim\limits_{n\to\infty} q_n = \infty$ and such that $0 \leq \frac{p_n}{q_n} - r_2 \leq \frac{1}{q_n^2}$. Since $\bar{y}$ is in the interior of S, there exists $\epsilon > 0$ such that if $||x - \bar{y}||_2 \leq \epsilon$, then $x \in int(S)$.

But then, pick n large enough such that $1/q_n < \epsilon$ and $p_n > |\bar{y}_2|$, $q_n > |\bar{y}_1|$. Notice that the point $w = (\bar{y}_1 + q_n, \bar{y}_2 + p_n)$ is a nonnegative integer point. Moreover, $w = x + q_n r$ where $x = (\bar{y}_1, \bar{y}_2 + p_n - r_2 q_n)$ and since $0 \leq \frac{p_n}{q_n} - r_2 \leq \frac{1}{q_n^2} \Rightarrow 0 \leq p_n - r_2 q_n \leq \frac{1}{q_n} < \epsilon$, we have that $||x - \bar{y}||_2 \leq \epsilon$ so $x \in int(S)$. This in turn implies that $w \in int(S)$, which contradicts the fact that S does not have nonnegative integer points in its interior. ∎

Lemmas 3.4, 3.5 and 3.6 show that in $\mathbb{R}^2$ any maximal positive lattice point free set that contains $f > 0$ in its interior is polyhedral. The following corollary follows immediately from the proofs of Lemmas 3.5 and 3.6 and will be used to bound the number of facets that such a maximal positive lattice point free set has.

Corollary 3.7 *If $S \subseteq \mathbb{R}^2$ is a full-dimensional maximal positive lattice point free set then S is either a maximal lattice-free convex set or $RC(S) \cap \mathbb{R}^2_+ = \{0\}$.*

We now use Corollary 3.7 to show that, when $m = 2$, maximal positive lattice point free sets have a nonnegative integer point in the relative interior of each of their facets, which will imply that there are at most 4 facets. Notice that the fact that a polyhedral maximal positive lattice point free set has at most 2^m facets for general m can be proven by adapting the proof of a theorem in Schrijver [21] (credited to Bell [4], Doignon [11] and Scarf [19]) directly, without using this fact. However, this fact is helpful in identifying when a positive lattice point free set is not maximal and will be used in Section 4 where we are concerned with strengthening inequalities that are defined by non-maximal positive lattice point free sets.

Lemma 3.8 *Let $B = \{x \in \mathbb{R}^2 : a_i^T x \leq b_i, \forall i = 1, \ldots, k\}$ be a full-dimensional maximal positive lattice point free set. Then there exists a nonnegative integer point in the relative interior of each one of its facets.*

Proof. If B is a maximal lattice-free convex set, then the result was proven by Bell [4], Doignon [11] and Scarf [19], so we may assume that B is not maximal lattice-free convex set and hence, by Corollary 3.7, $RC(B) \cap \mathbb{R}^2_+ = \{0\}$. Without loss of generality we assume the inequality description of B is minimal and each inequality describes a facet. We also assume that $b_i \in \mathbb{Q}$ for all $i \in I = \{1, \ldots, k\}$. Consider the face $F_j = B \cap \{x \in \mathbb{R}^2 : a_j^T x = b_j\}$ defined by the jth inequality and assume that F_j does not contain a nonnegative integer point in its relative interior. Let $F_j^+ = F_j \cap \mathbb{R}^2_+$. Notice that $RC(F_j^+) \subseteq RC(B)$ and hence $RC(F_j^+) = \{0\}$ so F_j^+ is bounded. We next consider 2 cases.

<u>*Case 1:*</u> $a_j \in \mathbb{Q}^2$. In this case, let τ be such that $\tau a_j \in \mathbb{Z}^2$ and consider replacing $a_j^T x \leq b_j$ in the description of B with $\tau a_j^T x \leq \tau b_j + 1/2$. Clearly, the new set contains B strictly and is positive lattice point free, a contradiction.

<u>*Case 2:*</u> $a_j \notin \mathbb{Q}^2$. Since F_j^+ is bounded, there exists vectors $l, u \in \mathbb{R}^n_+$ be such that for $B = \{x \in \mathbb{R}^m : u_j \geq x_j \geq l_j\}$ we have

$$\Delta = \{x \in \mathbb{R}^m_+ : a_i^T x \leq b_i \forall i \in I \setminus \{j\},\ a_j^T x > b_j,\ a_j^T x \leq b_j + 1\} \subseteq B.$$

Let $T = \{x \in B : a_j^T x > b_j\} \cap \mathbb{Z}^m$ and note that T is finite. Notice that Δ gives the points that will be included in B if b_j is increased by 1 in the description of B, and, T contains all nonnegative integer point that would be contained in B if b_j is increased by 1. If $T = \emptyset$ then replacing b_j in the description of B with $b_j + 1$ gives a strictly larger positive lattice point free set, a contradiction. If $T \neq \emptyset$ then let $\hat{b}_j = \min_{x \in T}\{a_j^T x\} > b_j$ and note that in this case, we can replacing b_j in the description of B with $\hat{b}_j$ to obtain a strictly larger positive lattice point free set, again a contradiction. ∎

From Lemma 3.8 it is straightforward to obtain a bound on the number of facets of maximal positive lattice point free sets by the following simple argument due to Bell [4] (also see Borozan and Cornuéjols [5]).

Lemma 3.9 *Let $B \in \mathbb{R}^2$ be a full-dimensional maximal positive lattice point free set. Then it is a polyhedron with at most 4 facets.*

Proof. Each facet F of B has a point x^F in its relative interior. If there are more than 4 facets, two nonnegative integral points x^F and $x^{F'}$ must be identical modulo 2. Then their middle point $\frac{1}{2}(x^F + x^{F'})$ is integral, nonnegative and interior, which is a contradiction. ∎

The following lemma is true for any arbitrary number of rows, and is just stated here for completeness of the characterization of maximal positive lattice point free sets in $\mathbb{R}^2$.

Lemma 3.10 *Let S be a maximal positive lattice point free set that is not full-dimensional. Then S is an irrational hyperplane.*

Proof. If S is not full dimensional then all $x \in S$ satisfy $ax = b$ for some $b \in \mathbb{R}$ and $a \in \mathbb{R}^m$. Therefore $S \subseteq \{x \in \mathbb{R}^m : ax = b\}$ and as S is maximal positive lattice point free, $S = \{x \in \mathbb{R}^m :$

$ax = b\}$. If b is not integral, it is possible to rewrite the equation defining S as $(1/b)ax = 1$ and therefore, without loss of generality, we assume that $b \in \mathbb{Z}$. Now, if a is rational there exists a large enough $\tau \in \mathbb{Z}$ such that $\tau a \in \mathbb{Z}^m$. In this case, $S \subset \{x \in \mathbb{R}^m : \tau ax \geq \tau b,\ \tau ax \leq \tau b + 1\}$ which contradicts the maximality of S. Therefore, $a \notin \mathbb{Q}^m$, and S indeed is an irrational hyperplane. ∎

3.2 Minimal valid functions for $m = 2$

We are finally ready to characterize minimal valid functions for P_f^+ by relating them to maximal positive lattice point free sets, as stated in the following theorem.

Theorem 3.11 *Let $\psi : \mathbb{Q}^2 \to \mathbb{R}$ be a minimal valid function for P_f^+. If $S(\psi, f)$ contains f in its interior, then $S(\psi, f)$ is a maximal positive lattice point free set.*

Proof. Let $B = S(\psi, f)$ and note that as ψ is a valid function, B is positive lattice point free. Also remember that, by Lemma 3.1, $\psi = \psi_B$. For the sake of contradiction, assume that B is not maximal, and let B' be a maximal positive lattice point free set strictly containing B. If $int(RC(B)) = \emptyset$, then $\psi_{B'}$ dominates ψ_B, a contradiction. Therefore, we assume that $RC(B)$ is full-dimensional.

Let $RC = \{x \in \mathbb{R}^2 : c_i x \leq 0, i \in J\}$ be a minimal description of the recession cone of B and let $d_i = \sup\{c_i x : x \in B\}$ for $i \in J$. Notice that $B \subseteq C = \{x \in \mathbb{R}^2 : c_i x \leq d_i, i \in J\}$. Now let $B'' = B' \cap C$ and note $RC(B'') = RC$ and therefore $\psi_B(r) = \psi_{B''}(r)$ for all $r \in RC$. Furthermore, as $B'' \supset B$, by Lemma 2.4, $\psi_{B''}(r) \leq \psi_B(r)$ for all $r \notin RC$. As ψ is minimal, $\psi = \psi_{B''}$ and hence $B = B''$. Therefore, B is polyhedral as $B = B' \cap C$ where both B' and C are polyhedral.

Let $B = \{x : a_i x \leq b_i, i \in I\}$. By Lemma 3.8, a polyhedral set is maximal positive lattice point free, if and only if, there exists a nonnegative integer point in the relative interior of each one of its facets. As B is not maximal, for some $t \in I$, the facet $\mathcal{F}_t$ defined by $a_t x \leq b_t$ does not contain any nonnegative integer points in its relative interior. If $\mathcal{F}_t$ is bounded, that is if $a_t x \leq 0$ does not define a facet of RC, then for some $\epsilon > 0$, the set $\bar{B} = \{x : a_i x \leq b_i, i \in I \setminus \{t\}\ ;\ a_t x \leq b_t + \epsilon\}$ is positive lattice point free. Therefore, $\psi_{\bar{B}}$ is a valid function and as $\bar{B} \supset B$, $\psi_{\bar{B}}$ dominates ψ_B, a contradiction. Hence, we assume that $\mathcal{F}_t$ is unbounded and $a_t x \leq 0$ defines a facet of RC.

We now argue that there is another facet of RC defined by $a_k x \leq 0$ where a_k and a_t are linearly independent. If there is no such a_k then $RC = \{x \in \mathbb{R}^2 : a_t x \leq 0\}$. As $f = (f_1, f_2)^T \notin \mathbb{Z}^2$, not all of the following four points are the same

$$p_1 = \begin{pmatrix} \lfloor f_1 \rfloor \\ \lfloor f_2 \rfloor \end{pmatrix}, p_2 = \begin{pmatrix} \lfloor f_1 \rfloor \\ \lceil f_2 \rceil \end{pmatrix}, p_3 = \begin{pmatrix} \lceil f_1 \rceil \\ \lfloor f_2 \rfloor \end{pmatrix}, p_4 = \begin{pmatrix} \lceil f_1 \rceil \\ \lceil f_2 \rceil \end{pmatrix}$$

and since $f > 0$, they are non-negative and integral. Furthermore, $f \in conv(p_1, p_2, p_3, p_4)$ and hence, there exist two distinct points $p', p'' \in \{p_1, p_2, p_3, p_4\}$ such that $a_t p' \leq a_t f \leq a_t p''$. This implies that $a_t(p' - f) \leq 0$ and hence $r' = p' - f \in RC(B)$. As $f \in int(B)$, we have that $p' = f + r' \in int(B)$, contradicting the fact that B is positive lattice point free.

Therefore, RC indeed has a facet defined by $a_k x \leq 0$ where a_k and a_t are linearly independent. This also implies that B has an unbounded facet defined by $a_k x \leq b_k$. Now, in the linear description of B, replace $a_t x \le b_t$ by $(a_t + \epsilon a_k)x \le b_t + \epsilon b_k$ for some small $\epsilon > 0$ and call the resulting set B^ϵ. Clearly, $B^\epsilon \supset B$. In addition, if ϵ is small enough the new inequality is facet defining for B^ϵ and also it induces a facet of $RC(B^\epsilon)$.

We next show that, if $\epsilon > 0$ is sufficiently small, then B^ϵ would be positive lattice point free. To see this, note that, by Lemma 3.5, B does not have nonnegative rays in its recession cone RC, and therefore, there exists $\epsilon' > 0$ such that for every $\epsilon < \epsilon'$ we have that B^ϵ also has no nonnegative rays in its recession cone. Therefore, if $\epsilon > 0$ is small enough, $B^\epsilon \cap R^2_+$ is bounded and therefore $B^\epsilon \cap Z^2_+$ is finite. Let $U = (B^\epsilon \setminus B) \cap Z^2_+$ and note that for all points $x \in U$ we have (i) $a_t x > b_t$ and (ii) $a_k x < b_k$. Let

$$\beta = \min_{x \in U}\{a_t x - b_t\} \qquad \text{and } \alpha = \max_{x \in U}\{b_k - a_k x\}$$

and reduce ϵ, if necessary, so that $\epsilon < \beta/\alpha$. If B^ϵ is not positive lattice point free, there is a nonnegative integer point $y \in int(B^\epsilon)$. As $\mathcal{F}_t$, the face of B defined by $a_t x \le b_t$, has no integer points by assumption, $a_t y > b_t$ and therefore $y \in U$. But then,

$$(a_t + \epsilon a_k)y < b_t + \epsilon b_k \Rightarrow a_t y - b_t < \epsilon(b_k - a_k y) < \epsilon\alpha < \beta \le a_t y - b_t.$$

This is a contradiction and therefore $int(B^\epsilon) \cap Z^2_+ = \emptyset$ and B^ϵ is positive lattice point free.

As the final step, we will next show that ψ_{B^ϵ} dominates ψ_B which will imply that ψ_B can not be minimal, a contradiction. First note that as B^ϵ is larger than B, we have $\psi_{B^\epsilon}(r) \le \psi_B(r)$ for all $r \notin RC(B^\epsilon)$. Moreover, $RC(B^\epsilon) \supsetneq RC$ and therefore $\psi_{B^\epsilon}(r) < \psi_B(r)$ for all $r \in RC(B^\epsilon) \setminus int(RC)$. Finally, for $r \in int(RC)$, first note that

$$\psi_B(r) = \max\left\{\gamma, \frac{r^T a_t}{b_t - f^T a_t}\right\} \text{ and } \psi_{B^\epsilon}(r) = \max\left\{\gamma, \frac{r^T(a_t + \epsilon a_k)}{(b_t + \epsilon b_k) - f^T(a_t + \epsilon a_k)}\right\}$$

where $\gamma = \max_{i \in J \setminus \{t\}} \{r^T \hat{c}_i\}$. First note that as $f \in int(B)$,

$$(b_t + \epsilon b_k) - f^T(a_t + \epsilon a_k) = (b_t - f^T a_t) + \epsilon(b_k - f^T a_k) < b_t - f^T a_t.$$

In addition for $r \in int(RC)$ we have $r^T a^k < 0$ and therefore $r^T(a_t + \epsilon a_k) < r^T a_t$ implying that $\psi_B(r) \ge \psi_{B^\epsilon}(r)$. Therefore, ψ_{B^ϵ} indeed dominates ψ_B which contradicts the starting assumption that ψ_B is minimal. ∎

3.3 Geometry of positive lattice point free sets in $\mathbb{R}^2$

So far in this section we established a strong relationship between minimal functions and maximal positive lattice point free sets for $m = 2$. In particular, Theorem 3.11 shows that any minimal function is generated by a maximal positive lattice point free set, which by Theorem 3.2 is polyhedral and therefore, by Lemma 3.8, has at most 4 facets. In other words, if $\psi : \mathbb{Q}^2 \to \mathbb{R}$ is a minimal valid function for P_f^+ , then $\psi = \psi_B$ where B is full-dimensional and has a minimal description

$$B = \{x \in \mathbb{R}^2 : a_i^T x \le b_i, \forall i = 1, \ldots, k\}$$

with $k \le 4$. We next show that if $k = 4$ then B is a maximal lattice point free set.

Lemma 3.12 *Let B be a maximal positive lattice point free set in $\mathbb{R}^2$ that contains a point $f > 0$ in its interior. If B has 4 facets, then it contains no lattice points in its interior and therefore it is a maximal lattice point free set. Furthermore, B is bounded.*

Proof. Assume that B contains lattice points in its interior and let $\bar{x}$ be one such point with the property that $d(x) = \max\{-x_1, -x_2\}$ is smallest. As B is positive lattice point free, $x \notin \mathbb{R}^2_+$ and $d(x) > 0$. As every facet of B has to have a positive lattice point in its relative interior by Lemma 3.8, B has to contain, on its boundary, 4 positive lattice points with all 4 possible odd/even parity. Let $y \in B$ be a positive lattice point that has the same odd/even parity as x and notice that $z = x/2 + y/2$ is integral and $z \in int(B)$ and therefore z is not a positive lattice point. But then, as $y \geq 0$ we have $d(z) \leq d(x/2) < d(x)$, a contradiction. Therefore, B is a maximal lattice point free set with 4 facets and as such it has to be a quadrilateral (see [2]) and therefore, it is bounded. ∎

Remember that P_f is the relaxation of P_f^+ where integer variables are not required to be non-negative, see [5]. Also remember that minimal valid inequalities for P_f are defined by maximal lattice point free sets. More precisely, if B is a maximal lattice point free set, then ψ_B is a minimal valid function for P_f, and if ψ is a minimal valid function for P_f, then $S(\psi, f)$ is a maximal lattice point free set.

From a practical point of view, Lemma 3.3 implies that any minimal valid inequality $\psi : \mathbb{Q}^2 \to \mathbb{R}$ for P_f^+ is also valid and minimal for P_f provided that $S(\psi, f)$ is a quadrilateral. The converse, however, is not true as minimal valid inequalities for P_f that are associated with quadrilaterals might need to be strengthened to obtain minimal valid inequalities for P_f^+. To see this point notice that the maximal lattice point free quadrilateral $Q = conv\{A, B, C, D\}$ shown in Figure 2(a) is strictly contained in the maximal positive lattice point free cone K defined as the convex hull of the rays $\overrightarrow{EF}$ and $\overrightarrow{EG}$. The maximal lattice point free quadrilateral $Q' = conv\{H, I, J, K\}$ shown in Figure 2(b), on the other hand, is both maximal lattice point free and maximal positive lattice point free and therefore the function $\psi_{Q'}$ is a minimal valid function for both sets P_f^+ and P_f.

The cone K shown in Figure 2(a) also shows another difference between maximal positive lattice point free and maximal lattice point free sets. In the case of maximal lattice point free sets, if a set is full dimensional and unbounded, then it is a split which does not have a full-dimensional recession cone. On the contrary, as shown in Figure 2(a), maximal positive lattice point free sets can have full-dimensional recession cones.

When a maximal positive lattice point free set has 3 facets it can be a bounded set (triangle) or an unbounded set. In both cases, the set might contain lattice points and therefore might lead to minimal valid inequalities that are not valid for P_f. Figure 3 shows these cases. Finally, if a maximal positive lattice point free set has 2 facets, it can be a split, in which case the set is also maximal lattice point free, or, it might be a translated cone as shown in Figure 2(a).

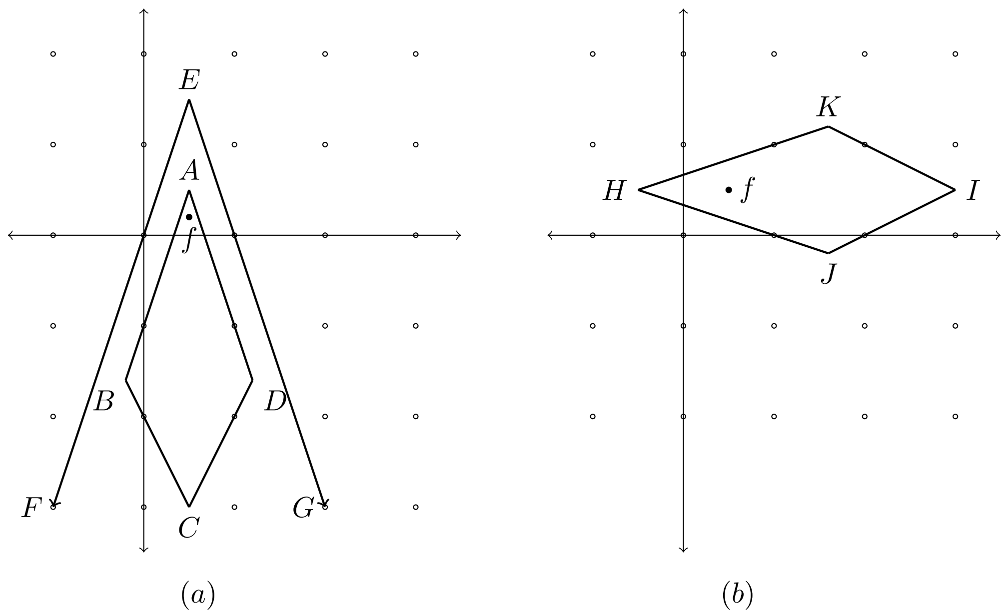

Figure 2: (a) A maximal lattice free quadrilateral contained in a positive lattice free cone, and,
. (b) a maximal lattice free quadrilateral which is also maximal positive lattice point free.

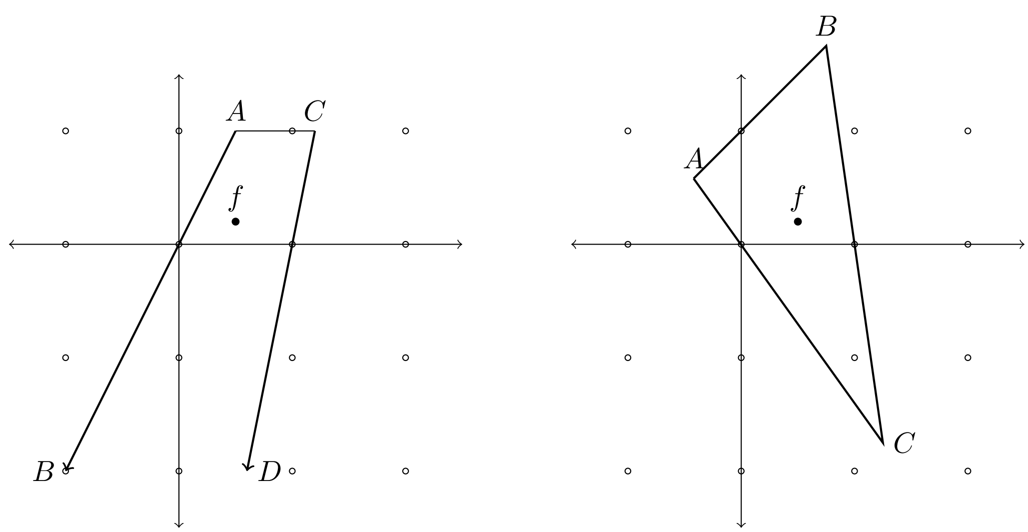

Figure 3: Bounded and unbounded maximal positive lattice point free sets with 3 facets

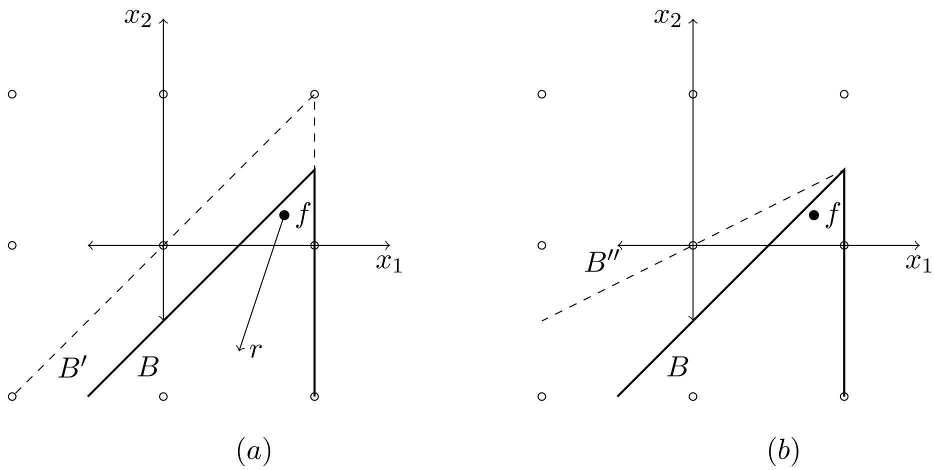

Figure 4: Two positive lattice point free sets B' and B'' that contain B. $\psi_{B'} \not\leq \psi_B$ whereas $\psi_{B''} \leq \psi_B$.

4 Strengthening valid inequalities for P_f^+ .

In the previous section we showed that when $m = 2$, it is sufficient to consider polyhedral positive lattice point free sets to obtain all minimal valid functions for X_+ . This result motivates the following question addressed in this section: given a polyhedral positive lattice point free set $B \subset \mathbb{R}^m$ which is not maximal, how can one obtain a positive lattice point free set $B' \supsetneq B$ such that $\psi_{B'} \leq \psi_B$? One possibility is to start with a maximal lattice-free convex set as in Example 1.1 and try to obtain a positive lattice point free set that strictly contains it. It is important to note, however, that $B' \supsetneq B$ does not imply $\psi_{B'} \leq \psi_B$. In other words, larger sets do not necessarily lead to better valid inequalities. This is an important difference between the relaxation P_f^+ studied in this paper and relaxation P_f studied in [5]. The following example illustrates this fact.

Example 4.1 *Let* $f = (0.8, 0.2)$ *and consider the following two positive lattice point free sets* $B = \{x \in \mathbb{R}^2 : -x_1 + x_2 \leq -1/2 \ ; \ x_1 \leq 1\}$ *and* $B' = \{x \in \mathbb{R}^2 : -x_1 + x_2 \leq 0 \ ; \ x_1 \leq 1\}$. *Figure 4(a) illustrates this example. Notice that* $B' \supsetneq B$ *and both sets contain* f *in their interior. For* $r = (-0.3, -0.9)$ *we have that*

$$\psi_B(r) = \max\{\frac{(-1,1)^T(-0.3,-0.9)}{-0.5-(-1,1)^T(0.8,0.2)}, \frac{(1,0)^T(-0.3,-0.9)}{1-(1,0)^T(0.8,0.2)}\} = \max\{-6,-1.5\} = -1.5, \text{ and}$$

$$\psi_{B'}(r) = \max\{\frac{(-1,1)^T(-0.3,-0.9)}{0-(-1,1)^T(0.8,0.2)}, \frac{(1,0)^T(-0.3,-0.9)}{1-(1,0)^T(0.8,0.2)}\} = \max\{-1,-1.5\} = -1$$

and therefore $\psi_B(r) < \psi_{B'}(r)$ *even though* B' *contains* B. *As all minimal functions are associated with maximal sets in* $\mathbb{R}^2$*, there exists a different maximal positive lattice point free set* $B'' \supsetneq B$ *that gives a stronger valid inequality.*

The set $B'' = \{x \in \mathbb{R}^2 : -1/2x_1 + x_2 \leq 0 \ ; \ x_1 \leq 1\} \supset B$ *shown in Figure 4(b), on the other hand, gives a valid inequality that dominates* ψ_B. *The fact that* $\psi_{B''} \leq \psi_B$ *follows from Lemma 4.2.*

The following Lemma gives sufficient conditions under which $B' \supseteq B$ implies that $\psi_{B'}$ dominates ψ_B (i.e. $\psi_{B'} \leq \psi_B$ and $\psi(\bar{r}) < \psi_B(\bar{r})$ for some $\bar{r} \in \mathbb{Q}^m$.) We assume that all polyhedral descriptions given are minimal, in other words, all inequalities given define facets of the corresponding polyhedra.

Lemma 4.2 *Let $B = \{x \in \mathbb{R}^m : a^i x \leq b^i, i \in I\}$ and $RC(B) = \{x \in \mathbb{R}^m : a^i x \leq 0, i \in I^c\}$ where $I^c \subseteq I$. Assume B is positive lattice point free and $0 < f \in int(B)$. Let $B' \supsetneq B$. If one of the following conditions hold, then $\psi_{B'}$ dominates ψ_B.*
(i) $int(RC(B)) = \emptyset$.
(ii) B' is obtained from B by dropping a constraint, i.e., $B' = \{x \in \mathbb{R}^m : a^i x \leq b^i, i \in I \setminus \{k\}\}$.
(iii) B' is obtained from B by relaxing a constraint that does not give a facet of the recession cone, i.e., $B' := \{x \in \mathbb{R}^m : a^i x \leq b^i, i = I \setminus \{k\}\ ;\ a^k x \leq b^k + \epsilon\}$ where $k \in I \setminus I^c$ and $\epsilon > 0$.
(iv) B' is obtained from B by rotating a facet-defining inequality of B using another one, i.e., $B' := \{x \in \mathbb{R}^m : (a^l + \epsilon a^k)x \leq b^l + \epsilon b^k\ ;\ a^i x \leq b^i, i \in I \setminus \{l\}\}$ where $l, k \in I$ and $\epsilon > 0$.

Proof. Let $\bar{x} \in B' \setminus B$ and define $\bar{r} = \bar{x} - f$ so that $f + \bar{r} \in B' \setminus B$. Note that $\bar{r} \notin RC(B)$ as $f + \bar{r} \notin B$. By Lemma 2.4, for this choice of $\bar{r}$ we have $\psi_{B'}(\bar{r}) < \psi_B(\bar{r})$. We next consider each case separately and show that $\psi_{B'} \leq \psi_B$ also holds.
(i) Follows directly from Lemma 2.4 as $\psi_B \geq 0$ when $int(RC(B)) = \emptyset$.
(ii) Follows from the definition of ψ as $\psi_B(r) = \max_{i \in I}\left\{r^T \hat{a}_i\right\}$ and $\psi_{B'}(r) = \max_{i \in I \setminus \{k\}}\left\{r^T \hat{a}_i\right\}$.
(iii) As $B' \supseteq B$, Lemma 2.4 implies that $\psi_{B'}(r) \leq \psi_B(r)$ for all $r \notin int(RC(B))$. In addition, for $r \in int(RC(B))$ by Lemma 2.9, $\psi_B(r) = \hat{a}^j r$ for some $j \in I^c$ and as $RC(B) = RC(B')$, we have $\psi_{B'}(r) = \psi_B(r)$.
(iv) Let $c = (a^l + \epsilon a^k)$ and $d = b^l + \epsilon b^k$. We will show that $cr/(d - cf) \leq \max\{\hat{a}^l r, \hat{a}^k r\}$. This fact, together with the fact that $\psi_{B'}(r) = \max\{cr/(d - cf), \max_{I \setminus \{l\}} \hat{a}^i r\}$ implies that $\psi_{B'}(r) \leq \psi_B(r)$. Suppose, for the sake of contradiction that $\hat{c}r = cr/(d - cf) > \max\{\hat{a}^l r, \hat{a}^k r\}$. Then $\hat{c}r > \hat{a}^l r$ implies

$$\frac{a^l r + \epsilon a^k r}{(b^l - a^l f) + \epsilon(b^k - a^k f)} > \frac{a^l r}{b^l - a^l f}.$$

As all the denominators are positive, we have

$$a^l r(b^l - a^l f) + \epsilon a^k r(b^l - a^l f) > a^l r(b^l - a^l f) + \epsilon a^l r(b^k - a^k f)$$

and hence $a^k r(b^l - a^l f) > a^l r(b^k - a^k f)$. Similarly, $\hat{c}r > \hat{a}^k r$ implies $a^k r(b^l - a^l f) < a^l r(b^k - a^k f)$, which is a contradiction and thus the result follows. ■

Figure 1 shows an example of B and B' satisfying conditions (i) and (ii) of Lemma 4.2, while Figure 5 shows an example of B and B' that satisfy conditions (iii) and (iv). Also note that the set B' in Figure 4(a) does not satisfy condition (iii) as the relaxed constraint of B is associated with a facet of $RC(B)$. The set B'' in Figure 4(b), however, satisfies condition (iv), as it is obtained by rotating one facet defining inequality of B is using another facet-defining inequality.

Notice that Lemma 4.2 states conditions under which $B' \supseteq B$ gives a function $\psi_{B'} \leq \psi_B$. However, such a function $\psi_{B'}$ is not useful for generating valid inequalities for P_f^+ unless B' is

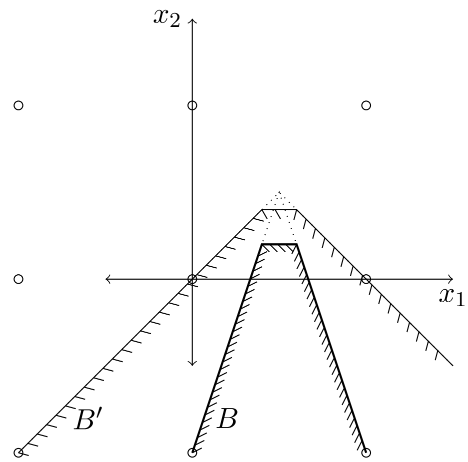

Figure 5: Example of a positive lattice point free set B' that contains another positive lattice point free set B such that $\psi_{B'} \leq \psi_B$.

positive lattice point free. Therefore, one needs to be able to check if B' is positive lattice point free in order to apply Lemma 4.2 to strengthen a valid inequality for P_f^+. In general, checking this condition can be as difficult as solving an IP in m dimensions. However, there are some sufficient conditions that can be checked that guarantee that B' is positive lattice point free.

We next identify simple conditions under which dropping a constraint from B leads to a positive lattice point free set. We are not able to establish easily checkable conditions for the remaining operations described in Lemma 4.2. Formally, let $B = \{x \in \mathbb{R}^m : a^i x \leq b^i, \forall i = 1, \ldots, k\}$ be a polyhedral positive lattice point free set that contains $f > 0$ in its interior (we also assume that all inequalities describing B define facets). Let

$$B^j = \{x \in \mathbb{R}^m : a^i x \leq b^i, \forall i \in \{1, \ldots, k\} \setminus \{j\}\}$$

denote the polyhedron obtained by dropping the jth inequality and let

$$F^j = \{x \in B : a^j x = b^j\}$$

denote the facet defined by the jth inequality of B. It is easy to see that B^j can not be positive lattice point free if F^j contains a nonnegative integer point in its relative interior. The following observation establishes the reverse condition.

Lemma 4.3 *Assume that F^k does not contain a nonnegative integer point in its relative interior. In addition, if $\mathbb{Z}^m_+ \cap int(B^k) \subseteq \{x : a^k x \leq b^k\}$, then $\mathbb{Z}^m_+ \cap int(B^k) = \emptyset$, that is, B^k is positive lattice point free.*

Proof. If $\mathbb{Z}^m_+ \cap int(B^k) \neq \emptyset$, let $y \in \mathbb{Z}^m_+ \cap int(B^k)$ and note that by assumption $a^k y \leq b^k$. If $a^k y < b^k$, then $y \in int(B)$, which contradicts the fact that B is positive lattice point free. Hence $a^k y = b^k$ and y has to be in the relative interior of F^k, again a contradiction. ∎

Note that $\mathbb{Z}^m_+ \cap int(B^k) \subseteq \mathbb{R}^m_+ \cap B^k$. Based on this observation and Lemma 4.3, we next present two conditions that can be checked easily to verify that B^k is positive lattice point free.

Corollary 4.4 *Assume that F^k does not contain a nonnegative integer point in its relative interior. Then B^k is a positive lattice point free provided that $\mathbb{R}^m_+ \cap B^k \subseteq \{x : a^k x \leq b^k\}$.*

Also note that if $a^k \leq 0$ and $b^k \geq 0$, then $\mathbb{R}^m_+ \subseteq \{x : a^k x \leq b^k\}$ and the above condition holds trivially. Another condition that can be checked is the following.

Lemma 4.5 *If $F^k \cap \mathbb{R}^m_+ = \emptyset$ then B^k is positive lattice point free.*

Proof. Suppose not. Then there exists $y \in \mathbb{Z}^m_+$ such that $a^k y > b^k$ and $a^j y < b^j$ for all $j = 1, \ldots, k-1$. In addition, as $f > 0$ is in the interior of B, we have that $a^j f < b^j$ for all $j = 1, \ldots, k$. But then for all $\lambda \in [0, 1]$ we have that $x^\lambda = \lambda f + (1-\lambda)y$ satisfies $a^j x^\lambda < b^j$ for all $j = 1, \ldots, k-1$ and $x^\lambda \geq 0$. Moreover there exists λ such that $a^k x^\lambda = b^k$, but this contradicts the assumption that $F^k \cap \mathbb{R}^m_+ = \emptyset$ ∎

Remember Example 1.1 and note that the inequality that was dropped to obtain the maximal positive lattice point free set satisfies the conditions of both Corollary 4.4 and Lemma 4.5. Also note that in order to apply Lemma 4.3 or Corollary 4.4, one needs to check if $int(F^k)$ contains nonnegative integer points, which requires solving an integer program in $\mathbb{R}^m$. The condition $F^k \cap \mathbb{R}^m_+ = \emptyset$ in Lemma 4.5, however, can be checked by solving a linear program.

5 Conclusion

In this paper, we defined a new relaxation for mixed-integer sets and studied valid inequalities associated with it. Our relaxation can be seen as a tightening of the relaxation defined by Borozan and Cornuéjols [5] and Andersen et. al. [2]. The difference between the two relaxations is the presence of non-negativity constraints in our set. In this respect, the difference between the two relaxations is similar to the difference between the master equality polyhedron [7] that we studied recently and the cyclic group polyhedron of Gomory. In both cases, exploiting non-negativity leads to stronger inequalities.

Even though some of our results generalize easily for $m > 2$ constraints, there are others that we were not able to extend. For instance, for $m > 2$, are maximal positive lattice point free sets polyhedral? If so, do they always have a nonnegative integer point in the relative interior of each of their facets? Moreover, can there be minimal functions that arise from non-maximal positive lattice-free convex sets? Even though we only derived a one-to-one correspondence between maximal positive lattice point free sets and minimal functions for $m = 2$, we believe such correspondence also exists for $m > 2$.

Finally, notice that the nonnegativity on the integer variables is an arbitrary choice of constraints. In principle, one could impose any additional set of constraints to the integer variables and use this additional information to strengthen the inequalities obtained. A case of particular interest is when the basic variables are all between given bounds $[0, u]$ (for example binary variables) and hence we only need to focus on convex sets that don't have integer points in $[0, u]$ in their interior. We believe, for instance that Theorems 3.2 and 3.11 can be generalized for such cases.

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References

[1] K. Andersen, personal communication (2009).

[2] K. Andersen, Q. Louveaux, R. Weismantel and L. A. Wolsey, Inequalities from two rows of a simplex tableau, IPCO 2007, 1–15.

[3] K. Andersen, Q. Louveaux,and R. Weismantel, Geometric study of mixed-integer sets from two rows of from two adjavent simplex bases, *Math. Programming*, to appear.

[4] D.E. Bell, A theorem concerning the integer lattice. *Studies in Applied Mathematics* **56**, 187–188 (1977).

[5] V. Borozan and G.Cornuéjols, Minimal Valid Inequalities for Integer Constraints. *Mathematics of Operations Research* (2009) to appear.

[6] G. Cornuejols and F. Margot, On the Facets of Mixed Integer Programs with Two Integer Variables and Two Constraints, *Mathematical Programming*, to appear.

[7] S. Dash, R. Fukasawa, and O. Günlük. On a generalization of the master cyclic group polyhedron, *Math. Programming*, to appear.

[8] S. Dey, personal communication (2009).

[9] S. Dey and J. P. Richard, Two Families of Facets for the Two-Dimensional Mixed Integer Infinite Group Problem, *Mathematics of Operations Research*, to appear.

[10] S.S. Dey, J. P. Richard, Y. Li, and L. A. Miller, On the extreme inequalities of infinite group problems, *Mathematical Programming*, to appear.

[11] J.-P. Doignon. Convexity in cristallographical lattices, *Journal of Geometry* **3**, 71–85 (1973).

[12] D. Espinoza, Computing with multi-row Gomory cuts, IPCO 2008, Bertinoro, Italy, LNCS 5035 214-224 (2008).

[13] R.E. Gomory, Some polyhedra related to combinatorial problems, Journal of Linear Algebra and its Applications **2**, 451–558 (1969).

[14] R.E. Gomory and E. L. Johnson, Some continuous functions related to corner polyhedra I, *Mathematical Programming* **3**, 23–85 (1972).

[15] R.E. Gomory and E. L. Johnson, T-space and cutting planes, *Mathematical Programming* **96**, 341–375 (2003).

[16] E. L. Johnson, On the group problem for mixed integer programming, *Mathematical Programming study*, 137–179 (1974).

[17] L. Lovász, Geometry of numbers and integer programming, *Mathematical Programmin: Recent Developments and Applications*, M. Iri and K. Tanabe eds., Kluwer (1989) 177–201.

[18] R. T. Rockefeller, Convex analysis, Princeton Univ. Press, Princeton, NJ (1970).

[19] H.E. Scarf. An observation on the structure of production sets with indivisibilities, *Proceedings of the National Academy of Sciences of the United States of America* **74**, 3637–3641 (1977).

[20] W. M. Schmidt, Diophantine approximation, Springer-Verlag, Berlin (1980).

[21] A. Schrijver, Theory of linear and integer programming, Wiley-Interscience series in Discrete Mathematics and Optimization, Chichester, England (1986).

[22] G. Zambelli, On Degenerate Multi-row Gomory Cuts, technical report, April 2008, to appear in *Operations Research Letters.*