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On a Connection between Facility Location and Perfect Graphs

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ON A CONNECTION BETWEEN FACILITY LOCATION AND PERFECT GRAPHS

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ABSTRACT. We characterize the graphs for which the linear relaxation of a facility location problem defines a polytope with all integral extreme points. We use a transformation to a stable set problem in perfect graphs.

1. INTRODUCTION

Let $D = (V, A)$ be a directed graph, not necessarily connected, where each arc and each node has a weight associated with it. We study a *facility location problem* as follows. A set of nodes is selected, usually called *centers*, and then each non selected node has to be assigned to a center. The weight of a node is the cost of building the facility. The weight of an arc (i, j) is the cost of assigning the location i to the location j . The goal is to minimize the sum of the weights of the selected nodes plus the sum of the weights yielded by the assignment. The linear system below defines a linear programming relaxation.

$$\begin{aligned} (1) \quad & \min \sum w(u, v)x(u, v) + \sum w(v)y(v) \\ (2) \quad & \sum_{(u, v) \in A} x(u, v) + y(u) = 1 \quad \forall u \in V, \\ (3) \quad & x(u, v) \leq y(v) \quad \forall (u, v) \in A, \\ (4) \quad & 0 \leq y(v) \leq 1 \quad \forall v \in V, \\ (5) \quad & x(u, v) \geq 0 \quad \forall (u, v) \in A. \end{aligned}$$

For each node u , the variable $y(u)$ should take the value one if the node u is selected and zero otherwise. For each arc (u, v) the variable $x(u, v)$ should take the value one if u is assigned to v and zero otherwise. Equations (2) express the fact that either node u is selected or it should be assigned to another node. Inequalities (3) indicate that if a node u is assigned to a node v then this last node should be selected.

Let $P(D)$ be the polytope defined by (2)-(5), and let $LP(D)$ be the convex hull of $P(D) \cap \{0, 1\}^{|A|+|V|}$. Clearly

$$LP(D) \subseteq P(D).$$

In this paper we characterize the directed graphs D for which $LP(D) = P(D)$. More precisely, we show that $LP(D) = P(D)$ if and only if D does not contain certain type of “odd” cycles. In [2] we studied a similar question but when equations (2) are replaced by inequalities. We could not adapt the techniques of [2] to the present study, here we had to use a transformation to a problem in perfect graphs; it seems difficult to give a proof without this transformation.

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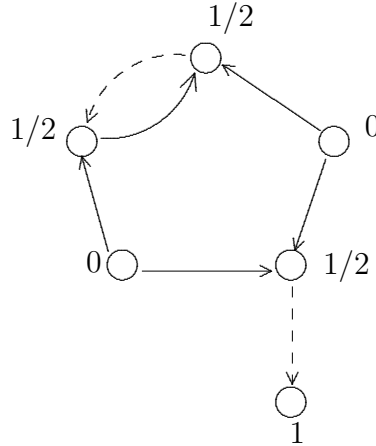


FIGURE 1. A g -odd Y -cycle. The arcs in the Y -cycle appear with solid lines.

To describe our main result we need some definitions and notations. A simple *cycle* C is an ordered sequence $v_0, a_0, v_1, a_1, \dots, a_{t-1}, v_t = v_0$, where all v_i 's, for $i = 0, \dots, t - 1$, are distinct and such that for $0 \leq i \leq t - 1$: either v_i is the tail of a_i and v_{i+1} is the head of a_i , or v_i is the head of a_i and v_{i+1} is the tail of a_i . Let $V(C)$ and $A(C)$ denote the nodes and the arcs of C , respectively. By setting $a_t = a_0$, we associate with C three more sets as below.

- We denote by \hat{C} the set of nodes v_i , such that v_i is the head of a_{i-1} and also the head of a_i , $1 \leq i \leq t$.
- We denote by \dot{C} the set of nodes v_i , such that v_i is the tail of a_{i-1} and also the tail of a_i , $1 \leq i \leq t$.
- We denote by \tilde{C} the set of nodes v_i , such that either v_i is the head of a_{i-1} and also the tail of a_i , or v_i is the tail of a_{i-1} and also the head of a_i , $1 \leq i \leq t$.

Notice that $|\hat{C}| = |\dot{C}|$. A cycle will be called *g -odd* if $|\tilde{C}| + |\hat{C}|$ is odd, otherwise it will be called *g -even*. A cycle C with $V(C) = \tilde{C}$ is a *directed* cycle. The notion of g -odd (g -even) cycles generalizes the notion of odd (even) directed cycles.

Definition 1. A simple cycle is called a *Y -cycle* if for every $v \in \hat{C}$ there is an arc (v, \bar{v}) in A , where \bar{v} is in $V \setminus \dot{C}$.

Note that when $\hat{C} = \emptyset$, then C is a directed cycle and also a Y -cycle. In Figure 1 we show a fractional extreme point of $P(D)$. The graph contains a g -odd Y -cycle, the values of the arc variables are all $1/2$ and those corresponding to the nodes are reported in the figure. It turns out that g -odd Y -cycles are the only configurations that should be forbidden in order to have a polytope with all integral extreme points as it is stated below. Our main result is the following.

Theorem 2. Let $D = (V, A)$ be a directed graph, then $P(D)$ is a polytope with all integer extreme points if and only if D does not contain a g -odd Y -cycle.

De Simone and Mannino [14] studied the optimization problem (1)-(5), transformed it into a maximum stable set problem in an undirected graph, and proved the perfect graph theorem for this class of graphs. We are going to use the same transformation. Avella and Sassano [1] had used the same transformation to study the p -median problem.

The facets of the uncapacitated facility location polytope have been studied in [16], [13], [6], [7], [5]. The uncapacitated facility location problem has also been studied from the point of view of approximation algorithms in [18], [8], [19], [4], [20] and others. Other references on this problem are [12] and [17]. The relationship between location polytopes and the stable set polytope has been studied in [13], [6], [7], [14], and others.

This paper is organized as follows. In the next section we give some definitions and introduce perfect graphs. In Section 3 we start giving the proof of Theorem 2. In Section 4 we continue the proof giving a reduction to the stable set problem in a new graph. In Section 5 we show that the new graph cannot have an antihole of size greater than seven. Section 6 is devoted to show that the new graph cannot have an antihole of size seven. In Section 7 we show that the new graph cannot have an odd hole. We discuss the recognition of this class of graphs in Section 8.

2. FURTHER DEFINITIONS

The definition of a *path* P is similar to the one of a cycle, but with $v_0 \neq v_p$. In a similar way we define \dot{P} , \hat{P} and \tilde{P} , excluding v_0 and v_p . The nodes v_1, \dots, v_{p-1} are called *internal*.

For $W \subset V$, we denote by $\delta^+(W)$ the set of arcs $(u, v) \in A$, with $u \in W$ and $v \in V \setminus W$. Also we denote by $\delta^-(W)$ the set of arcs (u, v) , with $v \in W$ and $u \in V \setminus W$. We write $\delta^+(v)$ and $\delta^-(v)$ instead of $\delta^+(\{v\})$ and $\delta^-(\{v\})$, respectively. If there is a risk of confusion we use δ_G^+ and δ_G^- .

Given a ground set U , and $S \subseteq U$, the *incidence vector* x^S of S is defined by $x^S(v) = 1$ if $v \in S$ and $x^S(v) = 0$ if $v \in U \setminus S$. For a vector $x : U \rightarrow \mathbb{R}$, and $T \subseteq U$ we use $x(T)$ to denote $x(T) = \sum_{v \in T} x(v)$.

A *polyhedron* P is defined by a set of linear inequalities, i.e., $P = \{x \mid Ax \leq b\}$. A *polytope* is a bounded polyhedron. A polyhedron whose extreme points are integer is called an *integral polyhedron*.

2.1. Perfect graphs and the stable set polytope. In this sub-section $G = (V, E)$ is an undirected graph. A *hole* is an induced cycle of length at least four. An *antihole* is an induced subgraph isomorphic to the complement of a hole. A *clique* is a set $C \subseteq V$ of pairwise adjacent nodes, and a *stable set* is a set of pairwise non adjacent nodes. The size of a maximum clique is denoted by $\omega(G)$ and the size of a maximum stable set is denoted by $\alpha(G)$. The *chromatic number* of G , denoted by $\chi(G)$ is the minimum number k such that V can be partitioned into k stable sets. Clearly $\chi(G) \geq \omega(G)$. A graph is called *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G . Berge [3] conjectured in 1961 the following theorem; this was proved in 2002 by Chudnovsky et al. [10]. This result had been proved before for many special classes of graphs; for the class of graphs that we treat in this paper it was proved by de Simone and Mannino [14].

Theorem 3. *A graph is perfect if and only if it has no odd hole and no odd antihole.*

The *stable set polytope* is the convex hull of the incidence vectors of the stable sets of G . Fulkerson [15] and Chvátal [11] proved the following polyhedral characterization of perfect graphs.

Theorem 4. *A graph G is perfect if and only if its stable set polytope is defined by the system below.*

$$\begin{aligned} x(C) &\leq 1 && \text{for each clique } C, \\ x(v) &\geq 0 && \text{for each } v \in V. \end{aligned}$$

3. PROOF OF THEOREM 2

First we assume that D has a g -odd Y -cycle C and we are going to show that $LP(D) \neq P(D)$.

We define a fractional vector $(\bar{x}, \bar{y}) \in P(G)$ as follows:

$$\begin{aligned} \bar{y}(u) &= 0 && \text{for all nodes } u \in \hat{C}, \\ \bar{y}(u) &= 1/2 && \text{for all nodes } u \in V(C) \setminus \hat{C}, \\ \bar{x}(a) &= 1/2 && \text{for } a \in A(C), \\ \bar{x}(a) &= 1/2 && \text{for each } v \in \hat{C}, \text{ and } a = (v, \bar{v}) \text{ is one arc with } \bar{v} \in V \setminus \hat{C}, \\ \bar{y}(v) &= 1 && \text{for all other nodes } v \notin V(C), \\ \bar{x}(a) &= 0 && \text{for all other arcs.} \end{aligned}$$

Below we show an inequality that separates from $LP(G)$ the vector defined above.

Lemma 5. *The following inequality is valid for $LP(G)$.*

$$(6) \quad \sum_{a \in A(C)} x(a) - \sum_{v \in \hat{C}} y(v) \leq \frac{|\tilde{C}| + |\hat{C}| - 1}{2}.$$

Proof. From inequalities (2)-(5) we obtain

$$\begin{aligned} x(u, v) + x(\delta^+(v)) &\leq 1, && \text{for every arc } (u, v) \in A(C), v \notin \hat{C}, \\ x(u, v) - y(v) &\leq 0, && \text{for every arc } (u, v) \in A(C), v \in \hat{C}, \\ x(\delta^+(v)) &\leq 1, && \text{for } v \in \hat{C}. \end{aligned}$$

Their sum gives

$$2 \sum_{a \in A(C)} x(a) - 2 \sum_{v \in \hat{C}} y(v) + \sum_{v \in \hat{C}} x(\delta^+(v) \setminus A(C)) + \sum_{v \in \hat{C}} x(\delta^+(v) \setminus A(C)) \leq |A(C)| - 2|\hat{C}| + |\hat{C}|.$$

which implies

$$2 \sum_{a \in A(C)} x(a) - 2 \sum_{v \in \hat{C}} y(v) \leq |\tilde{C}| + |\hat{C}|.$$

dividing by 2 and rounding down the right hand side, as in Chvátal's procedure [8], we obtain

$$\sum_{a \in A(C)} x(a) - \sum_{v \in \hat{C}} y(v) \leq \frac{|\tilde{C}| + |\hat{C}| - 1}{2}.$$

□

For the vector (\bar{x}, \bar{y}) we have

$$\sum_{a \in A(C)} \bar{x}(a) - \sum_{v \in \hat{C}} \bar{y}(v) = \frac{|\tilde{C}| + |\hat{C}|}{2}.$$

Therefore $LP(D) \neq P(D)$.

In the second part of the proof we assume that D has no g-odd Y -cycle. The rest of the proof is the subject of the following four sections. First we reduce the problem to a maximum weight stable set problem in a related graph, then we show that the non existence of a g-odd Y -cycle implies that the new graph is perfect. This implies that $P(D)$ is an integral polytope.

4. REDUCTION TO THE STABLE SET PROBLEM

Consider problem (1)-(5) with the additional constraint that all variables should take integer values. Using equations (2) we can eliminate the variables y , and the problem becomes

$$(7) \quad \max \sum w'(u, v) x(u, v)$$

$$(8) \quad \sum_{(u,v) \in A} x(u, v) \leq 1 \quad \forall u \in V,$$

$$(9) \quad x(u, v) + \sum_{(v,t) \in A} x(v, t) \leq 1 \quad \forall (u, v) \in A,$$

$$(10) \quad x(u, v) \geq 0 \quad \forall (u, v) \in A,$$

$$(11) \quad x(u, v) \text{ integer} \quad \forall (u, v) \in A.$$

We have that (2)-(5) defines an integral polytope if and only if (8)-(10) defines an integral polytope.

We can build an undirected graph $L(D)$ where for each arc of D there is a node in $L(D)$. Let a_1 and a_2 be two arcs in D , we put an edge between the two corresponding nodes of $L(D)$ in the following two cases:

- a_1 and a_2 have the same tail, or
- the tail of one coincides with the head of the other.

Thus problem (7)-(11) is equivalent to a maximum weight stable set problem in $L(D)$. In order to use the linear system given in Theorem 4 we should identify the maximal cliques of $L(D)$.

Consider a maximal clique with n elements, this corresponds to a set of n arcs of D . We have the following four cases:

- If $n \geq 4$ then either $n - 1$ arcs share the same tail, and this is the head of the remaining arc; or the n arcs have the same tail. Thus we have an inequality of type (8) or (9).
- If $n = 3$ and the three corresponding arcs form a directed cycle we have a contradiction with the non existence of a g-odd Y -cycle, so at least two of them share the same tail, and we have an inequality of type (8) or (9).
- If $n = 2$ we have an inequality of type (8) or (9).
- If $n = 1$ we have an inequality of type (8).

Thus all maximal clique inequalities are included in (8)-(9). It remains to prove that if D has no g-odd Y -cycle then $L(D)$ is perfect. For that we use Theorem 3. We prove this in the next three sections.

5. $L(D)$ CANNOT HAVE AN ANTIHOLE OF SIZE GREATER THAN SEVEN

In this section we show that $L(D)$ does not contain an antihole with eight nodes or more. Assume the contrary. Then $L(D)$ must contain as a subgraph the graph H of Figure 2.

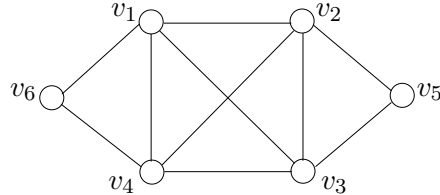


FIGURE 2. H , a subgraph in any antihole of size ≥ 8 .

The clique $C = \{v_1, v_2, v_3, v_4\}$ of H can be originated from the three subgraphs of D shown in Figure 3. Because of the symmetry, any arc can be associated with v_1 , we label that arc in the figure.

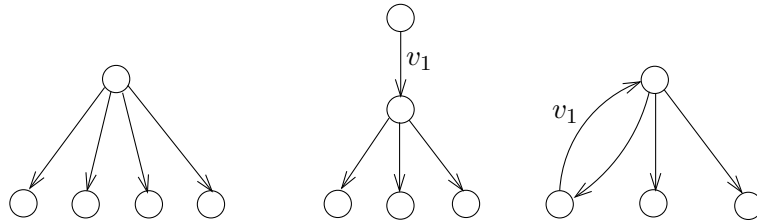


FIGURE 3

In this paragraph we say *the arc v_i* instead of *the arc associated with v_i* . If the head or the tail of the arc v_5 is the tail of the arc v_2 , then v_5 must be adjacent to v_4 which is not possible. Thus the tail of v_5 must be the head of v_2 , but then v_5 cannot be adjacent to v_3 .

6. THE ANTIHOLE WITH SEVEN NODES

In this section we show that if $L(D)$ has an antihole with seven nodes, then D contains a directed cycle with three nodes and we have a contradiction.

The antihole with seven nodes is in Figure 4. Consider the clique induced by the nodes v_1, v_2 and v_3 . This can be originated from the four configurations in D shown in Figure 5.

We first study configuration (a). The node v_5 is adjacent to v_1 and v_3 but not to v_2 . For this configuration, it is not possible to add an arc in D such that the corresponding node in $L(D)$ satisfies the above conditions. Thus configuration (a) cannot arise.

Configuration (b) already contains a g-odd Y -cycle, so it remains to study configurations (c) and (d).

6.1. Configuration (c). Here we have to study the three possibilities in Figure 6.

Case (1). This is depicted in Figure 7. Consider the arc in D associated with v_7 . The node v_7 is adjacent to v_1 and v_2 , and not to v_3 . So this arc cannot be incident to a .

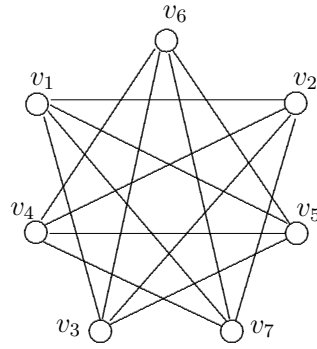


FIGURE 4. The antihole with seven nodes.

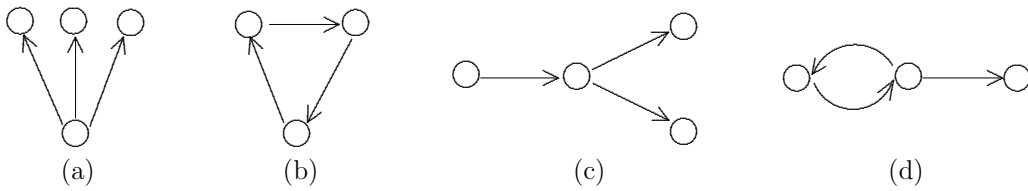


FIGURE 5

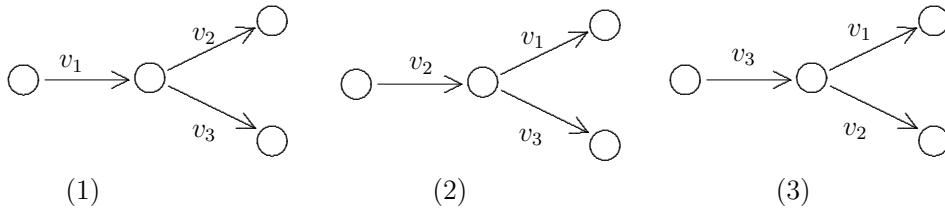


FIGURE 6

Thus the only possibility is an arc from b to c , and we have a directed cycle with three nodes.

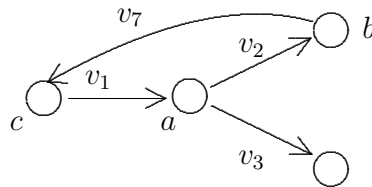


FIGURE 7

Case (2). This is shown in Figure 8. The arc in D associated with v_7 should go from b to c , and we have a directed cycle with three nodes.

Case (3). This is shown in Figure 9. The arc associated with v_5 should go from b to c .

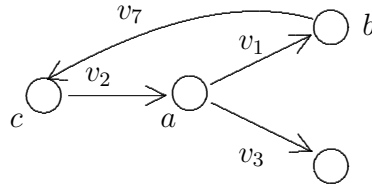


FIGURE 8

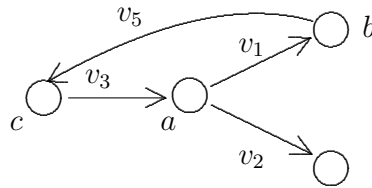


FIGURE 9

6.2. **Configuration (d).** Since v_2 and v_3 play a symmetric role, we have to study the three possibilities in Figure 10.

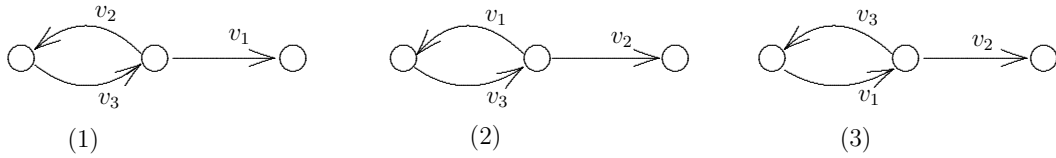


FIGURE 10

Case (1). In this case the arc associated with v_4 cannot be placed.

Case (2). This is depicted in Figure 11. The arc associated with v_4 should go from c to d . Then for the arc associated with v_7 there are two possibilities. If it goes from d to b we have directed cycle of size three. Otherwise it goes from c to b . In this case the arc associated with v_5 cannot be added.

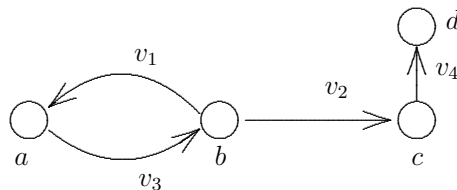


FIGURE 11

Case (3). This is shown in Figure 12. The arc associated with v_7 should go from c to a . Then there is a directed cycle of size three.

Thus we have proved that $L(D)$ cannot have an antihole of size seven.

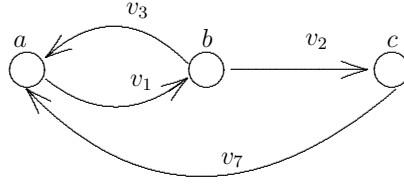


FIGURE 12

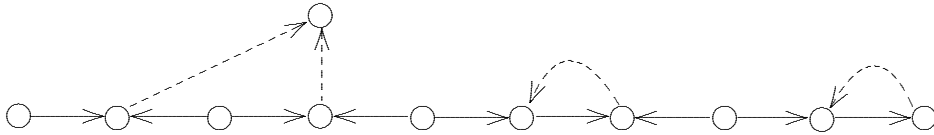
7. ODD HOLES AND g -ODD Y -CYCLES

We remind the reader that in this section we assume that D has no g -odd Y -cycle, in particular no directed cycle of size three. Suppose that $L(D)$ has an odd hole H with node set $\{u_1, \dots, u_{2p+1}\}$, $2p+1 \geq 5$. We are going to prove that H corresponds to a g -odd Y -cycle in D , this gives a contradiction. For that we use implicitly the fact that if u_i and u_j , ($i < j$), are adjacent then $j = i+1$ if $2 \leq i \leq 2p$, and $j = 2$ or $j = 2p+1$ if $i = 1$.

Definition 6. A Y -path P' is a subgraph of D with $P' = (V \cup V', A \cup A' \cup A'')$, $V \cap V' = \emptyset$; A , A' and A'' form a partition of the arc-set of P' . P' has the following properties:

- P' is connected.
- The subgraph $P = (V, A)$ is a path, and the first node of P is the tail of the first arc of P' . Also the first node of P has no other arc in P' incident to it.
- If $v \in \hat{P}$ then exactly one of the following two possibilities holds:
 - there is an arc $(v, v') \in A'$ with $v' \in V'$, or
 - if (u, v) and (w, v) are the two arcs incident to v in P , then either (v, u) or (v, w) is in A'' . In the first case $u \in \tilde{P}$, and in the second case $w \in \tilde{P}$.
- If v is the last node of P and $(u, v) \in P$, then at most one of the two following cases is possible:
 - (v, u) could be in A'' . If that is the case then $u \in \tilde{P}$.
 - (v, v') could be in A' , with $v' \in V'$ and $|\delta_{P'}^-(v')| \geq 2$.
- $\delta_{P'}^+(v) = \emptyset$ for all $v \in V'$.
- If $(u, v) \in A''$, then $(v, u) \in P$ and $v \in \tilde{P}$; and $u \in \hat{P}$ or u is the last node of P .
- If $(u, v) \in A'$, then $v \in V'$; and $u \in \hat{P}$, or u is the last node of P and the head of the last arc of P .

In Figure 13 we show a Y -path.

FIGURE 13. A Y -path. The arcs in $A' \cup A''$ are drawn with dashed lines.

We plan to build a Y -path by adding one by one the arcs corresponding to the nodes $\{u_i\}$.

Lemma 7. We may assume that the arcs corresponding to u_1 and u_2 are (a, b) and (b, c) respectively, $a \neq c$.

Proof. The sequence u_1, u_2, u_3 corresponds in D to one of the configurations of Figure 14. We just have to change the numbering of the nodes to satisfy the hypothesis. \square

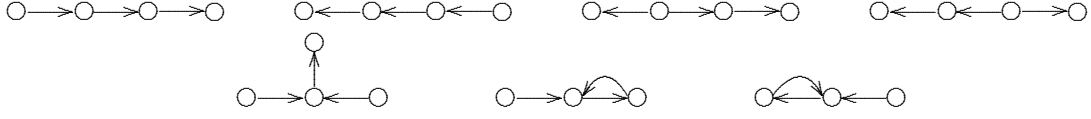


FIGURE 14

Lemma 8. *Consider the arc-set corresponding to $\{u_1, \dots, u_k\}$, $2 \leq k < 2p + 1$. We may assume that this defines a Y -path P' in D . Moreover, the sequence $\{u_1, \dots, u_k\}$ can be obtained by traversing the path P and performing sequentially, each time, one of following three steps:*

- *Take the current arc of P .*
- *Whenever we encounter a node $v \in \hat{P}$ we take the arc in $A' \cup A''$ whose tail is v .*
- *At the end, if v is the last node of P and there is an arc in $A' \cup A''$ whose tail is v , take this arc.*

Proof. The proof is by induction on k . For $k = 2$, the preceding lemma shows that we may assume that the sequence u_1, u_2 corresponds to a directed path with two arcs.

Now we assume that for $\{u_1, \dots, u_k\}$ we have a Y -path with the above properties, and we add u_{k+1} , $k < 2p$. Assume that u_{k+1} corresponds to (v, w) and u_k corresponds to (r, s) . We have the following cases:

Case 1. If $(r, s) \in P$ and s is the last node of P . Notice that we cannot have an arc of $A' \cup A''$ incident to s . We have the following sub-cases:

- $v = s$ and $w \notin V \cup V'$. In this case (v, w) is added to P and w becomes the last node of P .
- $v = s$ and $w = r$. We should have that $w \in \tilde{P}$. Then (v, w) becomes an element of A'' .
- $v = s$ and $w \in V'$. In this case (v, w) is added to A' .
- $w = r$ and $v \notin V \cup V'$. We should have that $r \in \tilde{P}$. In this case (r, s) becomes an element of A' , (v, w) is added to P and v becomes the last node of P .

The sub-cases below are not possible. For each of them we would have that u_{k+1} is adjacent to some node u_i , with $i < k$.

- $v = s$, $w \in V$, $w \neq r$.
- $w = r$ and $v \in V \cup V'$, $v \neq s$.
- $v = r$.

Case 2. If $(r, s) \in P$ and r is the last node of P , then there is no arc in $A' \cup A''$ incident to r . We have the sub-cases below:

- If $w = r$ and $v \notin V \cup V'$, then (v, w) is added to P and v becomes the last node of P .
- If $v = r$ and $w \notin V \cup V'$, then (v, w) is added to P , and w becomes the last node of P .

The following sub-cases are not possible. We would have that u_{k+1} is adjacent to some node u_i , with $i < k$.

- $w = r$ and $v \in V \cup V'$.
- $v = r$ and $w \in V$.

– $v = s$.

It remains to consider the sub-case $v = r$ and $w \in V'$. Consider the next node u_{k+2} and let (p, q) be the corresponding arc. If $p = v$ or $q = v$ we would have that u_{k+2} is adjacent to u_k and u_{k+1} . If $p = w$, since $|\delta_{P'}^-(w)| \geq 2$ we would have that u_{k+2} is adjacent to u_{k+1} and to another node u_i , with $i < k$. So this sub-case is not possible.

Case 3. If $(r, s) \in A'$ and r is the last node of P . In this case $|\delta_{P'}^-(s)| \geq 2$. The following sub-cases are impossible because u_{k+1} would be adjacent to a node u_i , with $i < k$.

– $v = r$.

– $v = s$. Notice that $|\delta_{P'}^-(s)| \geq 2$.

– $w = r$ and $v \in V \cup V'$.

So the only possibility is $w = r$ and $v \notin V \cup V'$. Then (v, w) is added to P , and v becomes the last node of P .

Case 4. If $(r, s) \in A''$ and r is the last node of P . The following sub-cases are impossible for the same reasons as above.

– $v = r$.

– $v = s$.

– $w = r$ and $v \in V \cup V'$.

So we should have $w = r$ and $v \notin V \cup V'$. Then (v, w) is added to P and v becomes the last node of P .

These are all possible cases, so the proof is complete. \square

Lemma 9. *When u_{2p+1} is added we obtain a Y -cycle.*

Proof. Here we show how to modify P to obtain a Y -cycle C . Suppose that u_{2p+1} corresponds to (v, w) , u_1 corresponds to (r, s) , and u_{2p} corresponds to (a, b) . We have two cases:

Case 1. Assume that $w = r$. We have the following sub-cases:

– (a, b) is the last arc of P and b is the last node of P . Then $v = b$, otherwise u_{2p+1} would be adjacent to a third node u_i . Then C is obtained by adding (v, w) to P .

– (a, b) is the last arc of P and a is the last node of P . Then for the same reasons as above we should have $v = a$. Also C is obtained by adding (v, w) to P .

– If $(a, b) \in A'$ then if v coincides with one endnode of (a, b) we would have that u_{2p+1} is adjacent to a third node u_i . Notice that $|\delta_{P'}^-(b)| \geq 2$. Thus this sub-case is not possible.

– If $(a, b) \in A''$ then if v coincides with one endnode of (a, b) we would have that u_{2p+1} is adjacent to a third node u_i . So this sub-case is not possible.

Case 2. Assume that $v = r$. We have four sub-cases:

– (a, b) is the last arc of P and b is the last node of P . Notice that there is no arc in $A' \cup A''$ that is incident to b . Thus $w = a$. We should have that $a \in \tilde{P}$, otherwise u_{2p+1} would be adjacent to a third node u_i . Here C is obtained by removing (a, b) from P and adding (v, w) . The arc (a, b) becomes an arc outside C whose tail is a node in \hat{C} .

– (a, b) is the last arc of P and a is the last node of P . Then $w = a$. The arc (u, v) is added to P to produce C .

– If $(a, b) \in A'$ then $w = a$. Also the arc (u, v) is added to P to produce C .

– If $(a, b) \in A''$ then $w = a$. And (u, v) is added to P to obtain C .

□

At any stage of the procedure that builds the Y -path P' , for any node $v \in \hat{P}$, there is exactly one arc in $A' \cup A''$ whose tail is v . Thus when we obtain the Y -cycle C , there is a set of arcs S with $S \cap A(C) = \emptyset$. Each arc $(v, w) \in S$ is such that $v \in \hat{C}$ and either $w \notin V(C)$ or $w \in \tilde{C}$, and for each node $v \in \hat{C}$ there is exactly one arc in S whose tail is v . Then $|S| = |\hat{C}|$. Since $|A(C)| + |\hat{C}| = |A(C)| + |S| = 2p + 1$, we have that C is a g -odd Y -cycle.

This shows that if D has no g -odd Y -cycle then $L(D)$ does not contain an odd hole. In Sections 5 and 6 we have shown that $L(D)$ does not contain an odd antihole, therefore $L(D)$ is a perfect graph. Thus the proof of Theorem 2 is complete.

8. FINAL REMARKS

We have given a characterization of the graphs for which (2)-(5) defines a polytope with all integral extreme points. In order to recognize this class of graphs, one can first check the existence of a directed cycle with three nodes, if such a cycle does not exist then one can construct the graph $L(D)$ and check whether $L(D)$ is perfect using the algorithm of [9]. In [2] we gave an algorithm for finding a g -odd cycle if there is one, however we have not been able to derive a direct algorithm for finding a g -odd Y -cycle.

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