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On Control of Networks of Dynamical Systems

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***Abstract*—We consider a network of dynamical systems whose trajectories we wish to control by applying stimuli to a subset of systems. We study the minimum number of systems to control and which systems to control and provide sufficient conditions and necessary conditions for successful control. These conditions are given in terms of graph theoretical properties of the underlying network. For instance, we show that for the cycle graph, the best way to achieve control is by applying control to systems that are approximately equally spaced apart.**

I. Introduction

In recent years, there is much research activity to study synchronization in complex networks of nonlinear dynamical systems [1]–[5]. In these studies criteria are derived that ensure all dynamical systems are synchronized to the same behavior. An important area of study is how these criteria are related to the topology of the network [6]–[9]. In this case there is generally no external forcing; the dynamical systems only interact with each other. A related research area is the problem of control in such complex networks [10]–[16], where forcing is applied to a subset of dynamical systems in order to bring the entire network to follow a specific trajectory. In particular, it was shown that by forcing the behavior of a few systems control can be achieved. In this paper we continue this investigation and present new results. In particular, we attempt to answer the following questions; how many systems should be controlled and which systems should be controlled?

II. Notations and definitions

We consider weighted graphs (V, E, W) where each edge $e \in E$ has a positive weight $0 < w_e \in W$. The Laplacian matrix of a graph is defined as a zero row sum matrix L such that $L_{ij} = -w_{ij}$ where w_{ij} is the weight of the edge (i, j) and $L_{ij} = 0$ for all other $i \neq j$. This implies that L_{ii} is the (weighted) degree of vertex i. We will also consider augmented weighted graphs defined as (V, E, W, C) where there is a value $c_i \in C$ associated to each vertex i. Such graphs are depicted in Fig. 1 where we denote an augmented graph (V, E, W, C) by assigning a label w_i to each edge and adding an arrow with label c_i pointing into each vertex i. The Laplacian of an augmented graph is defined as $L' = L + \mathrm{diag}(c_1, c_2, \cdots, c_n)$ where L is the Laplacian matrix of (V, E, W). For a Hermitian matrix A, we ordered its eigenvalues as: $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$. For matrices A and B, we write $A \succeq B$ is $A - B$ is positive semidefinite.

We consider a network of n coupled dynamical systems whose state equations are written in the following form:

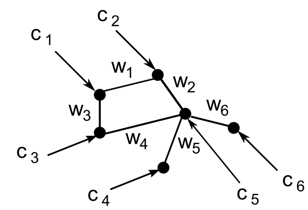

Fig. 1. Augmented weighted graph.

$$\frac{dx_i}{dt} = f(x_i, t) - \sum_j A_{ij} D(t)(x_i - x_j) \tag{1}$$

where x_i is the state vector of the i-th system. The scalar $A_{ij} \geq 0$, $i \neq j$ denotes the coupling coefficient between the i-th and the j-th system. The total number of systems is denoted by n (i.e. $1 \leq i \leq n$). The matrix $D(t)$ describes the linear coupling between two systems which is the same between any pair of systems. By setting $L_{ij} = -A_{ij}$ for $i \neq j$ and $L_{ii} = \sum_j A_{ij}$, this can be rewritten as:

$$\frac{dx_i}{dt} = f(x_i, t) - \sum_j L_{ij} D(t) x_j \tag{2}$$

Note that the matrix $L = \{L_{ij}\}$ is a zero row sum matrix with nonpositive off-diagonal elements. The underlying topology of the network is expressed as the weighted graph such that A and L are its adjacency matrix and Laplacian matrix respectively.

We say the system in Eq. (1) synchronizes (globally) if $\|x_i - x_j\| \to 0$ as $t \to \infty$. Conditions for global and local synchronization have been obtained using a variety of techniques [17]–[21]. In many cases, the synchronization conditions depend on the nonzero eigenvalues of L.

III. Control in Networks of Dynamical Systems

We consider the scenario where in order to control the network in Eq. (2), forcing terms are applied to a subset of systems to drive the entire network to follow a prescribed trajectory. In particular, we consider linear control of the form:

$$\frac{dx_i}{dt} = f(x_i, t) - \sum_j L_{ij} D(t) x_j - c_i D(t)(x_i - u(t)) \tag{3}$$

where $u(t)$ is the desired target trajectory and $c_i > 0$ if control is applied to the i-th system and $c_i = 0$ otherwise. We define P as the set of systems where such control is applied, i.e., $i \in P \Leftrightarrow c_i > 0$. We call P the set of *controlled* systems and denote the number of controlled systems as $p = |P|$. We write $C = \mathrm{diag}(c_1, \ldots, c_n)$.

When a large control signal is applied with $c_i \to \infty$ for $i \in P$, this implies that $x_i \to u(t)$, i.e. the states of the i-th system is forced to approach the trajectory $u(t)$.

Let us assume that $u(t)$ is a trajectory of the individual dynamical system in the network, i.e.

$$\frac{du(t)}{dt} = f(u(t), t) \tag{4}$$

Then Eq. (4) is a virtual system [13] and by setting $x_{n+1}(t) = u(t)$, we obtain a network of $n+1$ dynamical systems with state equations

$$\frac{dx_i}{dt} = f(x_i, t) - \sum_j \tilde{L}_{ij} D(t) x_j \tag{5}$$

where $\tilde{L}$ is related to L as

$$\tilde{L} = \begin{pmatrix} L_{11}+c_1 & L_{12} & & \dots & L_{1n} & -c_1 \\ L_{21} & L_{22}+c_2 & L_{23} & \dots & L_{2n} & -c_2 \\ \vdots & & & & & \vdots \\ & & & \dots & L_{nn}+c_n & -c_n \\ 0 & & & & & 0 \end{pmatrix}$$

and the control problem is reduced to a synchronization problem. Control is achieved in Eq. (3), i.e. every system's state vector x_i follows the trajectory $u(t)$ if the extended system in Eq. (5) synchronizes. We next look at how properties of L and $L+C$ are useful in deriving a criterion for achieving control in Eq. (3).

IV. Criteria for achieving control and graph topology

In [15] the following was shown using the results in [22]:

Theorem 1: If the following conditions are satisfied:

1) L is symmetric,
2) $(y-z)^T V(f(y,t) - f(z,t) - \alpha D(t)(y-x)) \leq -\mu\|y-z\|^2$ for some $\alpha, \mu > 0$, a symmetric positive definite matrix V and all y, z, t.
3) $VD(t)$ is symmetric positive definite for all t, and
4) $\lambda_{\min}(L+C) \geq \alpha$,

then control is achieved, i.e. $x_i(t) \to u(t)$ for all i as $t \to \infty$.

Let us assume that the first 3 conditions in Theorem 1 are satisfied and focus on the last condition: $\lambda_{\min}(L+C) \geq \alpha$. In the rest of this paper, we will use $\lambda_m = \lambda_{\min}(L+C)$ to denote the *effectiveness* of the control and study what properties of L and C contributes to maximizing (or minimizing) λ_m. Note that if L is the Laplacian matrix of the graph (V, E, W), then $L+C$ is the Laplacian matrix of the augmented graph (V, E, W, C).

In particular, we are interested in the following formulation. For a given network of dynamical system (Eq. (3)), where the underlying topology is expressed as a weighted graph with Laplacian matrix L, we choose how many systems to apply a control signal, which systems to apply it to and how large the control gain c_i is. We describe this by specifying the coupling matrix $C = \mathrm{diag}(c_1, \dots c_n)$.

What can be say about the matrix C such that control of the network in Eq.(3) is achieved? Based on the discussion above, we can attack this problem by looking at conditions for C such that $\lambda_{\min}(L+C) \geq \alpha$. Of particular interest is how it depends on the values of p and P.

The following simple Lemma establishes the monotonicity of $\lambda_{\min}$, i.e. adding more edges or more control will not decrease λ_m.

Lemma 1: Consider two augmented graphs $\mathcal{G}_1 = (V, E_1, W, C)$ and $\mathcal{G}_2 = (V, E_2, U, F)$ where $C = (c_1, \dots c_n)$, $F = (f_1, \dots f_n)$, $W = (w_1, \dots w_n)$, and $U = (u_1, \dots u_n)$. If $c_i \leq f_i$ and $w_i \leq u_i$ for all i then $\lambda_{\min}(L_1) \leq \lambda_{\min}(L_2)$ where L_1 and L_2 are the Laplacian matrices of the augmented graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively.

Proof: First note that $L_2' - L_1'$ is symmetric positive semidefinite where L_1' and L_2' are the Laplacian matrices of the weighted graphs (V, E_1, W) and (V, E_2, U) respectively. The result then follows from the fact that L_1' is symmetric and thus by Courant-Fischer theorem $\lambda_{\min}(L_1'+C) = \min_{x\neq 0} \frac{x^T(L_1'+C)x}{x^Tx} \leq \min_{x\neq 0} \frac{x^T(L_2'+F)x}{x^Tx} = \lambda_{\min}(L_2'+F)$. □

Let $c = \min_{c_i \neq 0} c_i$. Consider the case $p = n$. As $C \succeq cI$, this implies that $\lambda_{\min}(L+C) \geq \lambda_{\min}(L+cI) = \lambda_{\min}(L)+c = c$. Therefore λ_m can be made arbitrarily large by choosing c large. This means that if all systems are controlled, control can be achieved by making the control gains $c_i > 0$ large enough.

However, the scenario is very different if $c_i = 0$ for some i, i.e. some systems do not receive any control and $p < n$. As we show below, even if the nonzero c_i are arbitrarily large, $\lambda_{\min}(L+C)$ will still remain bounded. The following Lemma lists lower and upper bounds for $\lambda_{\min}$.

Lemma 2:

$$\frac{\lambda_2(L)}{\left(1+\sqrt{1+\frac{\lambda_2(L)}{\sum_i c_i}}\right)^2 n + 1} \leq \lambda_{\min}(L+C) \leq \frac{\sum_i c_i}{n} \tag{6}$$

If $p < n$, then

$$\lambda_{\min}(L+C) \leq \lambda_{p+1}(L) \tag{7}$$

Proof: The proof of Eq. (6) can be found in [23]. Eq. (7) is a consequence of Weyl's eigenvalue interlacing inequality [24] and the fact that $\lambda_i(C) = 0$ for $i > p$. □

Eq. (7) suggests that if some system did not receive any control ($p < n$), then control may not be possible if the eigenvalues of L are small, even if the nonzero control gain coefficients c_i are arbitrarily large. This gives us some information on how many systems forcing need to applied to. For instance, if $\lambda_{\min}(L+C)$ needs to be larger than a value γ in order to achieve control, where $\gamma \geq \lambda_{p+1}(L)$, then it is necessary to apply control to at least p systems.

Definition 1: The *isoperimetric ratio* $r(V')$ of a subset of vertices $V' \subset V$ is defined as the number of edges between V' and $V \backslash V'$ divided by the number of vertices in V'.

Theorem 2: If $\emptyset \neq V' \subset V \backslash P$, then $\lambda_{\min}(L+C) \leq r(V')$.

Proof: Let v be a vector with $v_i = 1$ for $i \in V'$ and $v_i = 0$ otherwise. Then $\lambda_{\min}(L+C) \leq \frac{v^T(L+C)v}{v^Tv} = r(V')$. □

Corollary 1: If $p < n$, then $\lambda_{\min}(L+C) \leq p$.

Proof: Let $V' = V \backslash P$. Then $|V'| = n-p$ and the number of edges between V' and $V \backslash V'$ is at most $p(n-p)$. $\square$

We will show in Section V that this bound is achieved for the complete graph, when $c \to \infty$.

Corollary 2: If $p = n-1$, i.e. only one c_i is equal to zero, then $\lambda_{\min}(L+C) \leq \delta$, where δ is the degree of the vertex i such that $c_i = 0$.

Corollary 3: If $p < n$, then $\lambda_{\min}(L+C) \leq \delta_{\max}$ where $\delta_{\max}$ is the maximal vertex degree.

The first inequality of Lemma 2 relates $\lambda_{\min}(L+C)$ to the algebraic connectivity λ_2 of the underlying graph with Laplacian matrix L. The next result relates $\lambda_{\min}$ of the Laplacian matrix of an augmented weighted graph to the algebraic connectivity λ_2 of a related weighted graph. Consider an augmented graph $\mathcal{G}$. Construct a graph $\mathcal{H}$ by taking two copies of $\mathcal{G}$ minus the augmented edges C and add a new vertex v_0. For each $i \in P$, add an edge of weight c_i from v_0 to vertex v_i of each copy of $\mathcal{G}$. This is shown schematically in Fig. 2.

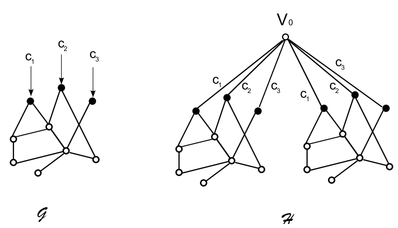

Fig. 2. The weighted graph $\mathcal{H}$ is generated from the augmented weighted graph $\mathcal{G}$ by connecting two copies of $\mathcal{G}$ via an additional vertex v_0.

Theorem 3:

$$\lambda_{\min}(L(\mathcal{G})) \geq \lambda_2(L(\mathcal{H}))$$

Proof: Let v be the unit norm vector that minimizes $L(\mathcal{G})$, i.e $v^T L(\mathcal{G}) v = \min_{x \neq 0} \frac{x^T L(\mathcal{G}) x}{x^T x} = \lambda_{\min}(L(\mathcal{G}))$. Let $w^T = (v^T, 0, -v^T)$, where v and $-v$ corresponds to the two copies of $\mathcal{G}$ and 0 corresponds to vertex v_0. Since $\sum_i w_i = 0$ and $w^T w = 2$, the Courant-Fischer theorem shows that $w^T L(\mathcal{H}) w \geq 2\lambda_2(L(\mathcal{H}))$. It is easy to see that $w^T L(\mathcal{H}) w = 2v^T L(\mathcal{G}) v$ and the conclusion follows. $\square$

Corollary 4: For an augmented graph $\mathcal{G}$ with n vertices and $P \neq \emptyset$,

$$\lambda_{\min}(L(\mathcal{G})) = \lambda_{\min}(L+C) \geq 2c_m \left(1 - \cos\left(\frac{\pi}{2n+1}\right)\right)$$

where $c_m = \min_{i \in P}\{c_i, 1\}$.

Proof: We only need to look at the case $c_i = 1$ for all $i \in P$, as the other cases are similar. In this case $\mathcal{H}$ is a graph with $2n+1$ vertices. In [25] it was shown that for a graph with n vertices, $\lambda_2(L) \geq 2\left(1 - \cos\left(\frac{\pi}{n}\right)\right)$ and the result follows. $\square$

The bound in Corollary 4 is tight. For instance, for the path graph P_n with $c_1 = 1$, $c_i = 0$ for $i > 1$, we have $\lambda_{\min}(L+C) = 2\left(1 - \cos\left(\frac{\pi}{2n+1}\right)\right)$ (see e.g. [26], [27]).

For the complete graph with Laplacian matrix L, Lemma 2 shows[1] that if $\sum_i c_i$ is bounded for all n, then $\lambda_{\min}(L+C) \to \frac{\sum_i c_i}{n}$ as $n \to \infty$.

[1]See also [23].

V. Localization of control as $c \to \infty$

For a fixed $1 \leq p < n$, we study how the choice of P, i.e. the set of vertices for which c_i is nonzero, affects $\lambda_{\min}(L+C)$. In particular, we are interested in the configuration that maximizes or minimizes $\lambda_{\min}(L+C)$. In [16] some preliminary studies were done to determine these configurations and it was shown that in many cases a configuration that minimizing the (average) distance from P to $V \backslash P$ also maximizes $\lambda_{\min}(L+C)$. In particular, for $p = 1$ and $c = 10$, it was shown that this is true for all graphs with 6 or less vertices, with counterexamples in graphs with 7 vertices.

In this section we study the scenario as $c \to \infty$ since it allows us to explicitly find configurations P which maximizes or minimizes $\lambda_{\min}(L+C)$ for certain classes of graphs.

For a fixed set of indices P, let $C(P,c)$ be the diagonal matrix such that $c_i = c$ for $i \in P$ and $c_i = 0$ otherwise. Define $\kappa(P) = \lim_{c \to \infty} \lambda_{\min}(L + C(P,c))$, $\eta_{\max}(p) = \sup_{|P|=p} \kappa(P)$ and $\eta_{\min}(p) = \inf_{|P|=p} \kappa(P)$.

Theorem 4: If $P \neq \emptyset$, then $\kappa(P) \geq \eta_{\min}(P) \geq \frac{\lambda_2(L)}{4n+1}$.

Proof: Follows from Eq. (6). $\square$

Lemma 3: Let L' be the principal submatrix corresponding to the indices $V \backslash P$. Then $\kappa(P) = \lambda_{\min}(L')$.

Proof: Let v be a unit eigenvector of $L + C(P,c)$ corresponding to $\lambda_{\min}(L + C(P,c))$. It is clear that for $i \in P$, v_i vanishes as $c \to \infty$. Let w be the subvector of v restricted to $V \backslash P$. Then $w^T(L + C(P,c))w = w^T L' w$ and this also minimizes $w^T L' w$ among all unit vectors w and thus is equal to $\lambda_{\min}(L')$. $\square$

A. Optimal configurations

1) Cycle graphs:

Theorem 5: For a cycle graph of n vertices and $p < n$, $\eta_{\max}(p) = 2 - 2\cos\left(\frac{\pi}{\lceil \frac{n}{p} \rceil}\right)$ and $\eta_{\min}(p) = 2 - 2\cos\left(\frac{\pi}{n-p+1}\right)$.

Proof: We show that the configuration P which attains $\eta_{\max}$ and $\eta_{\min}$ is the configuration which spreads out the most and the least respectively. It is clear that L' is block diagonal with the block submatrices of the form:

$$\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{pmatrix} \tag{8}$$

This is a Toeplitz matrix whose smallest eigenvalue is equal to $2 - 2\cos\left(\frac{\pi}{m+1}\right)$ where m is the order of the matrix. Thus the largest (smallest) value for $\kappa(P)$ is achieved when these submatrices are as small (large) as possible. This implies that to maximize $\kappa(P)$, P should be as dispersed as possible in order to "cut" the cycle graph into as many small pieces as possible. Since $|P| = p$, it will cut the graph into p pieces. If P is placed as evenly around the cycle as possible, then the largest piece is of length $\lceil \frac{n-p}{p} \rceil = \lceil \frac{n}{p} \rceil - 1$. The corresponding Toeplitz matrix has its smallest eigenvalue equal to $\eta_{\max}(p) = 2 - 2\cos\left(\frac{\pi}{\lceil \frac{n}{p} \rceil}\right)$. The submatrix is the largest possible if all elements of P are adjacent on the cycle graph, in which case the submatrix is of order $n-p$. $\square$

2) Path graphs:

Theorem 6: For a path graph of n vertices, $\eta_{\max}(p) = 2 - 2\cos\left(\frac{\pi}{\lceil \frac{n}{p} \rceil}\right)$ and $\eta_{\min}(p) = 2 - 2\cos\left(\frac{\pi}{2(n-p)+1}\right)$ if $p < n$.

Proof: The proof is similar to that of Theorem 5, except that in this case L' is block diagonal with blocks of the form Eq. (8) and of the form (perhaps after a simultaneous row and column permutation)

$$\begin{pmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{pmatrix} \tag{9}$$

Note that for P being internal vertices of the path graph (i.e. those vertices with degree 2), this splits L' into $p+1$ blocks with $p-1$ blocks of the form Eq. (8) and 2 blocks of the form Eq. (9). For matrices of the form Eq. (9) the smallest eigenvalue is equal to $2 - 2\cos\left(\frac{\pi}{2m+1}\right)$ (see [26], [27]), i.e. the same as a matrix of the form Eq. (8) of order $2m$. Thus the optimal splitting into submatrices is such that the blocks of the form Eq. (9) are about half the size as the blocks of the form Eq. (8). This means that η_{max} is obtained for a configuration that splits it into $p-1$ blocks of the form Eq. (8) of order $\approx \frac{n-p}{p}$ and 2 blocks of the form Eq. (9) of order $\approx \frac{n-p}{2p}$. As for $\eta_{\min}$ the biggest block is created when all vertices of P is on one side of the graph, resulting in a single block of the form Eq. (9) of size $n-p$. □

It is interesting to note that $\eta_{\max}(p)$ is the same for path graphs and cycle graphs. Computer experiments show that $\eta_{\max}(1)$ is attained for a configuration such that the vertex in P minimizes the distance to $V \backslash P$ [2] for all graphs of 7 vertices or less. For graphs with 8 vertices, there is a graph where the $\eta_{\max}$-maximizing configuration P is not in the graph center.

3) Complete graphs: It is clear that for the complete graph, only the cardinality p of the set P and not the set of P itself affects $\lambda_{\min}(L+C)$. It is easy to see that the principal submatrix corresponding to $V \backslash P$ is $L^K_{n-p} + pI$ where L^K_{n-p} is the Laplacian matrix of the complete graph of $n-p$ vertices and thus $\eta_{\max}(p) = \eta_{\min}(p) = \kappa(P) = \lambda_{min}(L^K_{n-p} + pI) = p$ for $p < n$.

VI. Future research

Consider the problem where the goal is to find the matrix C such that $\lambda_{\min}(L+C)$ is maximized under the constraint that $\sum_i c_i = B$ for some constant B. First note that $\lambda_{\min}(L+C) \leq \frac{B}{n}$ by Lemma 2. If p is not fixed, then the answer is clear: set $p = n$ and $c_i = \frac{B}{n}$ for all i. In this case $\lambda_{\min}(L+C) = \lambda_{\min}(L) + \frac{B}{n} = \frac{B}{n}$. An interesting question is to determine how to allocate and assign c_i when the set P is fixed (with $|P| < n$) in order to maximize (or minimize) $\lambda_{\min}(L+C)$.

VII. Acknowledgements

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[2] i.e. the $\eta_{\max}$-maximizing configuration $P = \{i\}$ is a subset of the graph center.

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