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On Explicit Substitution with Names

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Abstract This paper recounts the origins of the λx family of calculi of explicit substitution with proper variable names, including the original result of preservation of strong β-normalization based on the use of special reductions for garbage collection. We then discuss the properties of a variant of the calculus which is also confluent for "open" terms (with meta-variables), and verify that a version with garbage collection preserves strong β-normalization (as is the state of the art). Finally, we summarize the relationship with other efforts on using names and garbage collection rules in explicit substitution.

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1 Introduction

The λ calculus (Church 1941) permits substitution of all occurrences of a bound variable with a copy of a term as an atomic operation, however, it was always clear that, if used as a programming language construct, it should be assigned an operational complexity related to the size of the term in which the substitution is performed. Curry and Feys (1958) claimed that the reduction step count of *combinatory logic* would be a more faithful complexity measure for substitution, however, the resulting measure seems to overshoot the reality of the amount of work needed. The first realistic measure of the amount of work involved in substitution was offered by the SECD machine of Landin (1964), however, such abstract machines use an environment component and thus sacrifice an important property of the λ calculus, namely the ability to reason about the equivalence of terms because subterms that occur inside abstractions cannot be reduced. The solution of Curien (1991) was to encode the notion of environment as a special term of the calculus; this was then generalized to the *explicit substitution* calculus $\lambda\sigma$ by Abadi, Cardelli, Curien, and Lévy (1991). Since this development started with abstract machines, which use indices into a stack to represent the values that are bound to variables, it was natural for the initial explicit substitution calculi to use de Bruijn (1972) indices in a similar way, which in finitely described terms leads to the use of sequences like $(\uparrow\circ\ldots\circ\uparrow\circ\mathrm{id})$ to represent variable references.

However, it soon turned out that the $\lambda\sigma$ calculus did not get it quite right: it does not permit reasoning about open terms by being confluent on terms with meta-variables (Curien et al 1996), and the substitution rules in some cases allow infinite reduction even of terms that are strongly normalizing in the original λ calculus (Melliès 1995). The effect was an opening of Pandora's box resulting in a flurry of calculi, each with slightly different properties than $\lambda\sigma$; Kesner (2009) gives a nice account of the multitude of explicit substitution calculi that resulted.

The approach proposed with λx of Bloo and Rose (1995), the subject of this paper, was to keep the generic "strategy free" rewriting principle of λ-calculus while retaining named variables using Barendregt's (1984) *variable convention* to ensure that rewriting does not create naming problems, and to augment the calculus with explicit *garbage collection* to manage the possibilities for run-away reductions. The resulting calculus is somewhat simpler and notationally much closer to the original λ calculus, making it easy to understand how λx can be used to reason about λ calculus reduction. And, indeed, the rôle of λx has been to be a basis for understanding the specifics of the formal properties of explicit substitution with minimal distraction from the independent challenge of an explicit representation of variable binding.

After introducing appropriate rewriting preliminaries in Sec. 2, we present the original basic definition and key properties of λx in Sec. 3. In Sec. 4 we add *garbage collection* and explain how it allows an understanding of the requirements to ensure that the explicit substitution calculus is strongly normalizing for all terms that are strongly normalizing for the simulated λ calculus. In Sec. 5 we then discuss λxcw, an extension of λx designed to be confluent also for open terms by combining meta-terms with a weakening operator and a corresponding suitably constrained substitution composition rule. In Sec. 6 we discuss preservation of strong normalization for λxcw, proving that it holds with aggressive garbage collection. Finally, we conclude with an account of the provenance of λx and pointers to recent related work on garbage collection rules and variables in Sec. 7.

2 Preliminaries

Here we briefly summarize the specific formal details we shall use of rewriting (*e.g.*, Dershowitz and Jouannaud 1990; Klop 1992; Baader and Nipkow 1999; Bezem et al 2003) and λ calculus (Church 1941; Barendregt 1984) in our definitions. The only non-standard notation is "$\twoheadrightarrow\!\!|$" for evaluation to normal form.

Definition 2.1 A *rewriting relation* R over a set S is a subset of $S \times S$. Define

- $s \xrightarrow[R]{} s'$ (and $s' \xleftarrow[R]{} s$, *etc.*) means $(s,s') \in R$, $s \xrightarrow[R]{} s' \xrightarrow[R]{} s''$ is short for $s \xrightarrow[R]{} s'$ and $s' \xrightarrow[R]{} s''$, and a "$\cdot$" should be understood as an anonymous (existentially quantified) value, so $\xrightarrow[R]{} \cdot \xrightarrow[Q]{}$ denotes the *composition* of the two: $\{\,(s,s') \mid \exists s''\colon\ s \xrightarrow[R]{} s'' \xrightarrow[Q]{} s'\,\}$;
- R is *confluent* if $(\twoheadleftarrow_R \cdot \twoheadrightarrow_R) \subseteq (\twoheadrightarrow_R \cdot \twoheadleftarrow_R)$, *weakly confluent* if $(\xleftarrow[R]{} \cdot \xrightarrow[R]{}) \subseteq (\twoheadrightarrow_R \cdot \twoheadleftarrow_R)$, and *strongly confluent* if $(\xleftarrow[R]{} \cdot \xrightarrow[R]{}) \subseteq (\xrightarrow[R]{} \cdot \xleftarrow[R]{?})$,
- R is *strongly normalizing* if there are no infinite rewrites $s \xrightarrow[R]{} s' \xrightarrow[R]{} \cdots$ (and we then say that s is strongly R-normalizing); it is *convergent* if it is also confluent;
- $\xrightarrow[R]{}|_{S'}$ is the *restriction* of R to S': $R \cap (S' \times S')$;
- $\xrightarrow[R]{0}$, or $\equiv|_S$, is the identity relation on S, $\xrightarrow[R]{n}$ is the *n-composition* of R: $\xrightarrow[R]{n+1} = \xrightarrow[R]{n} \cdot \xrightarrow[R]{}$, $\xrightarrow[R]{+}$ is the *transitive closure* of R: $\bigcup_{n\geq 1}(\xrightarrow[R]{n})$, $\xrightarrow[R]{?}$ is the *reflexive closure* of R: $\xrightarrow[R]{0} \cup \xrightarrow[R]{}$, and $\twoheadrightarrow_R$ is the *transitive and reflexive closure* of R: $\xrightarrow[R]{?} \cup \xrightarrow[R]{+}$;
- $\mathrm{nf}(R)$ are the *normal forms* of R: $\{\,s \mid \nexists s'\colon\ s \xrightarrow[R]{} s'\,\}$; $\rightarrow\!\!|_R$ is the *restriction to terminating reductions* of R: $\xrightarrow[R]{} \cap (S \times \mathrm{nf}(R))$; and, when R is convergent, $\downarrow_R(s) = s'$ where $s \twoheadrightarrow\!\!|_R\, s'$.

When obvious from the context, we will just use $\longrightarrow$ or confuse R and $\xrightarrow[R]{}$. We use $=$ to compare sets and relations and $\equiv$ to compare terms.

Rewriting is rich with generic abstract theorems that we shall only mention when they are needed in proofs, except the following.

Lemma 2.2 (Yokouchi and Hikita) *Let $\xrightarrow[R]{}, \xrightarrow[S]{}, \xrightarrow[T]{}$ be three relations on the same set. If the six following conditions hold, then $\xrightarrow[T]{}$ is confluent.*[1]

YH1: $\xrightarrow[R]{}$ *is strongly normalizing,*
YH2: $\xrightarrow[R]{}$ *is weakly confluent,*
YH3: $\xrightarrow[S]{}$ *is strongly confluent,*
YH4: $(\xleftarrow[R]{} \cdot \xrightarrow[S]{}) \subseteq (\twoheadrightarrow_R \cdot \xrightarrow[S]{} \cdot \twoheadrightarrow_R \cdot \twoheadleftarrow_R)$,
YH5: $\xrightarrow[T]{} \subseteq (\twoheadrightarrow_R \cdot \xrightarrow[S]{} \cdot \twoheadrightarrow_R)$.
YH6: $(\twoheadrightarrow_R \cdot \xrightarrow[S]{} \cdot \twoheadrightarrow_R) \subseteq \twoheadrightarrow_T$.

Proof Yokouchi and Hikita (1990). □

Definition 2.3 When the set S is a set of *terms* formed by a notion of *context* where from a collection of terms $\vec{t}$ new terms $C[\vec{t}]$ can be constructed in a number of ways, then rewriting becomes *term rewriting* and we define the *compatible closure* of R over S as the relation satisfying that $s \xrightarrow[R]{} s'$ implies $C[s] \xrightarrow[R]{} C[s']$.

[1] In fact, this is Yokouchi and Hikita coupled with Hindley Rosen (fifth condition), since the original lemma was to show that $\twoheadrightarrow_R \cdot \xrightarrow[S]{} \cdot \twoheadrightarrow_R$ is confluent.

Definition 2.4 Assume the function symbols $f, g \in \mathscr{F}$ of a term rewrite system are ordered by the partial order $\triangleright$. Then the *lexicographic path ordering* $\succ$ *induced by* $\triangleright$ on terms over $\mathscr{F}$ is defined by $f(s_1, \ldots, s_m) \succ g(t_1, \ldots, t_n)$ if either of the following three conditions hold (assuming appropriate conditions to ensure that all indexings are well defined):

"Sub-term:" For some i, either $s_i = g(t_1, \ldots, t_n)$ or $s_i \succ g(t_1, \ldots, t_n)$,
"Decreasing heads:" $f \triangleright g$ and for all i, $f(s_1, \ldots, s_m) \succ t_i$, or
"Equal heads:" $f = g$ and for all i, $f(s_1, \ldots, s_m) \succ t_i$ and there exists some $k < m$ such that for all $j \leq k$, $s_j = t_j$, and $s_{k+1} \succ t_{k+1}$.

(The last condition is what makes this a lexicographic rather than just a recursive path ordering.)

Theorem 2.5 *Let $\succ$ be the lexicographic path ordering induced by $\triangleright$. Then $\succ$ is strongly normalizing if and only if $\triangleright$ is strongly normalizing.*

Proof Dershowitz (1987). □

Definition 2.6 The λ *terms*, denoted Λ, are the terms formed by

$$M, N, P, Q ::= x \mid \lambda x.M \mid MN \tag{Λ}$$

where the productions are called (variable) occurrence, abstraction, and application, respectively, as listed in order of increasing precedence with the convention that application is left recursive, so $\lambda x.MNP$ is the same as $\lambda x.((MN)P)$ and we use ()s to disambiguate as needed. In addition:

- The variable x of $\lambda x.M$ is set to be *bound* by the abstraction, and all occurrences of x in M are *bound variables* (each bound by the nearest encapsulating matching abstraction).
- λ-term identity is modulo *renaming* of bound variables: the variable bound by an abstraction can be renamed along with all its occurrences as long as it does not change which abstractions each variable is bound by.
- Variables that are not bound are said to be *free*, and we collect all free variables of a term M with the construct $\mathrm{fv}(M)$ defined inductively over terms by $\mathrm{fv}(x) \equiv \{x\}$, $\mathrm{fv}(\lambda x.M) \equiv \mathrm{fv}(M) \setminus \{x\}$, and $\mathrm{fv}(MN) \equiv \mathrm{fv}(M) \cup \mathrm{fv}(N)$.
- The *context bound* variables are defined for every context as $\mathrm{bv}([]) = \varnothing$, $\mathrm{bv}(\lambda x.C[]) = \{x\} \cup \mathrm{bv}(C[])$, and $\mathrm{bv}(C[]\,N) = \mathrm{bv}(M\,C[]) = \mathrm{bv}(C[])$.
- The *variable convention* (Barendregt 1984, 2.1.13) stipulates that when a group of terms occur in a mathematical context (like a definition, proof, *etc.*), then all variables bound in the group of terms must be distinct from each other as well as and different from all free variables (this is always achievable by renaming bound variables).

The λ *calculus* equips the λ terms with the rewriting relation $\underset{\beta}{\rightarrow}$ defined as the compatible closure of

$$(\lambda x.M)N \longrightarrow M[x := N] \tag{β}$$

where "$M[x := N]$" is meta-notation for the *(meta-)substitution* of all bound occurrences of x in M with N, defined by $x[y := N] \equiv x$ (with $x \neq y$), $y[y := N] \equiv N$, $(\lambda x.M)[y := N] \equiv \lambda x.M[y := N]$ (with $x \neq y$), and $(M_1 M_2)[y := N] \equiv M_1[y := N]\, M_2[y := N]$.

Theorem 2.7 $\underset{\beta}{\rightarrow}$ *is confluent.*

Proof Church and Rosser (1935). □

Lemma 2.8 (substitution lemma) $M[x := N][y := P] \equiv M[y := P][x := N[y := P]]$.

Proof Barendregt (1984, 2.1.16). □

3 Plain Names

If the task is to make substitution explicit rather than mimic enviroment machines, then the simplest and most immediate solution is to just introduce syntax for substitution and then reinterpret the classic equations for substitution as syntactic rewrites instead of meta-level equations. In this section we summarize how λx formalizes this and thus manages to stay very close to the classical definition of the λ calculus yet still avoid implicit unbounded complexity, and we show that λx has a terminating substitution calculus, is confluent (on ground terms), and simulates $\xrightarrow[\beta]{}$. We comment on the history of such notations in the discussion (Sec. 7.1).

Definition 3.1 (λx calculus) The λx *terms*, Λx, extend the λ terms to

$$M,N,P,Q ::= x \mid \lambda x.M \mid MN \mid M\langle x := N\rangle \tag{Λx}$$

where

- the last case is called an *(explicit) substitution* and is assigned highest precedence, so $\lambda x.MNP\langle y := Q\rangle$ is the same as $\lambda x.((MN)(P\langle y := Q\rangle))$;
- substitution $M\langle x := N\rangle$ binds x subject to the same constraints as $(\lambda x.M)N$ and extends the definition of *free variables* with $\mathrm{fv}(M\langle x := N\rangle) = (\mathrm{fv}(M) \setminus \{x\}) \cup \mathrm{fv}(N)$ and *context bound variables* with $\mathrm{bv}(C[\,]\langle x := M\rangle) = \{x\} \cup \mathrm{bv}(C[\,])$ and $\mathrm{bv}(M\langle x := C[\,]\rangle) = \mathrm{bv}(C[\,])$;
- meta-substitution is extended: $M\langle x := N\rangle[y := P] \equiv M[y := P]\langle x := N[y := P]\rangle$ $(x \neq y)$.
- terms with no substitutions (so in Λ) are called *pure* terms.

The associated notion of *reduction*, $\xrightarrow[bx]{}$, is defined as the compatible closure over Λx of the rewrite rules

$$(\lambda x.M)N \longrightarrow M\langle x := N\rangle \tag{b}$$

$$y\langle y := P\rangle \longrightarrow P \tag{xv}$$

$$x\langle y := P\rangle \longrightarrow x \qquad x \neq y \tag{xvgc}$$

$$(\lambda x.M)\langle y := P\rangle \longrightarrow \lambda x.M\langle y := P\rangle \tag{xab}$$

$$(MN)\langle y := P\rangle \longrightarrow M\langle y := P\rangle\, N\langle y := P\rangle \tag{xap}$$

with $\xrightarrow[\mathrm{b}]{}$ denoting the compatible closure obtained from just including the rule (b) and $\xrightarrow[\mathrm{x}]{}$ the closure of using just the *explicit substitution* rules (xv,xvgc,xab,xap).

Example 3.2 To illustrate the granularity of explicit substitution with λx, consider the λ term $(\lambda y.(\lambda z.z)y)x$. It permits the β-reductions shown in this small graph (with the redexes indicated):

$$\underline{(\lambda y.\,\underline{(\lambda z.z)y}\,)x} \;\overset{\beta}{\underset{\beta}{\rightrightarrows}}\; \underline{(\lambda z.z)x} \xrightarrow[\beta]{} x$$

In contrast, λx permits the reductions shown in Fig. 1, where we have framed the pure λ terms where they occur. Notice these points:

- The unique normal form is the same.
- Not all possible reductions to the normal form pass through the intermediate pure $(\lambda z.z)x$.
- After one b reduction step from a pure term one can always follow x (explicit substitution) reduction steps to a pure term.

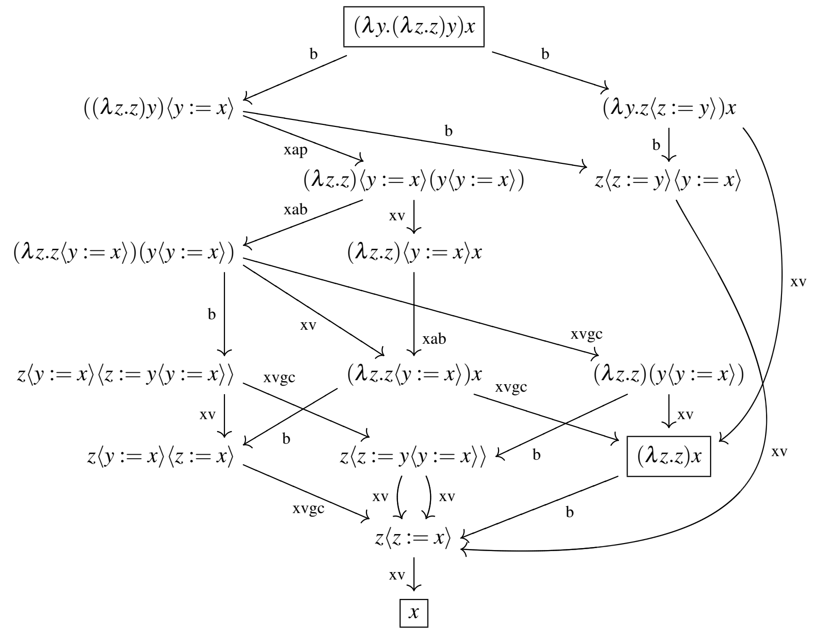


Fig. 1 λx reduction graph for $(\lambda y.(\lambda z.z)y)x$.

- Once both explicit substitutions have been introduced then the innermost must be eliminated first.

The calculus intentionally looks conspicuously like the λ calculus, and indeed it is easy to show the following basic results and correspondences (details in Rose 1992; Bloo and Rose 1995; Rose 1996b; Bloo 1997).

Proposition 3.3 $\underset{\mathrm{x}}{\rightarrow}$ *is strongly normalizing and properly eliminates all explicit substitutions:* $\downarrow_{\mathrm{x}}(M\langle x := N\rangle) \equiv \downarrow_{\mathrm{x}}(M)[x := \downarrow_{\mathrm{x}}(N)]$ *and* $\mathrm{nf(x)} = \Lambda$.

Proof Strong normalization is demonstrated with a map h from terms to integers such that for all $M \underset{\mathrm{x}}{\rightarrow} N$: $h(M) > h(N)$. One such is the inductively defined $h(x) = 1$, $h(MN) = h(M) + h(N) + 1$, $h(\lambda x.M) = h(M) + 1$, and $h(M\langle x := N\rangle) = h(M) \times (h(N) + 1)$. An easy induction over terms along with the observation that any λ-term can be reached shows that all explicit substitutions can be performed (on ground terms). □

Proposition 3.4 bx*-reduction simulates* β*-reduction steps and is complete for* β*-reduction:* $\underset{\beta}{\rightarrow} = (\underset{\mathrm{b}}{\rightarrow} \cdot \underset{\mathrm{x}}{\twoheadrightarrow})|_{\Lambda}$ *and* $\underset{\beta}{\twoheadrightarrow} = \underset{\mathrm{bx}}{\twoheadrightarrow}|_{\Lambda}$.

Proof Simulation follows from induction over terms involving the similar shape of implicit and explicit substitution in Definitions 2.6 and 3.1. Completeness follows by induction over the projection of rewrite sequences in both systems onto pure terms. □

Proposition 3.5 $\underset{\mathrm{bx}}{\longrightarrow}$ *is confluent on* Λx.

Proof By interpretation (Hardin 1989) into β-reduction using the previous results. □

Let a,b,y,y' be distinct variables. Define substitutions

$$S_0 \equiv \langle y := (\lambda y.a)b\rangle\ , \quad S_{n+1} \equiv \langle y := bS_n\rangle$$

and consider the following derivations (for simplicity we offend the variable convention but this is easily repaired):

$$\begin{aligned}
&(\lambda y.(\lambda y'.a)((\lambda y.a)b))\,((\lambda y.a)b)\\
&\quad \twoheadrightarrow_{\text{bx}} a\langle y' := (\lambda y.a)b\rangle\,\langle y := (\lambda y.a)b\rangle\\
&\quad \rightarrow_{\text{ss}} a\langle y' := ((\lambda y.a)b)\,\langle y := (\lambda y.a)b\rangle\rangle\\
&\quad \twoheadrightarrow_{\text{x}} a\langle y' := (\lambda y.a\langle y := (\lambda y.a)b\rangle)\,(b\langle y := (\lambda y.a)b\rangle)\rangle \equiv a\langle y' := (\lambda y.aS_0)\,(bS_0)\rangle\\
&\quad \rightarrow_{\text{b}} a\langle y' := aS_0\langle y := bS_0\rangle\rangle \qquad\qquad \equiv a\langle y' := aS_0S_1\rangle\ ,
\end{aligned}$$

$$\begin{aligned}
aS_0S_{m+1} &\equiv a\langle y := (\lambda y.a)b\rangle\,\langle y := bS_m\rangle \equiv a\langle y' := (\lambda y.a)b\rangle\,\langle y := bS_m\rangle\\
&\rightarrow_{\text{ss}} a\langle y' := ((\lambda y.a)b)\,\langle y := bS_m\rangle\rangle\\
&\twoheadrightarrow_{\text{x}} a\langle y' := (\lambda y.a\langle y := bS_m\rangle)\,(b\langle y := bS_m\rangle)\rangle \equiv a\langle y' := (\lambda y.aS_{m+1})\,(bS_{m+1})\rangle\\
&\rightarrow_{\text{b}} a\langle y' := aS_{m+1}\,\langle y := bS_{m+1}\rangle\rangle \qquad\qquad \equiv a\langle y := aS_{m+1}S_{m+2}\rangle\ ,
\end{aligned}$$

$$\begin{aligned}
aS_{m+1}S_{n+1} &\equiv a\langle y' := bS_m\rangle\,\langle y := bS_n\rangle\\
&\rightarrow_{\text{ss}} a\langle y' := bS_m\langle y := bS_n\rangle\rangle \qquad\qquad \equiv a\langle y' := bS_mS_{n+1}\rangle
\end{aligned}$$

which combine into an infinite derivation in the following schematic way:

$$\begin{aligned}
(\lambda y.(\lambda y'.a)((\lambda y.a)b))\,((\lambda y.a)b) &\twoheadrightarrow \ldots S_0S_1 \ldots \twoheadrightarrow \ldots S_1S_2 \ldots\\
&\twoheadrightarrow \ldots S_0S_2 \ldots \twoheadrightarrow \ldots S_2S_3 \ldots \twoheadrightarrow \ldots S_1S_3 \ldots\\
&\twoheadrightarrow \ldots S_0S_3 \ldots \twoheadrightarrow \ldots S_3S_4 \ldots \twoheadrightarrow \cdots\\
&\twoheadrightarrow \ldots
\end{aligned}$$

Fig. 2 Example infinite reduction of strongly normalizing term.

4 Preservation of Strong Normalization

Our goal in this section is to prove that $\xrightarrow[\text{bx}]{}$ preserves strong normalization of $\xrightarrow[\beta]{}$ (found as Corollary 4.8 below). We shall first study Melliès's (1995) counter-example as it would look for a slightly enhanced λx. It turns out that if λx is extended with the innocently looking rule

$$M\langle x := N\rangle\langle y := P\rangle \longrightarrow M\langle x := N\langle y := P\rangle\rangle \qquad\qquad y \notin \text{fv}(M)$$

(which still gives a terminating substitution calculus), then a variation of Melliès's counterexample (Bloo 1997; Bloo and Geuvers 1999), summarized in Fig. 2, can be used to demonstrate that the strongly normalizing λ term $(\lambda y.(\lambda y'.a)((\lambda y.a)b))\,((\lambda y.a)b)$ has infinite reductions. The key observation turns out to be that the infinite reduction involves *reducing garbage*, *i.e.*, using $\xrightarrow[\text{b}]{}$-reduction in subterms of substitutions that will never, in fact, be used, because their bound variable does not occur in the original λ-abstraction content. This suggests (Bloo and Rose 1995) that preservation of β-normalization (β-PSN) can be shown for λx by staging the proof in two steps:

1. Prove β-PSN for λx↓gc, a version of λx that does aggressive garbage collection.
2. Show that λx always keeps fragments copied into garbage substitutions strongly normalizing.

Below we generalize this to any so-called "garbage safe" calculus over Λx by first defining what we mean by garbage and then formalizing the necessary constraints that ensure β-PSN even with garbage reductions.

Definition 4.1 Define the *garbage collection* relation $\xrightarrow[\mathrm{gc}]{}$ as the compatible closure over Λx of

$$M\langle y := N\rangle \longrightarrow M \qquad y \notin \mathrm{fv}(M) \tag{gc}$$

Define, for every relation $\mathscr{R}$ on Λx, the following *garbage free $\mathscr{R}$-reduction* relation

$$\xrightarrow[\mathscr{R}\downarrow\mathrm{gc}]{} = \xrightarrow[\mathscr{R}]{} \cdot \xrightarrow[\mathrm{gc}]{}\!\!\!\!\twoheadrightarrow \tag{$\mathscr{R}\downarrow$gc}$$

We say that a reduction $M \longrightarrow N$ is *in garbage* (*outside garbage*) if $\downarrow_{\mathrm{gc}}(M) \equiv \downarrow_{\mathrm{gc}}(N)$ holds (does not not hold, respectively).

The key constraint is to ensure that no reduction creates a non-trivial term in garbage that permits run-away garbage reductions, as we saw in the counterexample. So we define a safety criterion that captures

- what "run-away" means, precisely, in terms of a term filtering criterion, and
- what the precise details of the substitution calculus are.

The first stage is captured by requiring the substitution calculus to be *garbage safe* followed by the second stage where it should additionally *preserve substitute strong normalization*.

Definition 4.2 The explicit substitution relation $\xrightarrow[\mathscr{X}]{}$ is *garbage safe* (over Λx) if

GS1: substitution terminates: $\xrightarrow[\mathscr{X}]{}$ is strongly normalizing;
GS2: pure terms have no substitutions: for pure M, $M \equiv \downarrow_{\mathscr{X}}(M)$;
GS3: the union with (b) simulates β: $(\xrightarrow[\mathrm{b}]{} \cdot \xrightarrow[\mathscr{X}]{}\!\!\!\!\twoheadrightarrow)|_{\Lambda} = \xrightarrow[\beta]{}$.
GS4: full substitution ignores garbage: if $M \xrightarrow[\mathrm{gc}]{} N$, then $\downarrow_{\mathscr{X}}(M) \equiv \downarrow_{\mathscr{X}}(N)$;
GS5: garbage-free b-contraction does real work: if $M \xrightarrow[\mathrm{b}]{} N$ and $\downarrow_{\mathrm{gc}}(M) \not\equiv \downarrow_{\mathrm{gc}}(N)$ then $\downarrow_{\mathscr{X}}(M) \xrightarrow[\beta]{+} \downarrow_{\mathscr{X}}(N)$;

Lemma 4.3 *Let $\xrightarrow[\mathscr{R}]{} = (\xrightarrow[\mathrm{b}]{} \cup \xrightarrow[\mathscr{X}]{})$ with $\xrightarrow[\mathscr{X}]{}$ garbage safe. Then $\xrightarrow[\mathscr{R}\downarrow\mathrm{gc}]{}$ preserves strong β-normalization.*

Proof We show that each $\xrightarrow[\mathscr{R}\downarrow\mathrm{gc}]{}$-reduction corresponds to a $\xrightarrow[\beta]{}$-reduction of comparable length. Assume P is pure and strongly normalizing for $\xrightarrow[\beta]{}$. Since P is pure, it has no $\xrightarrow[\mathscr{X}\downarrow\mathrm{gc}]{}$-redexes by GS2, and every $\xrightarrow[\mathscr{R}\downarrow\mathrm{gc}]{}\!\!\!\!\twoheadrightarrow$-reduction starting with P, whether finite or not, can be mapped to an equivalent $\xrightarrow[\beta]{}$-sequence as follows (using GS1-5):

$$\begin{array}{ccccccccccccccccc}
P & = & P & \xrightarrow[\mathrm{b}]{} & M_1 & \xrightarrow[\mathrm{gc}]{}\!\!\!\!\twoheadrightarrow & M_1' & \xrightarrow[\mathscr{X}\downarrow\mathrm{gc}]{}\!\!\!\!\twoheadrightarrow & M_1'' & \xrightarrow[\mathrm{b}]{} & M_2 & \xrightarrow[\mathrm{gc}]{}\!\!\!\!\twoheadrightarrow & M_2' & \xrightarrow[\mathscr{X}\downarrow\mathrm{gc}]{}\!\!\!\!\twoheadrightarrow & M_2'' & \xrightarrow[\mathrm{b}]{} & \cdots \\
 & & \| & & \mathscr{X}\downarrow\!\!\downarrow & & \mathscr{X}\downarrow\!\!\downarrow & & \mathscr{X}\downarrow\!\!\downarrow & & \mathscr{X}\downarrow\!\!\downarrow & & \mathscr{X}\downarrow\!\!\downarrow & & \mathscr{X}\downarrow\!\!\downarrow & & \\
 & & P & \xrightarrow[\beta]{+} & P_1 & = & P_1 & = & P_1 & \xrightarrow[\beta]{+} & P_2 & = & P_2 & = & P_2 & \xrightarrow[\beta]{+} & \cdots
\end{array}$$

It follows that if there is an infinite $\xrightarrow[\mathscr{R}\downarrow\mathrm{gc}]{}$ reduction then there must be an infinite $\xrightarrow[\beta]{}$ reduction, hence $\xrightarrow[\mathscr{R}\downarrow\mathrm{gc}]{}$ has β-PSN. □

Definition 4.4 A λx term is *substitute strongly normalizing* (SSN) if, for all subterms $M\langle x := N\rangle$, N is strongly normalizing. A relation $\longrightarrow$ on Λx *preserves substitute strong normalization* if $M \longrightarrow N$ for substitute strongly normalizing M implies that N is substitute strongly normalizing.

Lemma 4.5 *Let* $\xrightarrow[\mathscr{R}]{} = (\xrightarrow[\mathrm{b}]{} \cup \xrightarrow[\mathscr{X}]{})$ *with* $\xrightarrow[\mathscr{X}]{}$ *garbage safe and preserving substitute strong normalization. Then let* $g(M)$ *denote the maximal length of garbage-free* ($\xrightarrow[\mathscr{R}\downarrow\mathrm{gc}]{}$) *reduction starting with* $\downarrow_{\mathrm{gc}}(M)$*, using* ∞ *when there are infinite reductions. If* $g(M) < \infty$ *and* M *is substitute strongly normalizing, then* M *is strongly* $\mathscr{R}$*-normalizing.*

Proof By induction on $g(M)$. The base case, $g(M) = 0$, follows directly since by definition no reduction path of M can contain reductions outside garbage and pure terms are SSN.

For the induction step we assume $g(M) > 0$ and M SSN, which means we can construct

$$M \underbrace{\xrightarrow[\mathscr{R}]{} \cdots \xrightarrow[\mathscr{R}]{} M_m}_{\text{in garbage}} \underbrace{\xrightarrow[\mathscr{R}]{} M_{m+1}}_{\text{outside garbage}}$$

where m must be finite and M_m SSN because it only involves reductions inside garbage substitutions, already known to be strongly normalizing (SN). We determine that M_{m+1} is SSN by looking at the last reduction step:

- If $M_m \equiv C[(\lambda x.N)P] \xrightarrow[\mathrm{b}]{} C[N\langle x := P\rangle] \equiv M_{m+1}$ for some context C: We know that all bodies of substitutions of M_{m+1} except P are SN. P itself is SN because $g(P) < g((\lambda x.N)P) \leq g(M_m) = g(M)$, so the induction hypothesis applies and we have that M_{m+1} is SSN.
- If $M_m \xrightarrow[\mathscr{X}]{} M_{m+1}$ then preservation of SSN for $\xrightarrow[\mathscr{X}]{}$ applies.

Since by definition $g(M_m) > g(M_{m+1})$ we are done. □

Theorem 4.6 *The union of* $\xrightarrow[\mathrm{b}]{}$ *and an explicit substitution relation, which is garbage safe and preserves substitute strong normalization, preserves* β*-strong normalization.*

Proof Consider pure terms that are β-strongly normalizing. By Lemma 4.3 the longest garbage-free reduction is finite, so by Lemma 4.5 the longest reduction is finite. □

Theorem 4.7 $\xrightarrow[\mathrm{x}]{}$ *is garbage safe and preserves substitute strong normalization.*

Proof Prop. 3.3 is GS1 and GS2, and Prop. 3.4 is GS3. GS4 and GS5 both follow by induction over terms using Prop. 3.3 and careful attention to the free variables. SSN follows by the following analysis: Assume M is SSN and that $M \xrightarrow[\mathrm{x}]{} N$. If the reduction is inside a substitute then the containing substitute is already strongly normalizing. If the reduction is outside a substitute, then an investigation of the rules shows that no new substitutes are introduced. In both cases, N must then be substitute strongly normalizing. □

Corollary 4.8 λx *preserves strong* β*-normalization.*

Proof Follows from Theorems 4.6 and 4.7. □

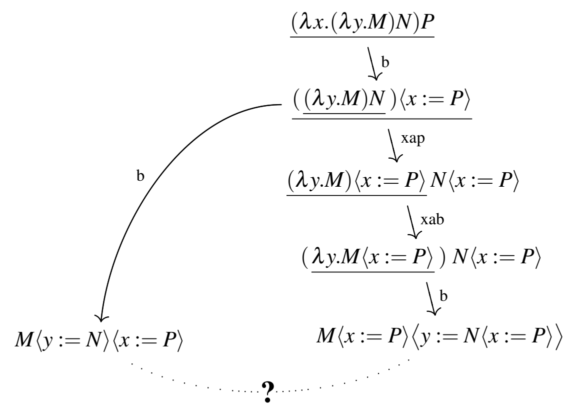


Fig. 3 A divergent reduction of open λx terms.

5 Confluence on Open Terms

In this section we define λxcw, an extension of λx, which permits "weak" composition of explicit substitutions such that it recovers confluence on open terms (Theorem 5.7). This is important, because to reason about reduction it is often useful to compute with terms that are "open" in that they have unknown parts, represented by meta-variables. The λx calculus does not support such reasoning properly because with meta-variables reduction does not converge, as illustrated by the two reductions in Fig. 3. Since the two terms of the shown critical pair are the terms of the substitution lemma (2.8), the solution is to somehow include the substitution lemma as a rule by directing it like

$$M\langle x := N\rangle\langle y := P\rangle \longrightarrow M\langle y := P\rangle\langle x := N\langle y := P\rangle\rangle$$

but this immediately leads to a non-terminating substitution calculus. The key observation is

> " *A composition between two substitutions is without purpose if the substitution variable does not occur free anywhere in the body.* "

Other calculi have exploited this by incorporating the garbage collection rule directly into the calculus (Lang and Rose 1998; Kesner and Lengrand 2007; Kesner 2009), however, we do not here want the calculus to dynamically check for free variables in subterms since that can be argued to be a non-local search and thus make the calculus less explicit as an operational model of computation.

The initial idea exploited here (Lang and Rose 1998) is to mark all substitution with their composition history and use a rule like

$$M\langle x := N\rangle_S\langle y := P\rangle_T \longrightarrow M\langle y := P\rangle_{T\cup\{x\}}\big\langle x := N\langle y := P\rangle_T\big\rangle_S \qquad y \notin S$$

where each substitution remembers the binders it has been composed with. We shall combine this idea with explicit weakening markers inspired by Ritter (1999) and similar to the ones used by Kesner and Lengrand (2007): The *weakening* $m^{x_1\cdots x_n}$ has a list of variables known to *not* occur free in m. We chose, however, to include weakening lists on *all* subterms and design the rules to continuously update these markers with information that can be obtained from the context while making sure that information about which substitutions have already been composed remains intact. The principal idea is that *knowledge of free variables travels outwards in the term* so that it may encounter and dissolve the appropriate substitutions, whereas *only composition history not derivable from free variables is maintained in the composition history*.

Definition 5.1 The λxcw *terms*, Λxcw, are defined inductively by

$$\begin{aligned} M,N,P,Q &::= m^A \\ m,n,p,q &::= x \mid \lambda x.M \mid MN \mid M\langle x := N\rangle_S \\ A,B,C,S,T,U &::= x_1 \dots x_n \end{aligned} \qquad (\Lambda\text{xcw})$$

The markings of form $x_1 \dots x_n$ are sets of variables, where we use (as meta-notation so terms only contain actual sets of variables) simple concatenation for union, & for intersection, $\setminus$ for difference, confuse individual variables x and their singleton set $\{x\}$, and, for conciseness, permit the special notation M^A to denote the λxcw term obtained by adding the A variables to the existing weakening variables of M (instead of writing m^B in the pattern and m^{AB} in the contraction). The superscript on all terms is called the *weakening* and the subscript on substitutions the *composition history*.

The *free variables* and *context bound variables* are unchanged (ignore markings).

The notion of reduction $\xrightarrow[\text{bxcw}]{}$ is given as the compatible closure generated by the rules

$$\big((\lambda x.M)^A\, N\big)^B \to M^A\langle x := N\rangle^B_{\varnothing} \qquad \text{(b)}$$

$$y^A\langle y := P\rangle^B_T \to P^B \qquad y \notin A \qquad \text{(xv)}$$

$$x^A\langle y := P\rangle^B_T \to x^{(A\setminus y)B} \qquad y \notin xA \qquad \text{(xvgc)}$$

$$(\lambda x.M)^A\langle y := P\rangle^B_T \to \big(\lambda x.M^A\langle y := P^x\rangle^{\varnothing}_{xT}\big)^B \qquad y \notin A \qquad \text{(xab)}$$

$$(MN)^A\langle y := P\rangle^B_T \to \big(M^A\langle y := P\rangle^{\varnothing}_T\, N^A\langle y := P\rangle^{\varnothing}_T\big)^B \qquad y \notin A \qquad \text{(xap)}$$

$$M\langle x := N\rangle^A_S\langle y := P\rangle^B_T \to M^A\langle y := P^x\rangle^{\varnothing}_{xT}\,\langle x := N^A\langle y := P\rangle^{\varnothing}_T\rangle^B_S \qquad y \notin AS \qquad \text{(c)}$$

$$m^A\langle y := P\rangle^B_T \to m^{(A\setminus y)B} \qquad y \in A \qquad \text{(wgc)}$$

$$(\lambda x.m^A)^B \to \big(\lambda x.m^{A\setminus x}\big)^{Bx} \qquad x \in A \setminus y \qquad \text{(wab)}$$

$$(m^A n^B)^C \to \big(m^{A\setminus x}\, n^{B\setminus x}\big)^{Cx} \qquad x \in A\&B \qquad \text{(wap)}$$

$$m^A\langle y := n^B\rangle^C_S \to m^{A\setminus x}\langle y := n^{B\setminus x}\rangle^{Cx}_{S\setminus x} \qquad x \in A\&B \qquad \text{(wx)}$$

with $\xrightarrow[\text{b}]{}$, $\xrightarrow[\text{x}]{}$, $\xrightarrow[\text{c}]{}$, and $\xrightarrow[\text{w}]{}$, denoting the subrelations involving only, (b), (xv,xvgc,xab,xap), (c), and (wab,wap,wx,wgc), respectively, and letter combinations correspond to unions, *e.g.*, $\xrightarrow[\text{xcw}]{} = (\xrightarrow[\text{x}]{} \cup \xrightarrow[\text{c}]{} \cup \xrightarrow[\text{w}]{})$.

Finally, we embed Λx (and Λ) into Λxcw by inserting empty markings everywhere (still referring to terms generated in this way from Λ as *pure*), and conversely project Λxcw terms into Λx and Λ by removing all markings.

We show in Fig. 4 how the divergent reduction discussed for λx in Fig. 3 is resolved by λxcw, which shows how the inner substitution variable of nested substitutions is recorded both on the substitution itself and inside the inner copy of the outer substitute body after composition. More formally, the figure illustrates the solution to the critical pair between

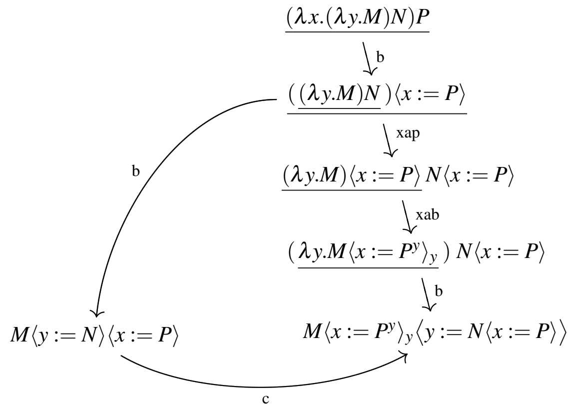


Fig. 4 Closed divergent reduction for λxcw (omitting empty markings).

(b) and (xap) for the λxcw term $((\lambda x.m^A)^B n^C)^D \langle y := p^E \rangle^F_S$ where $y \notin B$; for $y \in B$ the critical pair is closed with (wgc).

Notice that all the w rules effectively pick a variable to propagate outwards. One can instead use rules that propapate several variables at once, however, we have chosen to make the steps as small as possible. An important aspect of the w rules is that they are *monotonic* in that they reach a (local) fixed point; since it only migrates already existing markings it is possible to play a combinatoric game of finding the w normal form with as much information about the term as can be derived from the current markings. The limit of such information is naturally defined by the actual variables in the term having moved as far up in the term as the invariant allows.

Definition 5.2 A λxcw term is *well weakened* if the following holds for all (appropriate) subterms:

WW1: $C[x]^A$ implies $x \notin A$. free.
WW2: $C[M]^A$ and $C'[C[M], n^A]$ each imply that $\mathrm{bv}(C[])$ and A are disjoint, and similarly $C[M]\langle x := P \rangle^A_S$ and $C'[C[M], N\langle x := P \rangle^A_S]$ each imply that $\mathrm{bv}(C[])$ and S are disjoint.
WW3: for $C'[C[M\langle x := N \rangle^A_S]\langle y := P \rangle^B_T]$ we have $y \notin S$ implies $T \subseteq S$.
WW4: $C[C'[p^A]^B]$ implies that A and B are disjoint.
WW5: $C[M\langle x := N \rangle_S]^A$ implies S and A disjoint.

Informally, WW1 states that weakenings are correct and only contain variables that do not occur free. WW2 expresses that neither weakenings nor composition histories can escape their lexical scope. WW3 implies that when an inner substitution has been composed with an outer binder, then each intermediate substitution has been composed with at most one of the outer binder and the inner substitution; this is perhaps more easily seen by expanding the constraint to the equivalent constraint that for all $z \in \mathrm{bv}(C')$ one of the following conditions holds for $C'[C[M\langle x := N \rangle^A_S]\langle y := P \rangle^B_T]$:

- $y \in S$, *i.e.*, the inner substitution already crossed the outer one,
- $y \notin S$, $z \in S$, and $z \in T$, *i.e.*, the inner and outer substitutions did not cross each other but both crossed z, or
- $y \notin S$ and $z \notin T$, *i.e.*, the inner and outer substitutions did not cross each other and neither crossed z.

WW4 means that every variable occurs in at most one weakening on each branch of the term, and WW5 that variables in weakenings cannot occur in composition histories below that weakening. Finally, we observe that pure terms are trivially well weakened.

Proposition 5.3 *If M is well weakened and $M \xrightarrow[\text{bxcw}]{} N$ then N is well weakened.*

Proof Simple investigation of the rules. Details in Appendix A.1 (p. 21). □

Since we allow variable occurrences, the notion of open term needs to be carefully defined, following the tradition of higher-order rewriting of incorporating the list of all potentially occurring free variables into the meta-variable.

Definition 5.4 The *open* λxcw *terms* are defined as the Λxcw terms permitting meta-terms written $M\{x_1,\dots,x_n\}$, that is, using an explicit meta-variable in the term marked with a list of the free variables that may occur in it (this also avoids ambiguity with usual meta-variables). Free variables are extended to open terms by $\mathrm{fv}(M\{x_1,\dots,x_n\}) = \{x_1,\dots,x_n\}$. We use the meta-meta-variable $\mathscr{M}$ to denote an arbitrary meta-variable, where needed.

Now we can check the properties of the substitution calculus.

Lemma 5.5 $\xrightarrow[\text{xcw}]{}$ *is strongly normalizing for open* λxcw *terms.*

Proof We shall first establish that (c) by itself is strongly normalizing and then use this to construct a lexicographic path ordering over terms showing that the entire system is strongly normalizing.

Let V be the (finite) set of variables occurring in a term. Define the set of *composable substitutions* $\mathrm{cs}(M)$ of a subterm M by $\mathrm{cs}(\lambda x.M) = \mathrm{cs}(MN) = \mathrm{cs}(x) = \varnothing$ and $\mathrm{cs}(M\langle y := N\rangle_S) = V \setminus S$ and define the measure g as $g(x) = 1$, $g(\lambda x.M) = g(M)$, $g(MN) = g(M) + g(N)$, and

$$g(M\langle x := N\rangle_S) = \begin{cases} g(M) + g(N) & x \notin \mathrm{cs}(M) \\ g(M) \times g(N) \times \kappa(S) & x \in \mathrm{cs}(M) \end{cases}$$

where $\kappa(S) = |V \setminus S| + 2$. Now, when $R \xrightarrow[\text{c}]{} R'$, $g(R) > g(R')$, so $\xrightarrow[\text{c}]{}$ is strongly normalizing. Details in Appendix A.2 (p. 21).

Strong normalization of the full relation is then proved by a lexicographic path ordering induced by the well-founded symbol order $\cdots \rhd {}_{-}\langle_{-}\rangle^{n+1} \rhd {}_{-}\langle_{-}\rangle^{n} \rhd \cdots \rhd {}_{-}\langle_{-}\rangle^{0} \rhd {}_{-}@_{-} \rhd \lambda(_{-}), *$ (permitting any natural number for n). Open λxcw terms are projected into the ordered terms by

$$\begin{aligned} [\![x^A]\!] = [\![\,\mathscr{M}\{x_1,\dots,x_n\}\,]\!] &= * \\ [\![(\lambda x.M)^A]\!] &= \lambda([\![M]\!]) \\ [\![(MN)^A]\!] &= [\![M]\!]\,@\,[\![N]\!] \\ [\![M\langle x := N\rangle_S^A]\!] &= [\![M]\!]\langle[\![N]\!]\rangle^{c(M\langle x:=N\rangle_S^A)} \end{aligned}$$

with

$$c(M\langle x := N\rangle_S^A) = \max\{\, n \mid \exists P \in \ell(M\langle x := N\rangle_S^A) \colon P \xrightarrow[\text{c}]{n} \cdot\,\}$$

where

$$\ell(x^A) = \{x^A\}$$
$$\ell(\mathscr{M}\{x_1,\dots,x_n\}) = \{\mathscr{M}\{x_1,\dots,x_n\}\}$$
$$\ell((\lambda x.M)^A) = \ell(M)$$
$$\ell((MN)^A) = \ell(M) \cup \ell(N)$$
$$\ell(M\langle x := N\rangle^A_S) = \{\, P\langle y := Q\rangle^A_S \mid P \in \ell(M), Q \in \ell(N)\,\}$$

(the c marker counts the number of non-blocked (c) substitution compositions remaining in the substitution term by jotting them all together in all possible nestings and then trying to reduce as far as possible, known to be defined because $\xrightarrow[\text{c}]{}$ is strongly normalizing).

We then compute the terms for each rule of $\twoheadrightarrow_{\text{xcw}}$, show that the ordering is respected in the lexicographic path order, and use Theorem 2.5. Rules (xv,xvgc,wgc,wap,wab,wx) are straightforward after noting that for all M, N, A, A', and S, $c(M\langle x := N\rangle^A_S) = c(M\langle x := N\rangle^{A'}_S)$. For (xab) we note that $c(M\langle y := P\rangle^B_T) = c(M^A\langle y := P\rangle^{\varnothing}_T) \ge c(M^A\langle y := P\rangle^{\varnothing}_{xT})$ and for (xap) that $c((MN)^A\langle y := P\rangle^P_T) \ge c(Q^A\langle y := P\rangle^{\varnothing}_T)$ for $Q \equiv M,N$. Lastly, rule (c) is simple because $c(M\langle x := N\rangle^A_S\langle y := P\rangle^B_T)$ is larger than all three of $c(M^A\langle y := P^x\rangle^{\varnothing}_{xT}\langle x := N\langle y := P\rangle^{\varnothing}_T\rangle^B_S)$, $c(M^A\langle y := P^x\rangle^{\varnothing}_{xT})$, and $c(N^A\langle y := P\rangle^{\varnothing}_T)$. □

Lemma 5.6 $\twoheadrightarrow_{\text{xcw}}$ *is convergent for open* λxcw *terms.*

Proof Because of strong normalization we just verify weak confluence by closing all critical pairs. The foundational critical pair (b-xap) was illustrated in Fig. 4 (adding the general markings is straightforward). The critical pairs between the x and w rules, as well as between (c) and the x rules, are all closed by applying w rules to unify slight differences in the markings reached by the different reductions. Here well weakening is essential, especially that the w rules permit moving specific variables outwards, as we do not in the rules know all the variables. Details in Appendix A.3 (p. 22). □

We can now establish that the calculus was indeed designed properly for confluence.

Theorem 5.7 $\xrightarrow[\text{bxcw}]{}$ *is confluent on open* λxcw *terms.*

Proof We follow Curien et al (1996) and use Yokouchi and Hikita's Lemma 2.2 (with R, S, T, instantiated by xcw, b|| defined below, and bxcw, respectively). We already have YH1 by Lemma 5.5 and YH2 by Lemma 5.6. We will define a new relation $\xrightarrow[\text{b}\|]{}$ and show that it is strongly confluent (YH3), satisfies the Yokouchi and Hikita inclusion (YH4), and is properly squeezed in between $\xrightarrow[\text{bxcw}]{}$ and $\twoheadrightarrow_{\text{bxcw}}$ (YH5 and YH6).

Parallel b-*contraction* $\xrightarrow[\text{b}\|]{}$ is the relation that is provable using the axiom

$$M \xrightarrow[\text{b}\|]{} M \qquad \text{(b}\|\text{-ax)}$$

and inference rules

$$\frac{M \xrightarrow[\text{b}\|]{} M' \qquad N \xrightarrow[\text{b}\|]{} N'}{((\lambda x.M)^A N)^B \xrightarrow[\text{b}\|]{} M'^A\langle x := N'\rangle^B_{\varnothing}} \ \text{(b}\|\text{-b)} \qquad \frac{M \xrightarrow[\text{b}\|]{} M' \qquad N \xrightarrow[\text{b}\|]{} N'}{M\langle x := N\rangle^A_S \xrightarrow[\text{b}\|]{} M'\langle x := N'\rangle^A_S} \ \text{(b}\|\text{-x)}$$

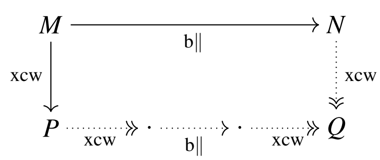
$$\frac{M \xrightarrow[\text{b}\|]{} M'}{(\lambda x.M)^A \xrightarrow[\text{b}\|]{} (\lambda x.M')^A} \ \text{(b}\|\text{-ab)} \qquad \frac{M \xrightarrow[\text{b}\|]{} M' \qquad N \xrightarrow[\text{b}\|]{} N'}{(MN)^A \xrightarrow[\text{b}\|]{} (M'N')^A} \ \text{(b}\|\text{-ap)}$$

It is easy to see that $\underset{\mathrm{b}\|}{\longrightarrow}$ is reflexive and strictly generalizes $\underset{\mathrm{b}}{\rightarrow}$ by permitting a single b-reduction using (b‖-b) in any context.

To prove strong confluence (YH3) of $\underset{\mathrm{b}\|}{\longrightarrow}$ consider open λxcw-terms M, M', M'' such that $M \underset{\mathrm{b}\|}{\longrightarrow} M'$ and $M \underset{\mathrm{b}\|}{\longrightarrow} M''$. We are looking for a λxcw-term N such that $M' \underset{\mathrm{b}\|}{\longrightarrow} N$ and $M'' \underset{\mathrm{b}\|}{\longrightarrow} N$. We proceed by induction on M:

- If $M \equiv P\langle x := Q\rangle^A_S$, then M' and M'' are respectively of the form $P'\langle x := Q'\rangle^A_S$ and $P''\langle x := Q''\rangle^A_S$ where $P \underset{\mathrm{b}\|}{\longrightarrow} A'$, $P \underset{\mathrm{b}\|}{\longrightarrow} P''$, $Q \underset{\mathrm{b}\|}{\longrightarrow} Q'$ and $Q \underset{\mathrm{b}\|}{\longrightarrow} Q''$. By the induction hypothesis, there exists P''' and Q''' such that $P' \underset{\mathrm{b}\|}{\longrightarrow} P'''$, $P'' \underset{\mathrm{b}\|}{\longrightarrow} P'''$, and idem for Q' and Q'' towards Q'''. Thus, $N \equiv P'''\langle x := Q'''\rangle^A_S$ fits.
- If $M \equiv (\lambda x.P)^A$, we can use the same technique as above.
- If $M \equiv x^A$, or M is a meta-variable, this is trivial as M can only reduce to itself, thus $M' \equiv M'' \equiv N \equiv M$.
- If $M \equiv (PQ)^A$, we have to consider several cases as M'' and N can be either applications or substitutions (when P is an abstraction). However, this does not introduce technical difficulty, and the proof is quite the same as above.

To verify YH4 we check that the following diagram closes (solid implies dotted) in all cases



which amounts to checking that all critical pairs between b‖ and xcw converge through the diagram, which is easy to verify: in most cases the single b‖-redex is merely duplicated or removed by the xcw-reduction; only in case of a (xap) reduction that overlaps with the context where (b‖-b) was used are a few extra xcw-reductions needed to push the appropriate substitutions inside the abstractions.

$\underset{\mathrm{bxcw}}{\twoheadrightarrow} \subseteq (\underset{\mathrm{xcw}}{\twoheadrightarrow} \cdot \underset{\mathrm{b}\|}{\longrightarrow} \cdot \underset{\mathrm{xcw}}{\twoheadrightarrow})$ (YH5) is simple, as every single xcw-reduction can be obtained on the right hand side by using that b‖ is reflexive, and every b-reduction by using the inference rules to pick just that context to use (b‖-b).

Finally, $(\underset{\mathrm{xcw}}{\twoheadrightarrow} \cdot \underset{\mathrm{b}\|}{\longrightarrow} \cdot \underset{\mathrm{xcw}}{\twoheadrightarrow}) \subseteq \underset{\mathrm{bxcw}}{\twoheadrightarrow}$ (YH6) follows by observing that any b‖-reduction can be achieved by multiple b-reductions. □

Finally, we mention the following properties that carry over from λx without change beyond the administrivia of inserting and erasing markings.

Definition 5.8 Given a relation R on Λxcw. The relation $\lfloor R \rfloor$ is the *projection* of R onto Λx consisting of starting with a λx term, inserting empty weakenings everywhere, using R, and erasing all weakenenings to extract a resulting λx term again.

Proposition 5.9

- $\underset{\mathrm{xcw}}{\twoheadrightarrow}$ *simulates meta-substitution (on ground* λxcw *terms),*
- $\lfloor \underset{\mathrm{bxcw}}{\longrightarrow} \cdot \underset{\mathrm{xcw}}{\twoheadrightarrow} \rfloor|_\Lambda = \underset{\beta}{\rightarrow}$.
- $\underset{\beta}{\twoheadrightarrow} = \lfloor \underset{\mathrm{bxcw}}{\twoheadrightarrow} \rfloor|_\Lambda$.

Proof Simulation of meta-substitution follows easily from the corresponding λx by observing that the x subcalculus of xcw never creates marks that block the innermost substitution thus all substitutions can be eliminated in the same way as for λx. Simulation of single β steps (ignoring marks) and completenes for full β reduction follow the same arguments as Prop. 3.4. □

6 On Confluence *and* Preservation of Normalization

Preservation of strong β-normalization (PSN) is inherently a property for ground terms because it relates to the λ calculus for which no reasonable notion of meta-variable can be defined. In spite of this, a calculus with both PSN and confluence on open terms would be useful as it allows implementations of common tools such as theorem provers where both reasoning at the meta-level and actual simulation of β reduction takes place.

However, the combination of the two properties has proven conspicuously elusive and noone has yet found a calculus with the following full complement of five properties.

Definition 6.1 An explicit substitution calculus is said to be *fully explicit* if it combines the following properties:

- Simulates β reduction.
- Substitution subcalculus identifiable, terminating, and simulates meta-substitution.
- Confluent on open terms (or, alternatively, allows composition of substitutions).
- Preserves strong β-normalization (PSN).
- Constrained to rules with only local side conditions, *i.e.*, side conditions that match information observed directly by the rule pattern (without depending on matched subterms).

For a thorough survey of all but the last of these properties, see Kesner (2009), where the calculus λex is presented, which uses explicit checks for free variables in the rules, thus making pattern matching non-local (discussed in the conclusion, Sec. 7.3). In this section we outline how the "first stage" of our PSN proof, Lemma 4.3, can be used to establish a similar result for the λxcw calculus with aggressive garbage collection based on a non-local test for free variables.

Definition 6.2 Define the *weakened garbage collection* relation $\xrightarrow[\text{gcw}]{}$ as the compatible closure over Λxcw of

$$M\langle y := N\rangle^{A}_{S} \longrightarrow M^{A} \qquad y \notin \mathrm{fv}(M) \tag{gcw}$$

Theorem 6.3 $\lfloor \xrightarrow[\text{bxcw}\downarrow\text{gcw}]{} \rfloor$ *preserves strong β-normalization.*

Proof First restate garbage safety modulo projection of pure terms and then prove that $\xrightarrow[\text{xcw}]{}\!\!\!\!\!\rightarrow$ is garbage safe (modulo projection). This amounts to prove GS1-5 from Def. 4.2 except modulo projection: GS1 follows from Prop. 5.5, and GS2-4 from Prop. 5.9. GS5 is shown in the same way as for λx, using substitution simulation. Now construct a β reduction sequence of comparable length for each $\xrightarrow[\text{bxcw}\downarrow\text{gcw}]{}$ reduction as in the proof of Lemma 4.3, adding projection in each mapping step. □

However, we do not know whether λxcw also possesses the substitute strong normalization property of Def. 4.4 needed to use Theorem 4.6 to prove PSN for the calculus, or whether a different strategy can be used to prove PSN for λxcw.

7 Conclusion & Discussion

We have reported the original definition and properties of λx as a calculus with explicit substitution using proper variable names that preserves strong β-normalization and explained the rôle of garbage collection rules for λx (Bloo and Rose 1995). The basic results for λx have certainly been surpassed and generalized by several authors (notably Ritter 1999; Kesner and Lengrand 2007; Kesner 2009), however, we still believe that the original formulation is of interest, as well as the new classification of garbage-safe reductions.

We have shown how λx can be generalized to λxcw, a calculus confluent on open terms inspired by Lang and Rose (1998) but using a simplified version of Ritter's (1999) weakening syntax; λxcw further benefits from only using local variable annotations in the rules, and while preservation of normalization in general remains unknown, we have shown that the aggressive garbage reduction version satisfies properties in line with similar recent results.

Finally, in future work we will attempt to see if the results reported here can be used to find a fully explicit substitution calculus (in the sense of Def. 6.1)

Below we first provide some details on where λx comes from before we comment specifically on related work on managing variable names and garbage collection.

7.1 Provenance

The first use of the definitional notation of substitution in a syntactic way, as in λx, seems to have been the "λ-calculus with lazy subsitution" by Lins (1986), which is used to prove a normalization property for weak reduction of categorical combinators. Lins identifies the rôle of renaming and the need for composable substitutions but does not study the rewrite properties of the calculus or the relationship with generic β-reduction, which may be why the presentation was not widely associated with explicit substitution until recently (Lins 2004). Instead the λx notation was obtained from the independently studied "explicit cyclic substitutions" (Rose 1992), where variables where chosen to permit recursive references in (simultaneous) substitutions.

The first attempt at understanding the stepwise nature of substitution applied to a strategy free λ calculus (with variable names) that we know of was the "axioms for the theory of λ-conversion" of Revesz (1985):

$$(\lambda x.x)Q \to Q \tag{$\beta 1$}$$

$$(\lambda x.y)Q \to y \qquad x \neq y \tag{$\beta 2$}$$

$$(\lambda x.\lambda y.P)Q \to \lambda y.(\lambda x.P)Q \tag{$\beta 3$}$$

$$(\lambda x.P_1P_2)Q \to (\lambda x.P_1)Q((\lambda x.P_2)Q) \tag{$\beta 4$}$$

The relation $\xrightarrow[\text{Revesz}]{}$ defined by these rules is related to λx by the equation

$$\xrightarrow[\text{Revesz}]{} = \xrightarrow[\text{b}]{} \cdot \xrightarrow[\text{x}]{} \cdot \twoheadleftarrow\!\!|_{\text{b}} \tag{Revesz-λx}$$

(where the rightmost arrow denotes *expansion* with (b) to normal form, *i.e.*, until no explicit substitutions remain). The problem with Revesz's calculus for our purpose, as also observed by Santo (2007), is that it does not make sense to talk about the substitution normal form: one cannot observe which substitutions are "in progress," an essential requirement in our formal treatment.

Another early discussion of names in explicit substitution is the named variant of $\lambda\sigma$ by Abadi et al (1991), here in λx style:

$$(\lambda x.M)N \longrightarrow M\langle(x := N)\cdot \mathrm{id}\rangle \qquad \text{(Beta)}$$

$$x\langle(x := N)\cdot s\rangle \longrightarrow N \qquad \text{(Var1)}$$

$$x\langle(y := N)\cdot s\rangle \longrightarrow x\langle s\rangle \qquad \text{(Var2)}$$

$$x\langle \mathrm{id}\rangle \longrightarrow x \qquad \text{(Var3)}$$

$$(NM)\langle s\rangle \longrightarrow (N\langle s\rangle)(M\langle s\rangle) \qquad \text{(App)}$$

$$(\lambda x.M)\langle s\rangle \longrightarrow \lambda y.(M\langle(x := y)\cdot s\rangle) \qquad \text{if } y \text{ does not occur in } M \qquad \text{(Abs)}$$

The rules correpond closely to those of λx except for the minor difference of using explicit lists of substitutions and the crucial difference that (Abs) introduces *explicit renaming*: the "names" are *not* variable names in the λ calculus sense but strings with explicit renaming insertion, with all the associated problems of allocation, *etc*., and hence difficult to describe formally, a conclusion also reached by Abadi et al, which influenced the initial focus on calculi with indices.

7.2 On the variable convention *vs* de Bruijn indices

In this paper we focus on describing explicit substitution with first class names. In the 1990s, presentations of this work were very often met with reactions in line of "if you do not use de Bruijn's (1972) indices then how can your results be formally sound?" The reason for this is that the variable convention is hard to formalize if constrained to a logic without names, as indeed witnessed by the multitude of concrete representations of de Bruijn indices used by explicit substitution calculi (Kamareddine and Nederpelt 1993; Lescanne 1994; Kesner 2000). However, it turns out that it is pretty simple to define *translations* in and out of de Bruijn formats, from which one can obtain theorems for de Bruijn calculi from the corresponding λx family ones (see, for example Rose 1996a, §2.2). The translations, however, are devised for each specific de Bruijn variation.

Recently, Urban (2008a) has formalized the variable convention in what is called "nominal" Isabelle (Urban 2008b). It will be interesting to see whether this formalization can lead to a systematic classification of de Bruijn-based calculi through the expressive power of the named calculus each corresponds to (Kesner 2007); indeed, it may be possible to restate the explicit substitution calculi with variables using the underlying Fraenkel-Mostowski set theory with atoms (Gabbay and Pitts 2002).

Finally, while de Bruijn indices certainly work great as a model for variables as used by abstract machines constrained to a single evaluation strategy, it remains unclear when variable names or de Bruijn indices are best suited for *reasoning* about reduction.

7.3 On Garbage Collection

As argued above, we have not included a garbage collection rule in our calculi as the involved non-local tests, in our view, ruins the explicitness of the calculus because the free variable constraint potentially requires a search through the term. Thus while the rule serves as a useful vehicle to prove preservation of strong normalization, it is not really itself appropriate in explicit calculi (the same can be said for the "explicification" of higher-order rewriting by Bloo and Rose 1996).

Several calculi of explicit substitution have been published that rely on some notion of garbage collection. Of special interest to λx is the λex calculus of Kesner (2009), evolved from the earlier λlxr (Kesner and Lengrand 2007). The equations and rules of λex would look as follows in λx's style:

$$M\langle x := N\rangle\langle y := P\rangle \equiv M\langle y := P\rangle\langle x := N\rangle \qquad y \notin \mathrm{fv}(N) \wedge x \notin \mathrm{fv}(P) \tag{C}$$

$$(\lambda x.M)N \to M\langle x := N\rangle \tag{B}$$

$$y\langle y := P\rangle \to P \tag{Var}$$

$$M\langle y := P\rangle \to M \qquad y \notin \mathrm{fv}(M) \tag{Gc}$$

$$(MN)\langle y := P\rangle \to M\langle y := P\rangle N\langle y := P\rangle \tag{App}$$

$$(\lambda x.M)\langle y := P\rangle \to \lambda x.M\langle y := P\rangle \tag{Lamb}$$

$$M\langle x := N\rangle\langle y := P\rangle \to M\langle y := P\rangle\langle x := N\langle y := P\rangle\rangle \qquad y \in \mathrm{fv}(N) \tag{Comp}$$

The effect is a hybrid of sets of simultaneous substitutions and λx-style nested substitutions (shown equivalent already by Kamareddine and Nederpelt 1993). Kesner shows preservation of strong normalization using a perpetual strategy (van Raamsdonk 1996) and a syntactic characterization of terms that depends on the tests for free variables. In this sense the calculus shares with the synthetic λxcw↓gcw calculus of Sec. 6 that garbage collection is used in an essential way, even if it is used less aggressively by λex, where only nested substitutions force a non-local search for free variables.

Finally, it is noteworthy that the garbage collection principle developed here (through Lang and Rose 1998) was the inspiration for Ritter's (1999) characterisation of calculi that preserve strong normalization leading back to the notion of weakening that we have used.

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A Detailed proofs

This appendix provides details for the proofs of Lemmas 5.3, 5.5, and 5.6, as these correspond to results that, in our experience, warrant detailed study.

A.1 Details for Lemma 5.3 (p.13)

Preservation of WW1 requires investigation of the rules that add a variable to a weakening. (xab,c) add a variable to subterms where it obviously does not occur free. (wab,wap,wx) add a variable to a term where the pattern effectively constrains the variable to be free in all appropriate subterms needed for it to be free in the entire term.

WW2 follows by a similar argument, observing that no weakening is ever moved outside any scope. Similarly, no composition history is ever moved outside any scope, except in rule (c), where composition history S is moved outside the scope of variable y. However, rule (c) requires that $y \notin S$, thus guaranteeing preservation of WW2.

WW3 requires a slightly more complex argument. First, we observe that both rules that insert variables in composition histories, (xab,c), do so immediately under the corresponding binder, which does not permit intermediate substitutions. Now consider the rules that change the substitution nesting:

(b) If the context of the reduction binds z, and M contains a substitution $\langle y := P\rangle_S$, then either the injected substitution satisfies WW3 because $x \in S$ or because $y \in S$ and the injected compostion history does not contain z.

(xab,c) The only problematic case for both is a substitution $\langle v := R\rangle_V^D$ inside M where $x \in V$ and $y \notin V$, which means that we create a new intermediate substitution under the binder x. The new substitution also has $x \in xT$ so the invariant is preserved.

(wx) Since the constraints imply that no substitution inside m can have x in a compostion history, the created substitution can safely remove x from the composition history because only the first and last conditions are possible.

Finally, WW4 and WW5 follow by simple observation of the scoping observed by the rules.

A.2 Details for Lemma 5.5 (p.13)

These are the details of the argument for strong normalization of (c). For simplicity, we omit weakenings, which have no influence on the argument (splitting the set in two does not make a difference). All terms are assumed to be well weakened.

As in the main text, we define V as the (finite) set of variables occurring in a term. The set of composable substitutions $cs(M)$ of a subterm M is defined by $cs(\lambda x.M) = cs(MN) = cs(x) = \varnothing$ and $cs(M\langle y := N\rangle_S) = V \setminus S$ and the measure g is defined by $g(x) = 1$, $g(\lambda x.M) = g(M)$, $g(MN) = g(M) + g(N)$, and

$$g(M\langle x := N\rangle_S) = \begin{cases} g(M) + g(N) & x \notin cs(M) \\ g(M) \times g(N) \times \kappa(S) & x \in cs(M) \end{cases}$$

where $\kappa(S) = |V \setminus S| + 2$. Observe that

$$g(M) \times g(N) \times \kappa(S) \geq g(M\langle x := N\rangle_S) \geq g(M) + g(N)$$

because $g(M) \geq 1$, $g(N) \geq 1$ and $\kappa(S) \geq 2$. We can show easily that if $y \notin S$ then $g(M\langle x := N\rangle_S\langle y := P\rangle_T) > g(M\langle y := P\rangle_{xT}\langle x := N\langle y := P\rangle_T\rangle_S)$. Indeed, with $a = g(M)$, $b = g(N)$, $c = g(P)$, and $d = \kappa(T)$, we have:

$$\begin{aligned} g(M\langle x := N\rangle_S\langle y := P\rangle_T) &= g(M\langle x := N\rangle_S) \times g(P) \times \kappa(T) \\ &\geq (g(M) + g(N)) \times g(P) \times \kappa(T) \end{aligned}$$

$$
\begin{aligned}
&= (a+b)cd \\
&= acd + bcd \\
&> acd - ac + bcd \\
&= ac(d-1) + bcd \\
&= g(M) \times g(P) \times \kappa(xT) + g(N) \times g(P) \times \kappa(T) \\
&\geq g(M\langle y := P\rangle_{xT}) + g(N\langle y := P\rangle_T) \\
&= g(M\langle y := P\rangle_{xT}\langle x := N\langle y := P\rangle_T\rangle_S)
\end{aligned}
$$

More generally, we must show that if $M \xrightarrow[c]{} M'$ then $g(M) > g(M')$. We proceed by induction on the depth n where the reduction occurs. For $n = 0$, the work has just been done. Then we assume that there exists $n \geq 0$ such that if $M \xrightarrow[c]{} M'$ at depth n then $g(M) > g(M')$, and show that this still holds at depth $n+1$. Let $M \xrightarrow[c]{} M'$ at depth $n+1$. We proceed by cases:

- If M has the form $\lambda x.M_0$, then M' has the form $\lambda x.M'_0$ and $M_0 \xrightarrow[c]{} M'_0$ at depth n, with $g(M_0) > g(M'_0)$ by the induction hypothesis. We have $g(\lambda x.M_0) = g(M_0) > g(M'_0) = g(\lambda x.M'_0)$.
- If M has the form M_1M_2 and the reduction performs in M_1 or in M_2, or if M has the form $M_1\langle z := M_2\rangle_U$ and the reduction performs in M_2, then the proof is similar.
- Let M have the form $M_1\langle z := M_2\rangle_U$ and the reduction perform in M_1, with $M_1 \xrightarrow[c]{} M'_1$ and $g(M_1) > g(M'_1)$ by the induction hypothesis. If $n > 1$ then the topmost symbols of M_1 and M'_1 are the same, and thus $cs(M_1) = cs(M'_1)$, and we can simply use the induction hypothesis to conclude. If $n = 1$, then M_1 is a redex for (c), *i.e.*, M has the form $M_0\langle x := N\rangle_S\langle y := P\rangle_T\langle z := M_2\rangle_U$ and N has the form $M_0\langle y := P\rangle_{xT}\langle x := N\langle y := P\rangle_T\rangle_S\langle z := M_2\rangle_U$, with $y \notin S$. We consider two cases:
  - If $z \in S$ then:

$$
\begin{aligned}
g(M) &\geq g(M_0\langle x := N\rangle_S\langle y := P\rangle_T) + g(M_2) \\
&> g(M_0\langle y := P\rangle_{xT}\langle x := N\langle y := P\rangle_T\rangle_S) + g(M_2) \qquad \text{(by the induction hypothesis)} \\
&= g(N)
\end{aligned}
$$

  - If $z \notin S$ then, since $y \notin S$, property WW3 implies that $z \notin T$ and therefore:

$$
\begin{aligned}
g(M) &= g(M_0\langle x := N\rangle_S\langle y := P\rangle_T) \times g(M_2) \times \kappa(U) \\
&> g(M_0\langle y := P\rangle_{xT}\langle x := N\langle y := P\rangle_T\rangle_S) \times g(M_2) \times \kappa(U) \qquad \text{(by the induction hypothesis)} \\
&= g(N)
\end{aligned}
$$

Since the measure g is a (finite) natural number strictly decreasing through reduction, we have shown that $\xrightarrow[c]{}$ is strongly normalizing.

A.3 Details for Lemma 5.6 (p.14)

We show how each of the critical pairs is closed. For each we provide the constraints imposed by the well weakening invariant and show the reductions.

b-xap critical pair with $y \in B$

- Term: $((\lambda x.m^A)^B\ n^C)^D\langle y := p^E\rangle^F_S$
- Condition of rule b: none
- Condition of rule xap: $y \notin D$
- WW1: none
- WW2: $x \notin B \wedge x \notin C \wedge x \notin D \wedge x \notin E \wedge y \notin E \wedge x \notin S \wedge x \notin F \wedge y \notin S \wedge y \notin F$
- WW3: none
- WW4: $y \notin A \wedge y \notin F \wedge y \notin D$
- WW5: none

$$
\begin{aligned}
((\lambda x.m^A)^B\ n^C)^D\langle y := p^E\rangle^F_S &\xrightarrow[b]{} m^{(AB)}\langle x := n^C\rangle^D_\emptyset\langle y := p^E\rangle^F_S \\
&\xrightarrow[c]{} m^{((AB)D)}\langle y := p^{(Ex)}\rangle^\emptyset_{(Sx)}\langle x := n^{(CD)}\langle y := p^E\rangle^\emptyset_S\rangle^F_\emptyset
\end{aligned}
$$

$$\xrightarrow[\text{wgc}]{} m^{((A(B\setminus y))D)}\langle x := n^{(CD)}\langle y := p^E\rangle^{\emptyset}_S\rangle^F_{\emptyset}$$

$$((\lambda x.m^A)^B\ n^C)^D\langle y := p^E\rangle^F_S \xrightarrow[\text{xap}]{} ((\lambda x.m^A)^{(BD)}\langle y := p^E\rangle^{\emptyset}_S\ n^{(CD)}\langle y := p^E\rangle^{\emptyset}_S)^F$$

$$\xrightarrow[\text{wgc}]{} ((\lambda x.m^A)^{((B\setminus y)D)}\ n^{(CD)}\langle y := p^E\rangle^{\emptyset}_S)^F$$

$$\xrightarrow[\text{b}]{} m^{(A((B\setminus y)D))}\langle x := n^{(CD)}\langle y := p^E\rangle^{\emptyset}_S\rangle^F_{\emptyset}$$

b-xap critical pair with $y \notin B$

- Term: $((\lambda x.m^A)^B\ n^C)^D\langle y := p^E\rangle^F_S$
- Condition of rule b: none
- Condition of rule xap: $y \notin D$
- WW1: none
- WW2: $x \notin B \wedge x \notin C \wedge x \notin D \wedge x \notin E \wedge y \notin E \wedge x \notin S \wedge x \notin F \wedge y \notin S \wedge y \notin F$
- WW3: none
- WW4: none
- WW5: none

$$((\lambda x.m^A)^B\ n^C)^D\langle y := p^E\rangle^F_S \xrightarrow[\text{b}]{} m^{(AB)}\langle x := n^C\rangle^D_{\emptyset}\langle y := p^E\rangle^F_S$$

$$\xrightarrow[\text{c}]{} m^{((AB)D)}\langle y := p^{(Ex)}\rangle^{\emptyset}_{(Sx)}\langle x := n^{(CD)}\langle y := p^E\rangle^{\emptyset}_S\rangle^F_{\emptyset}$$

$$((\lambda x.m^A)^B\ n^C)^D\langle y := p^E\rangle^F_S \xrightarrow[\text{xap}]{} ((\lambda x.m^A)^{(BD)}\langle y := p^E\rangle^{\emptyset}_S\ n^{(CD)}\langle y := p^E\rangle^{\emptyset}_S)^F$$

$$\xrightarrow[\text{xab}]{} ((\lambda x.m^{(A(BD))}\langle y := p^{(Ex)}\rangle^{\emptyset}_{(Sx)})^{\emptyset}\ n^{(CD)}\langle y := p^E\rangle^{\emptyset}_S)^F$$

$$\xrightarrow[\text{b}]{} m^{(A(BD))}\langle y := p^{(Ex)}\rangle^{\emptyset}_{(Sx)}\langle x := n^{(CD)}\langle y := p^E\rangle^{\emptyset}_S\rangle^F_{\emptyset}$$

wgc-c critical pair

- Term: $m^A\langle y := p^B\rangle^C_S\langle z := q^D\rangle^E_T$
- Condition of rule wgc: $y \in A$
- Condition of rule c: $z \notin S \wedge z \notin C$
- WW1: none
- WW2: $y \notin B \wedge y \notin S \wedge y \notin C \wedge y \notin D \wedge z \notin D \wedge y \notin T \wedge y \notin E \wedge z \notin T \wedge z \notin E$
- WW3: none
- WW4: $y \notin E \wedge y \notin C$
- WW5: none

$$m^A\langle y := p^B\rangle^C_S\langle z := q^D\rangle^E_T \xrightarrow[\text{wgc}]{} m^{((A\setminus y)C)}\langle z := q^D\rangle^E_T$$

$$m^A\langle y := p^B\rangle^C_S\langle z := q^D\rangle^E_T \xrightarrow[\text{c}]{} m^{(AC)}\langle z := q^{(Dy)}\rangle^{\emptyset}_{(Ty)}\langle y := p^{(BC)}\langle z := q^D\rangle^{\emptyset}_T\rangle^E_S$$

$$\xrightarrow[\text{wx}]{} m^{((A\setminus y)C)}\langle z := q^D\rangle^y_T\langle y := p^{(BC)}\langle z := q^D\rangle^{\emptyset}_T\rangle^E_S$$

$$\xrightarrow[\text{wgc}]{} m^{((A\setminus y)C)}\langle z := q^D\rangle^E_T$$

xv-c critical pair with $z \in A$

- Term: $y^A\langle y := p^B\rangle^C_S\langle z := q^D\rangle^E_T$
- Condition of rule xv: $y \notin A$
- Condition of rule c: $z \notin C \wedge z \notin S$
- WW1: $y \notin A$
- WW2: $y \notin B \wedge y \notin S \wedge y \notin C \wedge y \notin D \wedge z \notin D \wedge y \notin T \wedge y \notin E \wedge z \notin T \wedge z \notin E$
- WW3: none
- WW4: $z \notin E \wedge z \notin C$
- WW5: none

$$y^A\langle y := p^B\rangle^C_S\langle z := q^D\rangle^E_T \xrightarrow[\text{xv}]{} p^{(BC)}\langle z := q^D\rangle^E_T$$

$$y^A\langle y := p^B\rangle^C_S\langle z := q^D\rangle^E_T \xrightarrow[\text{c}]{} y^{(AC)}\langle z := q^{(Dy)}\rangle^{\emptyset}_{(Ty)}\langle y := p^{(BC)}\langle z := q^D\rangle^{\emptyset}_T\rangle^E_S$$

$$\xrightarrow[\text{wgc}]{} y^{((A\setminus z)C)}\langle y := p^{(BC)}\langle z := q^D\rangle^{\emptyset}_T\rangle^E_S$$

$$\xrightarrow[\text{xv}]{} p^{(BC)}\langle z := q^D\rangle^E_T$$

xv-c critical pair with $z \notin A$

- Term: $y^A \langle y := p^B \rangle_S^C \langle z := q^D \rangle_T^E$
- Condition of rule xv: $y \notin A$
- Condition of rule c: $z \notin C \wedge z \notin S$
- WW1: $y \notin A$
- WW2: $y \notin B \wedge y \notin S \wedge y \notin C \wedge y \notin D \wedge z \notin D \wedge y \notin T \wedge y \notin E \wedge z \notin T \wedge z \notin E$
- WW3: none
- WW4: none
- WW5: none

$$
\begin{aligned}
y^A \langle y := p^B \rangle_S^C \langle z := q^D \rangle_T^E &\xrightarrow[\text{xv}]{} p^{(BC)} \langle z := q^D \rangle_T^E \\
y^A \langle y := p^B \rangle_S^C \langle z := q^D \rangle_T^E &\xrightarrow[\text{c}]{} y^{(AC)} \langle z := q^{(Dy)} \rangle_{(Ty)}^{\emptyset} \langle y := p^{(BC)} \langle z := q^D \rangle_T^{\emptyset} \rangle_S^E \\
&\xrightarrow[\text{xvgc}]{} y^{(AC)} \langle y := p^{(BC)} \langle z := q^D \rangle_T^{\emptyset} \rangle_S^E \\
&\xrightarrow[\text{xv}]{} p^{(BC)} \langle z := q^D \rangle_T^E
\end{aligned}
$$

xvgc-c critical pair with $x \neq z \wedge z \in A$

- Term: $x^A \langle y := p^B \rangle_S^C \langle z := q^D \rangle_T^E$
- Condition of rule xvgc: $y \notin A$
- Condition of rule c: $z \notin C \wedge z \notin S$
- WW1: $x \notin A \wedge x \notin C \wedge x \notin E$
- WW2: $y \notin B \wedge y \notin S \wedge y \notin C \wedge y \notin D \wedge z \notin D \wedge y \notin T \wedge y \notin E \wedge z \notin T \wedge z \notin E$
- WW3: none
- WW4: $z \notin E \wedge z \notin C$
- WW5: none

$$
\begin{aligned}
x^A \langle y := p^B \rangle_S^C \langle z := q^D \rangle_T^E &\xrightarrow[\text{xvgc}]{} x^{(AC)} \langle z := q^D \rangle_T^E \\
&\xrightarrow[\text{wgc}]{} x^{(((A \setminus z)C)E)} \\
x^A \langle y := p^B \rangle_S^C \langle z := q^D \rangle_T^E &\xrightarrow[\text{c}]{} x^{(AC)} \langle z := q^{(Dy)} \rangle_{(Ty)}^{\emptyset} \langle y := p^{(BC)} \langle z := q^D \rangle_T^{\emptyset} \rangle_S^E \\
&\xrightarrow[\text{wgc}]{} x^{((A \setminus z)C)} \langle y := p^{(BC)} \langle z := q^D \rangle_T^{\emptyset} \rangle_S^E \\
&\xrightarrow[\text{xvgc}]{} x^{(((A \setminus z)C)E)}
\end{aligned}
$$

xvgc-c critical pair with $x \neq z \wedge z \notin A$

- Term: $x^A \langle y := p^B \rangle_S^C \langle z := q^D \rangle_T^E$
- Condition of rule xvgc: $y \notin A$
- Condition of rule c: $z \notin C \wedge z \notin S$
- WW1: $x \notin A \wedge x \notin C \wedge x \notin E$
- WW2: $y \notin B \wedge y \notin S \wedge y \notin C \wedge y \notin D \wedge z \notin D \wedge y \notin T \wedge y \notin E \wedge z \notin T \wedge z \notin E$
- WW3: none
- WW4: none
- WW5: none

$$
\begin{aligned}
x^A \langle y := p^B \rangle_S^C \langle z := q^D \rangle_T^E &\xrightarrow[\text{xvgc}]{} x^{(AC)} \langle z := q^D \rangle_T^E \\
&\xrightarrow[\text{xvgc}]{} x^{((AC)E)} \\
x^A \langle y := p^B \rangle_S^C \langle z := q^D \rangle_T^E &\xrightarrow[\text{c}]{} x^{(AC)} \langle z := q^{(Dy)} \rangle_{(Ty)}^{\emptyset} \langle y := p^{(BC)} \langle z := q^D \rangle_T^{\emptyset} \rangle_S^E \\
&\xrightarrow[\text{xvgc}]{} x^{(AC)} \langle y := p^{(BC)} \langle z := q^D \rangle_T^{\emptyset} \rangle_S^E \\
&\xrightarrow[\text{xvgc}]{} x^{((AC)E)}
\end{aligned}
$$

xvgc-c critical pair with $x = z$

- Term: $x^A \langle y := p^B\rangle^C_S \langle x := q^D\rangle^E_T$
- Condition of rule xvgc: $y \notin A$
- Condition of rule c: $x \notin C \wedge x \notin S$
- WW1: $x \notin A \wedge x \notin C$
- WW2: $y \notin B \wedge y \notin S \wedge y \notin C \wedge y \notin D \wedge x \notin D \wedge y \notin T \wedge y \notin E \wedge x \notin T \wedge x \notin E$
- WW3: none
- WW4: none
- WW5: none

$$
\begin{aligned}
x^A \langle y := p^B\rangle^C_S \langle x := q^D\rangle^E_T &\xrightarrow[\text{xvgc}]{} x^{(AC)} \langle x := q^D\rangle^E_T \\
&\xrightarrow[\text{xv}]{} q^{(DE)} \\
x^A \langle y := p^B\rangle^C_S \langle x := q^D\rangle^E_T &\xrightarrow[\text{c}]{} x^{(AC)} \langle x := q^{(Dy)}\rangle^{\emptyset}_{(Ty)} \langle y := p^{(BC)} \langle x := q^D\rangle^{\emptyset}_T\rangle^E_S \\
&\xrightarrow[\text{xv}]{} q^{(Dy)} \langle y := p^{(BC)} \langle x := q^D\rangle^{\emptyset}_T\rangle^E_S \\
&\xrightarrow[\text{wgc}]{} q^{(DE)}
\end{aligned}
$$

xab-c critical pair with $z \in B$

- Term: $(\lambda x.m^A)^B \langle y := p^C\rangle^D_S \langle z := q^E\rangle^F_T$
- Condition of rule xab: $y \notin B$
- Condition of rule c: $z \notin D \wedge z \notin S$
- WW1: none
- WW2: $x \notin B \wedge x \notin C \wedge y \notin C \wedge x \notin S \wedge x \notin D \wedge y \notin S \wedge y \notin D \wedge x \notin E \wedge y \notin E \wedge z \notin E \wedge x \notin T \wedge x \notin F \wedge y \notin T \wedge y \notin F \wedge z \notin T \wedge z \notin F$
- WW3: none
- WW4: $z \notin A \wedge z \notin F \wedge z \notin D$
- WW5: none

$$
\begin{aligned}
(\lambda x.m^A)^B \langle y := p^C\rangle^D_S \langle z := q^E\rangle^F_T &\xrightarrow[\text{xab}]{} (\lambda x.m^{(AB)} \langle y := p^{(Cx)}\rangle^{\emptyset}_{(Sx)})^D \langle z := q^E\rangle^F_T \\
&\xrightarrow[\text{xab}]{} (\lambda x.m^{(AB)} \langle y := p^{(Cx)}\rangle^D_{(Sx)} \langle z := q^{(Ex)}\rangle^{\emptyset}_{(Tx)})^F \\
&\xrightarrow[\text{c}]{} (\lambda x.m^{((AB)D)} \langle z := q^{((Ex)y)}\rangle^{\emptyset}_{((Tx)y)} \langle y := p^{((Cx)D)} \langle z := q^{(Ex)}\rangle^{\emptyset}_{(Tx)}\rangle^{\emptyset}_{(Sx)})^F \\
&\xrightarrow[\text{wgc}]{} (\lambda x.m^{((A(B\backslash z))D)} \langle y := p^{((Cx)D)} \langle z := q^{(Ex)}\rangle^{\emptyset}_{(Tx)}\rangle^{\emptyset}_{(Sx)})^F \\
&\xrightarrow[\text{wx}]{} (\lambda x.m^{((A(B\backslash z))D)} \langle y := p^{(CD)} \langle z := q^E\rangle^x_T\rangle^{\emptyset}_{(Sx)})^F \\
(\lambda x.m^A)^B \langle y := p^C\rangle^D_S \langle z := q^E\rangle^F_T &\xrightarrow[\text{c}]{} (\lambda x.m^A)^{(BD)} \langle z := q^{(Ey)}\rangle^{\emptyset}_{(Ty)} \langle y := p^{(CD)} \langle z := q^E\rangle^{\emptyset}_T\rangle^F_S \\
&\xrightarrow[\text{wgc}]{} (\lambda x.m^A)^{((B\backslash z)D)} \langle y := p^{(CD)} \langle z := q^E\rangle^{\emptyset}_T\rangle^F_S \\
&\xrightarrow[\text{xab}]{} (\lambda x.m^{(A((B\backslash z)D))} \langle y := p^{(CD)} \langle z := q^E\rangle^x_T\rangle^{\emptyset}_{(Sx)})^F
\end{aligned}
$$

xab-c critical pair with $z \notin B$

- Term: $(\lambda x.m^A)^B \langle y := p^C\rangle^D_S \langle z := q^E\rangle^F_T$
- Condition of rule xab: $y \notin B$
- Condition of rule c: $z \notin D \wedge z \notin S$
- WW1: none
- WW2: $x \notin B \wedge x \notin C \wedge y \notin C \wedge x \notin S \wedge x \notin D \wedge y \notin S \wedge y \notin D \wedge x \notin E \wedge y \notin E \wedge z \notin E \wedge x \notin T \wedge x \notin F \wedge y \notin T \wedge y \notin F \wedge z \notin T \wedge z \notin F$
- WW3: none
- WW4: none
- WW5: none

$$\begin{aligned}
(\lambda x.m^A)^B\langle y:=p^C\rangle^D_S\langle z:=q^E\rangle^F_T &\xrightarrow[\text{xab}]{} (\lambda x.m^{(AB)}\langle y:=p^{(Cx)}\rangle^{\emptyset}_{(Sx)})^D\langle z:=q^E\rangle^F_T\\
&\xrightarrow[\text{xab}]{} (\lambda x.m^{(AB)}\langle y:=p^{(Cx)}\rangle^D_{(Sx)}\langle z:=q^{(Ex)}\rangle^{\emptyset}_{(Tx)})^F\\
&\xrightarrow[\text{c}]{} (\lambda x.m^{((AB)D)}\langle z:=q^{((Ex)y)}\rangle^{\emptyset}_{((Tx)y)}\langle y:=p^{((Cx)D)}\langle z:=q^{(Ex)}\rangle^{\emptyset}_{(Tx)}\rangle^{\emptyset}_{(Sx)})^F\\
&\xrightarrow[\text{wx}]{} (\lambda x.m^{((AB)D)}\langle z:=q^{((Ex)y)}\rangle^{\emptyset}_{((Tx)y)}\langle y:=p^{(CD)}\langle z:=q^E\rangle^x_T\rangle^{\emptyset}_{(Sx)})^F
\end{aligned}$$

$$\begin{aligned}
(\lambda x.m^A)^B\langle y:=p^C\rangle^D_S\langle z:=q^E\rangle^F_T &\xrightarrow[\text{c}]{} (\lambda x.m^A)^{(BD)}\langle z:=q^{(Ey)}\rangle^{\emptyset}_{(Ty)}\langle y:=p^{(CD)}\langle z:=q^E\rangle^{\emptyset}_T\rangle^F_S\\
&\xrightarrow[\text{xab}]{} (\lambda x.m^{(A(BD))}\langle z:=q^{((Ey)x)}\rangle^{\emptyset}_{((Ty)x)})^{\emptyset}\langle y:=p^{(CD)}\langle z:=q^E\rangle^{\emptyset}_T\rangle^F_S\\
&\xrightarrow[\text{xab}]{} (\lambda x.m^{(A(BD))}\langle z:=q^{((Ey)x)}\rangle^{\emptyset}_{((Ty)x)}\langle y:=p^{(CD)}\langle z:=q^E\rangle^x_T\rangle^{\emptyset}_{(Sx)})^F
\end{aligned}$$

xap-c critical pair with $z \in C$

- Term: $(m^A\ n^B)^C\langle y:=p^D\rangle^E_S\langle z:=q^F\rangle^G_T$
- Condition of rule xap: $y \notin C$
- Condition of rule c: $z \notin E \wedge z \notin S$
- WW1: none
- WW2: $y \notin D \wedge y \notin S \wedge y \notin E \wedge y \notin F \wedge z \notin F \wedge y \notin T \wedge y \notin G \wedge z \notin T \wedge z \notin G$
- WW3: none
- WW4: $z \notin B \wedge z \notin A \wedge z \notin G \wedge z \notin E$
- WW5: none

$$\begin{aligned}
&(m^A\ n^B)^C\langle y:=p^D\rangle^E_S\langle z:=q^F\rangle^G_T\\
&\quad\xrightarrow[\text{xap}]{} (m^{(AC)}\langle y:=p^D\rangle^{\emptyset}_S\ n^{(BC)}\langle y:=p^D\rangle^{\emptyset}_S)^E\langle z:=q^F\rangle^G_T\\
&\quad\xrightarrow[\text{xap}]{} (m^{(AC)}\langle y:=p^D\rangle^E_S\langle z:=q^F\rangle^{\emptyset}_T\ n^{(BC)}\langle y:=p^D\rangle^E_S\langle z:=q^F\rangle^{\emptyset}_T)^G\\
&\quad\xrightarrow[\text{c}]{} (m^{((AC)E)}\langle z:=q^{(Fy)}\rangle^{\emptyset}_{(Ty)}\langle y:=p^{(DE)}\langle z:=q^F\rangle^{\emptyset}_T\rangle^{\emptyset}_S\ n^{(BC)}\langle y:=p^D\rangle^E_S\langle z:=q^F\rangle^{\emptyset}_T)^G\\
&\quad\xrightarrow[\text{wgc}]{} (m^{((A(C\setminus z))E)}\langle y:=p^{(DE)}\langle z:=q^F\rangle^{\emptyset}_T\rangle^{\emptyset}_S\ n^{(BC)}\langle y:=p^D\rangle^E_S\langle z:=q^F\rangle^{\emptyset}_T)^G\\
&\quad\xrightarrow[\text{c}]{} (m^{((A(C\setminus z))E)}\langle y:=p^{(DE)}\langle z:=q^F\rangle^{\emptyset}_T\rangle^{\emptyset}_S\ n^{((BC)E)}\langle z:=q^{(Fy)}\rangle^{\emptyset}_{(Ty)}\langle y:=p^{(DE)}\langle z:=q^F\rangle^{\emptyset}_T\rangle^{\emptyset}_S)^G\\
&\quad\xrightarrow[\text{wgc}]{} (m^{((A(C\setminus z))E)}\langle y:=p^{(DE)}\langle z:=q^F\rangle^{\emptyset}_T\rangle^{\emptyset}_S\ n^{((B(C\setminus z))E)}\langle y:=p^{(DE)}\langle z:=q^F\rangle^{\emptyset}_T\rangle^{\emptyset}_S)^G
\end{aligned}$$

$$\begin{aligned}
&(m^A\ n^B)^C\langle y:=p^D\rangle^E_S\langle z:=q^F\rangle^G_T\\
&\quad\xrightarrow[\text{c}]{} (m^A\ n^B)^{(CE)}\langle z:=q^{(Fy)}\rangle^{\emptyset}_{(Ty)}\langle y:=p^{(DE)}\langle z:=q^F\rangle^{\emptyset}_T\rangle^G_S\\
&\quad\xrightarrow[\text{wgc}]{} (m^A\ n^B)^{((C\setminus z)E)}\langle y:=p^{(DE)}\langle z:=q^F\rangle^{\emptyset}_T\rangle^G_S\\
&\quad\xrightarrow[\text{xap}]{} (m^{(A((C\setminus z)E))}\langle y:=p^{(DE)}\langle z:=q^F\rangle^{\emptyset}_T\rangle^{\emptyset}_S\ n^{(B((C\setminus z)E))}\langle y:=p^{(DE)}\langle z:=q^F\rangle^{\emptyset}_T\rangle^{\emptyset}_S)^G
\end{aligned}$$

xap-c critical pair with $z \notin C$

- Term: $(m^A\ n^B)^C\langle y:=p^D\rangle^E_S\langle z:=q^F\rangle^G_T$
- Condition of rule xap: $y \notin C$
- Condition of rule c: $z \notin E \wedge z \notin S$
- WW1: none
- WW2: $y \notin D \wedge y \notin S \wedge y \notin E \wedge y \notin F \wedge z \notin F \wedge y \notin T \wedge y \notin G \wedge z \notin T \wedge z \notin G$
- WW3: none
- WW4: none
- WW5: none

$$\begin{aligned}
&(m^A\ n^B)^C\langle y:=p^D\rangle^E_S\langle z:=q^F\rangle^G_T\\
&\quad\xrightarrow[\text{xap}]{} (m^{(AC)}\langle y:=p^D\rangle^{\emptyset}_S\ n^{(BC)}\langle y:=p^D\rangle^{\emptyset}_S)^E\langle z:=q^F\rangle^G_T
\end{aligned}$$

$$\xrightarrow[\text{xap}]{} (m^{(AC)}\langle y := p^{D}\rangle^{E}_{S}\langle z := q^{F}\rangle^{\emptyset}_{T}\; n^{(BC)}\langle y := p^{D}\rangle^{E}_{S}\langle z := q^{F}\rangle^{\emptyset}_{T})^{G}$$
$$\xrightarrow[\text{c}]{} (m^{((AC)E)}\langle z := q^{(Fy)}\rangle^{\emptyset}_{(Ty)}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{\emptyset}_{T}\rangle^{\emptyset}_{S}\; n^{(BC)}\langle y := p^{D}\rangle^{E}_{S}\langle z := q^{F}\rangle^{\emptyset}_{T})^{G}$$
$$\xrightarrow[\text{c}]{} (m^{((AC)E)}\langle z := q^{(Fy)}\rangle^{\emptyset}_{(Ty)}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{\emptyset}_{T}\rangle^{\emptyset}_{S}\; n^{((BC)E)}\langle z := q^{(Fy)}\rangle^{\emptyset}_{(Ty)}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{\emptyset}_{T}\rangle^{\emptyset}_{S})^{G}$$

$$(m^{A}\; n^{B})^{C}\langle y := p^{D}\rangle^{E}_{S}\langle z := q^{F}\rangle^{G}_{T}$$
$$\xrightarrow[\text{c}]{} (m^{A}\; n^{B})^{(CE)}\langle z := q^{(Fy)}\rangle^{\emptyset}_{(Ty)}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{\emptyset}_{T}\rangle^{G}_{S}$$
$$\xrightarrow[\text{xap}]{} (m^{(A(CE))}\langle z := q^{(Fy)}\rangle^{\emptyset}_{(Ty)}\; n^{(B(CE))}\langle z := q^{(Fy)}\rangle^{\emptyset}_{(Ty)})^{\emptyset}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{\emptyset}_{T}\rangle^{G}_{S}$$
$$\xrightarrow[\text{xap}]{} (m^{(A(CE))}\langle z := q^{(Fy)}\rangle^{\emptyset}_{(Ty)}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{\emptyset}_{T}\rangle^{\emptyset}_{S}\; n^{(B(CE))}\langle z := q^{(Fy)}\rangle^{\emptyset}_{(Ty)}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{\emptyset}_{T}\rangle^{\emptyset}_{S})^{G}$$

c-c critical pair with $z \in C$

- Term: $m^{A}\langle x := n^{B}\rangle^{C}_{S}\langle y := p^{D}\rangle^{E}_{T}\langle z := q^{F}\rangle^{G}_{U}$
- Condition of inner rule c: $y \notin S \wedge y \notin C$
- Condition of outer rule c: $z \notin E \wedge z \notin T$
- WW1: none
- WW2: $x \notin B \wedge x \notin S \wedge x \notin C \wedge x \notin D \wedge y \notin D \wedge x \notin T \wedge x \notin E \wedge y \notin T \wedge y \notin E \wedge x \notin F \wedge y \notin F \wedge z \notin F \wedge x \notin U \wedge x \notin G \wedge y \notin U \wedge y \notin G \wedge z \notin U \wedge z \notin G$
- WW3: none
- WW4: $z \notin B \wedge z \notin A \wedge z \notin G \wedge z \notin E$
- WW5: $z \notin S$

$$m^{A}\langle x := n^{B}\rangle^{C}_{S}\langle y := p^{D}\rangle^{E}_{T}\langle z := q^{F}\rangle^{G}_{U}$$
$$\xrightarrow[\text{c}]{} m^{(AC)}\langle y := p^{(Dx)}\rangle^{\emptyset}_{(Tx)}\langle x := n^{(BC)}\langle y := p^{D}\rangle^{\emptyset}_{T}\rangle^{E}_{S}\langle z := q^{F}\rangle^{G}_{U}$$
$$\xrightarrow[\text{c}]{} m^{(AC)}\langle y := p^{(Dx)}\rangle^{E}_{(Tx)}\langle z := q^{(Fx)}\rangle^{\emptyset}_{(Ux)}\langle x := n^{(BC)}\langle y := p^{D}\rangle^{E}_{T}\langle z := q^{F}\rangle^{\emptyset}_{U}\rangle^{G}_{S}$$
$$\xrightarrow[\text{c}]{} m^{((AC)E)}\langle z := q^{((Fx)y)}\rangle^{\emptyset}_{((Ux)y)}\langle y := p^{((Dx)E)}\langle z := q^{(Fx)}\rangle^{\emptyset}_{(Ux)}\rangle^{\emptyset}_{(Tx)}\langle x := n^{(BC)}\langle y := p^{D}\rangle^{E}_{T}\langle z := q^{F}\rangle^{\emptyset}_{U}\rangle^{G}_{S}$$
$$\xrightarrow[\text{wgc}]{} m^{((A(C\backslash z))E)}\langle y := p^{((Dx)E)}\langle z := q^{(Fx)}\rangle^{\emptyset}_{(Ux)}\rangle^{\emptyset}_{(Tx)}\langle x := n^{(BC)}\langle y := p^{D}\rangle^{E}_{T}\langle z := q^{F}\rangle^{\emptyset}_{U}\rangle^{G}_{S}$$
$$\xrightarrow[\text{wx}]{} m^{((A(C\backslash z))E)}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{x}_{U}\rangle^{\emptyset}_{(Tx)}\langle x := n^{(BC)}\langle y := p^{D}\rangle^{E}_{T}\langle z := q^{F}\rangle^{\emptyset}_{U}\rangle^{G}_{S}$$
$$\xrightarrow[\text{c}]{} m^{((A(C\backslash z))E)}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{x}_{U}\rangle^{\emptyset}_{(Tx)}\langle x := n^{((BC)E)}\langle z := q^{(Fy)}\rangle^{\emptyset}_{(Uy)}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{\emptyset}_{U}\rangle^{\emptyset}_{T}\rangle^{G}_{S}$$
$$\xrightarrow[\text{wgc}]{} m^{((A(C\backslash z))E)}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{x}_{U}\rangle^{\emptyset}_{(Tx)}\langle x := n^{((B(C\backslash z))E)}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{\emptyset}_{U}\rangle^{\emptyset}_{T}\rangle^{G}_{S}$$

$$m^{A}\langle x := n^{B}\rangle^{C}_{S}\langle y := p^{D}\rangle^{E}_{T}\langle z := q^{F}\rangle^{G}_{U}$$
$$\xrightarrow[\text{c}]{} m^{A}\langle x := n^{B}\rangle^{(CE)}_{S}\langle z := q^{(Fy)}\rangle^{\emptyset}_{(Uy)}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{\emptyset}_{U}\rangle^{G}_{T}$$
$$\xrightarrow[\text{wgc}]{} m^{A}\langle x := n^{B}\rangle^{((C\backslash z)E)}_{S}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{\emptyset}_{U}\rangle^{G}_{T}$$
$$\xrightarrow[\text{c}]{} m^{(A((C\backslash z)E))}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{x}_{U}\rangle^{\emptyset}_{(Tx)}\langle x := n^{(B((C\backslash z)E))}\langle y := p^{(DE)}\langle z := q^{F}\rangle^{\emptyset}_{U}\rangle^{\emptyset}_{T}\rangle^{G}_{S}$$

c-c critical pair with $z \notin C$

- Term: $m^{A}\langle x := n^{B}\rangle^{C}_{S}\langle y := p^{D}\rangle^{E}_{T}\langle z := q^{F}\rangle^{G}_{U}$
- Condition of rule c: $y \notin S \wedge y \notin C$
- Condition of rule c: $z \notin E \wedge z \notin T$
- WW1: none
- WW2: $x \notin B \wedge x \notin S \wedge x \notin C \wedge x \notin D \wedge y \notin D \wedge x \notin T \wedge x \notin E \wedge y \notin T \wedge y \notin E \wedge x \notin F \wedge y \notin F \wedge z \notin F \wedge x \notin U \wedge x \notin G \wedge y \notin U \wedge y \notin G \wedge z \notin U \wedge z \notin G$
- WW3: $z \notin S$
- WW4: none
- WW5: none

$$
\begin{aligned}
&m^A\langle x := n^B\rangle_S^C \langle y := p^D\rangle_T^E \langle z := q^F\rangle_U^G \\
&\quad\xrightarrow[\text{c}]{} m^{(AC)}\langle y := p^{(Dx)}\rangle_{(Tx)}^{\emptyset} \langle x := n^{(BC)}\langle y := p^D\rangle_T^{\emptyset}\rangle_S^E \langle z := q^F\rangle_U^G \\
&\quad\xrightarrow[\text{c}]{} m^{(AC)}\langle y := p^{(Dx)}\rangle_{(Tx)}^{E} \langle z := q^{(Fx)}\rangle_{(Ux)}^{\emptyset} \langle x := n^{(BC)}\langle y := p^D\rangle_T^E \langle z := q^F\rangle_U^{\emptyset}\rangle_S^G \\
&\quad\xrightarrow[\text{c}]{} m^{((AC)E)}\langle z := q^{((Fx)y)}\rangle_{((Ux)y)}^{\emptyset} \langle y := p^{((Dx)E)}\langle z := q^{(Fx)}\rangle_{(Ux)}^{\emptyset}\rangle_{(Tx)}^{\emptyset} \langle x := n^{(BC)}\langle y := p^D\rangle_T^E \langle z := q^F\rangle_U^{\emptyset}\rangle_S^G \\
&\quad\xrightarrow[\text{wx}]{} m^{((AC)E)}\langle z := q^{((Fx)y)}\rangle_{((Ux)y)}^{\emptyset} \langle y := p^{(DE)}\langle z := q^F\rangle_U^x\rangle_{(Tx)}^{\emptyset} \langle x := n^{(BC)}\langle y := p^D\rangle_T^E \langle z := q^F\rangle_U^{\emptyset}\rangle_S^G \\
&\quad\xrightarrow[\text{c}]{} m^{((AC)E)}\langle z := q^{((Fx)y)}\rangle_{((Ux)y)}^{\emptyset} \langle y := p^{(DE)}\langle z := q^F\rangle_U^x\rangle_{(Tx)}^{\emptyset} \langle x := n^{((BC)E)}\langle z := q^{(Fy)}\rangle_{(Uy)}^{\emptyset} \langle y := p^{(DE)}\langle z := q^F\rangle_U^{\emptyset}\rangle_T^{\emptyset}\rangle_S^G
\end{aligned}
$$

$$
\begin{aligned}
&m^A\langle x := n^B\rangle_S^C \langle y := p^D\rangle_T^E \langle z := q^F\rangle_U^G \\
&\quad\xrightarrow[\text{c}]{} m^{A}\langle x := n^B\rangle_S^{(CE)} \langle z := q^{(Fy)}\rangle_{(Uy)}^{\emptyset} \langle y := p^{(DE)}\langle z := q^F\rangle_U^{\emptyset}\rangle_T^G \\
&\quad\xrightarrow[\text{c}]{} m^{(A(CE))}\langle z := q^{((Fy)x)}\rangle_{((Uy)x)}^{\emptyset} \langle x := n^{(B(CE))}\langle z := q^{(Fy)}\rangle_{(Uy)}^{\emptyset}\rangle_S^{\emptyset} \langle y := p^{(DE)}\langle z := q^F\rangle_U^{\emptyset}\rangle_T^G \\
&\quad\xrightarrow[\text{c}]{} m^{(A(CE))}\langle z := q^{((Fy)x)}\rangle_{((Uy)x)}^{\emptyset} \langle y := p^{(DE)}\langle z := q^F\rangle_U^x\rangle_{(Tx)}^{\emptyset} \langle x := n^{(B(CE))}\langle z := q^{(Fy)}\rangle_{(Uy)}^{\emptyset} \langle y := p^{(DE)}\langle z := q^F\rangle_U^{\emptyset}\rangle_T^{\emptyset}\rangle_S^G
\end{aligned}
$$

wgc-wx critical pair

- Term: $m^A\langle y := p^B\rangle_S^C$
- Condition of rule wgc: $x \in A \wedge x \in B$
- Condition of rule wx: $y \in A$
- WW1: none
- WW2: $y \notin B \wedge y \notin S \wedge y \notin C$
- WW3: none
- WW4: $y \notin C \wedge x \notin C$
- WW5: none

$$
\begin{aligned}
m^A\langle y := p^B\rangle_S^C &\xrightarrow[\text{wx}]{} m^{(A\backslash x)}\langle y := p^{(B\backslash x)}\rangle_{(S\backslash x)}^{(Cx)} \\
&\xrightarrow[\text{wgc}]{} m^{(((A\backslash y)\backslash x)(Cx))} \\
m^A\langle y := p^B\rangle_S^C &\xrightarrow[\text{wgc}]{} m^{((A\backslash y)C)}
\end{aligned}
$$

xv-wx critical pair

- Term: $y^A\langle y := p^B\rangle_C^S$
- Condition of rule xv: $y \notin A$
- Condition of rule wx: $x \in A \wedge x \in B$
- WW1: $y \notin A$
- WW2: $y \notin B \wedge y \notin C \wedge y \notin S$
- WW3: none
- WW4: $x \notin S$
- WW5: none

$$
\begin{aligned}
y^A\langle y := p^B\rangle_C^S &\xrightarrow[\text{wx}]{} y^{(A\backslash x)}\langle y := p^{(B\backslash x)}\rangle_{(C\backslash x)}^{(Sx)} \\
&\xrightarrow[\text{xv}]{} p^{((B\backslash x)(Sx))} \\
y^A\langle y := p^B\rangle_C^S &\xrightarrow[\text{xv}]{} p^{(BS)}
\end{aligned}
$$

xvgc-wx critical pair

- Term: $z^A\langle y := p^B\rangle_S^C$
- Condition of rule xvgc: $y \notin A$
- Condition of rule wx: $x \in A \wedge x \in B$
- WW1: $z \notin A \wedge z \notin C$
- WW2: $y \notin B \wedge y \notin S \wedge y \notin C$
- WW3: none
- WW4: $x \notin C$
- WW5: none

$$\begin{aligned} z^A\langle y := p^B\rangle_S^C &\xrightarrow[\text{wx}]{} z^{(A\backslash x)}\langle y := p^{(B\backslash x)}\rangle_{(S\backslash x)}^{(Cx)} \\ &\xrightarrow[\text{xvgc}]{} z^{((A\backslash x)(Cx))} \\ z^A\langle y := p^B\rangle_S^C &\xrightarrow[\text{xvgc}]{} z^{(AC)} \end{aligned}$$

xab-wx critical pair

- Term: $(\lambda z.m^A)^B\langle y := p^C\rangle_S^D$
- Condition of rule xab: $y \notin B$
- Condition of rule wx: $x \in B \wedge x \in C$
- WW1: none
- WW2: $z \notin B \wedge z \notin C \wedge y \notin C \wedge z \notin S \wedge z \notin D \wedge y \notin S \wedge y \notin D$
- WW3: none
- WW4: $x \notin A \wedge x \notin D$
- WW5: none

$$\begin{aligned} (\lambda z.m^A)^B\langle y := p^C\rangle_S^D &\xrightarrow[\text{wx}]{} (\lambda z.m^A)^{(B\backslash x)}\langle y := p^{(C\backslash x)}\rangle_{(S\backslash x)}^{(Dx)} \\ &\xrightarrow[\text{xab}]{} (\lambda z.m^{(A(B\backslash x))}\langle y := p^{((C\backslash x)z)}\rangle_{((S\backslash x)z)}^{\emptyset})^{(Dx)} \\ (\lambda z.m^A)^B\langle y := p^C\rangle_S^D &\xrightarrow[\text{xab}]{} (\lambda z.m^{(AB)}\langle y := p^{(Cz)}\rangle_{(Sz)}^{\emptyset})^D \\ &\xrightarrow[\text{wx}]{} (\lambda z.m^{(A(B\backslash x))}\langle y := p^{((C\backslash x)z)}\rangle_{((Sz)\backslash x)}^{x})^D \\ &\xrightarrow[\text{wab}]{} (\lambda z.m^{(A(B\backslash x))}\langle y := p^{((C\backslash x)z)}\rangle_{((Sz)\backslash x)}^{\emptyset})^{(Dx)} \end{aligned}$$

xap-wx critical pair

- Term: $(m^A\ n^B)^C\langle y := p^D\rangle_S^E$
- Condition of rule xap: $y \notin C$
- Condition of rule wx: $x \in C \wedge x \in D$
- WW1: none
- WW2: $y \notin D \wedge y \notin S \wedge y \notin E$
- WW3: none
- WW4: $x \notin B \wedge x \notin A \wedge x \notin E$
- WW5: none

$$\begin{aligned} (m^A\ n^B)^C\langle y := p^D\rangle_S^E &\xrightarrow[\text{wx}]{} (m^A\ n^B)^{(C\backslash x)}\langle y := p^{(D\backslash x)}\rangle_{(S\backslash x)}^{(Ex)} \\ &\xrightarrow[\text{xap}]{} (m^{(A(C\backslash x))}\langle y := p^{(D\backslash x)}\rangle_{(S\backslash x)}^{\emptyset}\ n^{(B(C\backslash x))}\langle y := p^{(D\backslash x)}\rangle_{(S\backslash x)}^{\emptyset})^{(Ex)} \\ (m^A\ n^B)^C\langle y := p^D\rangle_S^E &\xrightarrow[\text{xap}]{} (m^{(AC)}\langle y := p^D\rangle_S^{\emptyset}\ n^{(BC)}\langle y := p^D\rangle_S^{\emptyset})^E \\ &\xrightarrow[\text{wx}]{} (m^{(A(C\backslash x))}\langle y := p^{(D\backslash x)}\rangle_{(S\backslash x)}^{x}\ n^{(BC)}\langle y := p^D\rangle_S^{\emptyset})^E \\ &\xrightarrow[\text{wx}]{} (m^{(A(C\backslash x))}\langle y := p^{(D\backslash x)}\rangle_{(S\backslash x)}^{x}\ n^{(B(C\backslash x))}\langle y := p^{(D\backslash x)}\rangle_{(S\backslash x)}^{x})^E \\ &\xrightarrow[\text{wap}]{} (m^{(A(C\backslash x))}\langle y := p^{(D\backslash x)}\rangle_{(S\backslash x)}^{\emptyset}\ n^{(B(C\backslash x))}\langle y := p^{(D\backslash x)}\rangle_{(S\backslash x)}^{\emptyset})^{(Ex)} \end{aligned}$$

c-wx critical pair with $x \neq y$ and outermost wx-redex

- Term: $m^A \langle z := n^B \rangle^C_W \langle y := p^D \rangle^E_T$
- Condition of rule c: $y \notin C \wedge y \notin W$
- Condition of rule wx: $x \in C \wedge x \in D$
- WW1: none
- WW2: $z \notin B \wedge z \notin W \wedge z \notin C \wedge z \notin D \wedge y \notin D \wedge z \notin T \wedge z \notin E \wedge y \notin T \wedge y \notin E$
- WW3: none
- WW4: $x \notin B \wedge x \notin A \wedge x \notin E$
- WW5: $x \notin W$

$$\begin{aligned}
m^A \langle z := n^B \rangle^C_W \langle y := p^D \rangle^E_T &\xrightarrow[\text{wx}]{} m^A \langle z := n^B \rangle^{(C\backslash x)}_W \langle y := p^{(D\backslash x)} \rangle^{(Ex)}_{(T\backslash x)} \\
&\xrightarrow[\text{c}]{} m^{(A(C\backslash x))} \langle y := p^{((D\backslash x)z)} \rangle^{\emptyset}_{((T\backslash x)z)} \langle z := n^{(B(C\backslash x))} \langle y := p^{(D\backslash x)} \rangle^{\emptyset}_{(T\backslash x)} \rangle^{(Ex)}_W \\
m^A \langle z := n^B \rangle^C_W \langle y := p^D \rangle^E_T &\xrightarrow[\text{c}]{} m^{(AC)} \langle y := p^{(Dz)} \rangle^{\emptyset}_{(Tz)} \langle z := n^{(BC)} \langle y := p^D \rangle^{\emptyset}_T \rangle^E_W \\
&\xrightarrow[\text{wx}]{} m^{(A(C\backslash x))} \langle y := p^{((D\backslash x)z)} \rangle^{x}_{((Tz)\backslash x)} \langle z := n^{(BC)} \langle y := p^D \rangle^{\emptyset}_T \rangle^E_W \\
&\xrightarrow[\text{wx}]{} m^{(A(C\backslash x))} \langle y := p^{((D\backslash x)z)} \rangle^{x}_{((Tz)\backslash x)} \langle z := n^{(B(C\backslash x))} \langle y := p^{(D\backslash x)} \rangle^{x}_{(T\backslash x)} \rangle^E_W \\
&\xrightarrow[\text{wx}]{} m^{(A(C\backslash x))} \langle y := p^{((D\backslash x)z)} \rangle^{\emptyset}_{((Tz)\backslash x)} \langle z := n^{(B(C\backslash x))} \langle y := p^{(D\backslash x)} \rangle^{\emptyset}_{(T\backslash x)} \rangle^{(Ex)}_W
\end{aligned}$$

c-wx critical pair with $x \neq y$ and innermost wx-redex

- Term: $m^A \langle z := n^B \rangle^C_W \langle y := p^D \rangle^E_T$
- Condition of rule c: $y \notin C \wedge y \notin W$
- Condition of rule wx: $x \in A \wedge x \in B$
- WW1: none
- WW2: $z \notin B \wedge z \notin W \wedge z \notin C \wedge z \notin D \wedge y \notin D \wedge z \notin T \wedge z \notin E \wedge y \notin T \wedge y \notin E$
- WW3: none
- WW4: $x \notin E \wedge x \notin C$
- WW5: none

$$\begin{aligned}
m^A \langle z := n^B \rangle^C_W \langle y := p^D \rangle^E_T &\xrightarrow[\text{wx}]{} m^{(A\backslash x)} \langle z := n^{(B\backslash x)} \rangle^{(Cx)}_W \langle y := p^D \rangle^E_T \\
&\xrightarrow[\text{c}]{} m^{((A\backslash x)(Cx))} \langle y := p^{(Dz)} \rangle^{\emptyset}_{(Tz)} \langle z := n^{((B\backslash x)(Cx))} \langle y := p^D \rangle^{\emptyset}_T \rangle^E_W \\
m^A \langle z := n^B \rangle^C_W \langle y := p^D \rangle^E_T &\xrightarrow[\text{c}]{} m^{(AC)} \langle y := p^{(Dz)} \rangle^{\emptyset}_{(Tz)} \langle z := n^{(BC)} \langle y := p^D \rangle^{\emptyset}_T \rangle^E_W
\end{aligned}$$

c-wx critical pair with $x = y$

- Term: $m^A \langle z := n^B \rangle^C_W \langle y := p^D \rangle^E_T$
- Condition of rule c: $y \notin C \wedge y \notin W$
- Condition of rule wx: $y \in A \wedge y \in B$
- WW1: none
- WW2: $z \notin B \wedge z \notin W \wedge z \notin C \wedge z \notin D \wedge y \notin D \wedge z \notin T \wedge z \notin E \wedge y \notin T \wedge y \notin E$
- WW3: none
- WW4: $y \notin E \wedge y \notin C$
- WW5: none

$$\begin{aligned}
m^A \langle z := n^B \rangle^C_W \langle y := p^D \rangle^E_T &\xrightarrow[\text{wx}]{} m^{(A\backslash y)} \langle z := n^{(B\backslash y)} \rangle^{(Cy)}_W \langle y := p^D \rangle^E_T \\
&\xrightarrow[\text{wgc}]{} m^{(A\backslash y)} \langle z := n^{(B\backslash y)} \rangle^{(CE)}_W \\
m^A \langle z := n^B \rangle^C_W \langle y := p^D \rangle^E_T &\xrightarrow[\text{c}]{} m^{(AC)} \langle y := p^{(Dz)} \rangle^{\emptyset}_{(Tz)} \langle z := n^{(BC)} \langle y := p^D \rangle^{\emptyset}_T \rangle^E_W \\
&\xrightarrow[\text{wgc}]{} m^{((A\backslash y)C)} \langle z := n^{(BC)} \langle y := p^D \rangle^{\emptyset}_T \rangle^E_W \\
&\xrightarrow[\text{wgc}]{} m^{((A\backslash y)C)} \langle z := n^{((B\backslash y)C)} \rangle^E_W
\end{aligned}$$

The diagram can be closed by applying rule (wx) repeatedly on the latter term, once for each element in C

wap-b critical pair

- Term: $((\lambda y.m^A)^B\ n^C)^D$
- Condition of rule wap: $x \in B \wedge x \in C$
- Condition of rule b: none
- WW1: none
- WW2: $y \notin B \wedge y \notin C \wedge y \notin D$
- WW3: none
- WW4: $x \notin A \wedge x \notin D$
- WW5: none

$$\begin{aligned} ((\lambda y.m^A)^B\ n^C)^D &\xrightarrow[\text{wap}]{} ((\lambda y.m^A)^{(B\backslash x)}\ n^{(C\backslash x)})^{(Dx)} \\ &\xrightarrow[\text{b}]{} m^{(A(B\backslash x))}\langle y := n^{(C\backslash x)}\rangle_{\emptyset}^{(Dx)} \\ ((\lambda y.m^A)^B\ n^C)^D &\xrightarrow[\text{b}]{} m^{(AB)}\langle y := n^C\rangle_{\emptyset}^{D} \\ &\xrightarrow[\text{wx}]{} m^{(A(B\backslash x))}\langle y := n^{(C\backslash x)}\rangle_{\emptyset}^{(Dx)} \end{aligned}$$

wap-wgc critical pair

- Term: $(m^A\ n^B)^C\langle y := p^D\rangle_S^E$
- Condition of rule wap: $x \in A \wedge x \in B$
- Condition of rule wgc: $y \in C$
- WW1: none
- WW2: $y \notin D \wedge y \notin S \wedge y \notin E$
- WW3: none
- WW4: $x \notin E \wedge x \notin C \wedge y \notin B \wedge y \notin A \wedge y \notin E$
- WW5: none

$$\begin{aligned} (m^A\ n^B)^C\langle y := p^D\rangle_S^E &\xrightarrow[\text{wap}]{} (m^{(A\backslash x)}\ n^{(B\backslash x)})^{(Cx)}\langle y := p^D\rangle_S^E \\ &\xrightarrow[\text{wgc}]{} (m^{(A\backslash x)}\ n^{(B\backslash x)})^{(((C\backslash y)x)E)} \\ (m^A\ n^B)^C\langle y := p^D\rangle_S^E &\xrightarrow[\text{wgc}]{} (m^A\ n^B)^{((C\backslash y)E)} \\ &\xrightarrow[\text{wap}]{} (m^{(A\backslash x)}\ n^{(B\backslash x)})^{(((C\backslash y)E)x)} \end{aligned}$$

wap-xap critical pair with $x \neq y$

- Term: $(m^A\ n^B)^C\langle y := p^D\rangle_S^E$
- Condition of rule wap: $x \in A \wedge x \in B$
- Condition of rule xap: $y \notin C$
- WW1: none
- WW2: $y \notin D \wedge y \notin S \wedge y \notin E$
- WW3: none
- WW4: $x \notin E \wedge x \notin C$
- WW5: none

$$\begin{aligned} (m^A\ n^B)^C\langle y := p^D\rangle_S^E &\xrightarrow[\text{wap}]{} (m^{(A\backslash x)}\ n^{(B\backslash x)})^{(Cx)}\langle y := p^D\rangle_S^E \\ &\xrightarrow[\text{xap}]{} (m^{((A\backslash x)(Cx))}\langle y := p^D\rangle_S^{\emptyset}\ n^{((B\backslash x)(Cx))}\langle y := p^D\rangle_S^{\emptyset})^E \\ (m^A\ n^B)^C\langle y := p^D\rangle_S^E &\xrightarrow[\text{xap}]{} (m^{(AC)}\langle y := p^D\rangle_S^{\emptyset}\ n^{(BC)}\langle y := p^D\rangle_S^{\emptyset})^E \end{aligned}$$

wap-xap critical pair with $x = y$

- Term: $(m^A\ n^B)^C \langle x := p^D \rangle_S^E$
- Condition of rule wap: $x \in A \wedge x \in B$
- Condition of rule xap: $x \notin C$
- WW1: none
- WW2: $x \notin D \wedge x \notin S \wedge x \notin E$
- WW3: none
- WW4: $x \notin E \wedge x \notin C$
- WW5: none

$$
\begin{aligned}
(m^A\ n^B)^C \langle x := p^D \rangle_S^E &\xrightarrow[\text{wap}]{} (m^{(A\backslash x)}\ n^{(B\backslash x)})^{(Cx)} \langle x := p^D \rangle_S^E \\
&\xrightarrow[\text{wgc}]{} (m^{(A\backslash x)}\ n^{(B\backslash x)})^{(CE)} \\
(m^A\ n^B)^C \langle x := p^D \rangle_S^E &\xrightarrow[\text{xap}]{} (m^{(AC)} \langle x := p^D \rangle_S^{\emptyset}\ n^{(BC)} \langle x := p^D \rangle_S^{\emptyset})^E \\
&\xrightarrow[\text{wgc}]{} (m^{((A\backslash x)C)}\ n^{(BC)} \langle x := p^D \rangle_S^{\emptyset})^E \\
&\xrightarrow[\text{wgc}]{} (m^{((A\backslash x)C)}\ n^{((B\backslash x)C)})^E
\end{aligned}
$$

The diagram can be closed by applying rule (wap) repeatedly on the latter term, once for each element in C

wab-b critical pair

- Term: $((\lambda y.m^A)^B\ n^C)^D$
- Condition of rule wab: $x \in A$
- Condition of rule b: none
- WW1: none
- WW2: $y \notin B \wedge y \notin C \wedge y \notin D$
- WW3: none
- WW4: $x \notin D \wedge x \notin B$
- WW5: none

$$
\begin{aligned}
((\lambda y.m^A)^B\ n^C)^D &\xrightarrow[\text{wab}]{} ((\lambda y.m^{(A\backslash x)})^{(Bx)}\ n^C)^D \\
&\xrightarrow[\text{b}]{} m^{((A\backslash x)(Bx))} \langle y := n^C \rangle_{\emptyset}^D \\
((\lambda y.m^A)^B\ n^C)^D &\xrightarrow[\text{b}]{} m^{(AB)} \langle y := n^C \rangle_{\emptyset}^D
\end{aligned}
$$

wab-wgc critical pair

- Term: $(\lambda y.m^A)^B \langle z := n^C \rangle_S^D$
- Condition of rule wab: $x \in A$
- Condition of rule wgc: $z \in B$
- WW1: none
- WW2: $y \notin B \wedge y \notin C \wedge z \notin C \wedge y \notin S \wedge y \notin D \wedge z \notin S \wedge z \notin D$
- WW3: none
- WW4: $x \notin D \wedge x \notin B \wedge z \notin A \wedge z \notin D$
- WW5: none

$$
\begin{aligned}
(\lambda y.m^A)^B \langle z := n^C \rangle_S^D &\xrightarrow[\text{wab}]{} (\lambda y.m^{(A\backslash x)})^{(Bx)} \langle z := n^C \rangle_S^D \\
&\xrightarrow[\text{wgc}]{} (\lambda y.m^{(A\backslash x)})^{(((B\backslash z)x)D)} \\
(\lambda y.m^A)^B \langle z := n^C \rangle_S^D &\xrightarrow[\text{wgc}]{} (\lambda y.m^A)^{((B\backslash z)D)} \\
&\xrightarrow[\text{wab}]{} (\lambda y.m^{(A\backslash x)})^{(((B\backslash z)D)x)}
\end{aligned}
$$

wab-xab critical pair with $x \neq z$

- Term: $(\lambda y.m^{A})^{B}\langle z := n^{C}\rangle_{S}^{D}$
- Condition of rule wab: $x \in A$
- Condition of rule xab: $z \notin B$
- WW1: none
- WW2: $y \notin B \wedge y \notin C \wedge z \notin C \wedge y \notin S \wedge y \notin D \wedge z \notin S \wedge z \notin D$
- WW3: none
- WW4: $x \notin D \wedge x \notin B$
- WW5: none

$$\begin{aligned}(\lambda y.m^{A})^{B}\langle z := n^{C}\rangle_{S}^{D} &\xrightarrow[\text{wab}]{} (\lambda y.m^{(A\setminus x)})^{(Bx)}\langle z := n^{C}\rangle_{S}^{D} \\ &\xrightarrow[\text{xab}]{} (\lambda y.m^{((A\setminus x)(Bx))}\langle z := n^{(Cy)}\rangle_{(Sy)}^{\emptyset})^{D} \\ (\lambda y.m^{A})^{B}\langle z := n^{C}\rangle_{S}^{D} &\xrightarrow[\text{xab}]{} (\lambda y.m^{(AB)}\langle z := n^{(Cy)}\rangle_{(Sy)}^{\emptyset})^{D}\end{aligned}$$

wab-xab critical pair with $x = z$

- Term: $(\lambda y.m^{A})^{B}\langle x := n^{C}\rangle_{S}^{D}$
- Condition of rule wab: $x \in A$
- Condition of rule xab: $x \notin B$
- WW1: none
- WW2: $y \notin B \wedge y \notin C \wedge x \notin C \wedge y \notin S \wedge y \notin D \wedge x \notin S \wedge x \notin D$
- WW3: none
- WW4: $x \notin D \wedge x \notin B$
- WW5: none

$$\begin{aligned}(\lambda y.m^{A})^{B}\langle x := n^{C}\rangle_{S}^{D} &\xrightarrow[\text{wab}]{} (\lambda y.m^{(A\setminus x)})^{(Bx)}\langle x := n^{C}\rangle_{S}^{D} \\ &\xrightarrow[\text{wgc}]{} (\lambda y.m^{(A\setminus x)})^{(BD)} \\ (\lambda y.m^{A})^{B}\langle x := n^{C}\rangle_{S}^{D} &\xrightarrow[\text{xab}]{} (\lambda y.m^{(AB)}\langle x := n^{(Cy)}\rangle_{(Sy)}^{\emptyset})^{D} \\ &\xrightarrow[\text{wgc}]{} (\lambda y.m^{((A\setminus x)B)})^{D}\end{aligned}$$

The diagram can be closed by applying rule (wab) repeatedly on the latter term, once for each element in B.