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On mixed-integer sets with two integer variables

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Abstract

We show that every facet-defining inequality of the convex hull of a mixed-integer polyhedral set with two integer variables is a crooked cross cut (which we defined recently in [3]). We then extend this observation to show that crooked cross cuts give the convex hull of mixed-integer sets with more integer variables provided that the coefficients of the integer variables form a matrix of rank 2. We also present an alternative characterization of the crooked cross cut closure of mixed-integer sets similar to the one about the equivalence of different definitions of split cuts presented in Cook, Kannan, and Schrijver [4]. This characterization implies that crooked cross cuts dominate the 2-branch split cuts defined by Li and Richard [6]. Finally, we extend our results to mixed-integer sets that are defined as the set of points (with some components being integral) inside a general convex set.

1 Introduction

Given a polyhedral mixed-integer set

$$P = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \ : \ Ax + Gy = b,\ y \geq 0\},$$

where A, G and b have m rows and rational components, let P^{LP} denote its continuous relaxation. For fixed $\pi_1, \pi_2 \in \mathbb{Z}^{n_1}$, and $\gamma_1, \gamma_2 \in \mathbb{Z}$, we define the sets

$$D_1(\pi_1, \pi_2, \gamma_1, \gamma_2) = \{x \in \mathbb{R}^{n_1} \ : \ \pi_1 x \leq \gamma_1,\ (\pi_2 - \pi_1)x \leq \gamma_2 - \gamma_1\}, \tag{1}$$

$$D_2(\pi_1, \pi_2, \gamma_1, \gamma_2) = \{x \in \mathbb{R}^{n_1} \ : \ \pi_1 x \leq \gamma_1,\ (\pi_2 - \pi_1)x \geq \gamma_2 - \gamma_1 + 1\}, \tag{2}$$

$$D_3(\pi_1, \pi_2, \gamma_1, \gamma_2) = \{x \in \mathbb{R}^{n_1} \ : \ \pi_1 x \geq \gamma_1 + 1,\ \pi_2 x \leq \gamma_2\}, \text{ and} \tag{3}$$

$$D_4(\pi_1, \pi_2, \gamma_1, \gamma_2) = \{x \in \mathbb{R}^{n_1} \ : \ \pi_1 x \geq \gamma_1 + 1,\ \pi_2 x \geq \gamma_2 + 1\}. \tag{4}$$

Note that $Z^{n_1} \subseteq \bigcup_{k \in \{1,2,3,4\}} D_k(\pi_1, \pi_2, \gamma_1, \gamma_2)$; we denote the latter set by $D(\pi_1, \pi_2, \gamma_1, \gamma_2)$. We define the extension of $D(\pi_1, \pi_2, \gamma_1, \gamma_2)$ as $\bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2) = \{(x, y) \in \mathbb{R}^{n_1+n_2} \ : x \in D(\pi_1, \pi_2, \gamma_1, \gamma_2)\}$ and call this set a *crooked cross (CC) disjunction* for $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$, and call the set $\mathbb{R}^{n_1+n_2} \setminus \bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2)$ the *CC set* associated with the disjunction. We similarly define extensions of the sets in (1)-(4), and call each such extension an *atom* of the disjunction $\bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2)$.

If a linear inequality is valid for $P^{LP} \cap \bar{D}_k(\pi_1, \pi_2, \gamma_1, \gamma_2)$ for $k = 1, \ldots, 4$ then it is called a *CC cut* for P obtained from the disjunction $\bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2)$, see [3]. Note that multiple cuts can be derived from the same disjunction. As $P \subseteq \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \subseteq \bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2)$, CC cuts are valid for all points in P. In [3], we showed that CC cuts dominate the "2-branch split cuts", defined in Li and Richard [6], when the matrix

A has full row-rank. A consequence of the results in this paper is that this dominance relationship holds for arbitrary A.

Define $P_{CC}(\pi_1, \pi_2, \gamma_1, \gamma_2)$ as the convex hull of $P^{LP} \cap \bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2)$, i.e.,

$$P_{CC}(\pi_1, \pi_2, \gamma_1, \gamma_2) = \text{conv}\left(\bigcup_{i \in \{1,2,3,4\}} P^{LP} \cap D_i(\pi_1, \pi_2, \gamma_1, \gamma_2) \right).$$

By definition, this set equals the convex hull of points in P^{LP} not contained in the CC set associated with the disjunction $\bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2)$, and is the set of points in P^{LP} satisfying all CC cuts from this disjunction. The *CC closure* of P, denoted by P_{CC}, is the set of points in P^{LP} that satisfy all CC cuts obtained from all possible disjunctions for P. Clearly,

$$P_{CC} = \bigcap_{\pi_1, \pi_2 \in \mathbb{Z}^{n_1}} \bigcap_{\gamma_1, \gamma_2 \in \mathbb{Z}} P_{CC}(\pi_1, \pi_2, \gamma_1, \gamma_2).$$

2 Mixed-integer sets with two integer variables

As noted in [3], CC cuts generalize *split cuts* [4], which are defined by Cook, Kannan, and Schrijver as inequalities valid for $P^{LP} \cap \{(x, y) : \pi x \leq \gamma\}$ and $P^{LP} \cap \{(x, y) : \pi x \geq \gamma + 1\}$ for some $\pi \in \mathbb{Z}^{n_1}$ and $\gamma \in \mathbb{Z}$. Thus the CC closure of P is contained in its split closure. Moreover, it is known that when $n_1 = 1$, the split closure of P is integral. We next prove a similar result for the facet-defining inequalities of P when $n_1 = 2$.

Lemma 2.1. *If $n_1 = 2$, then any valid inequality for conv(P) is a CC cut and consequently $P_{CC} =$ conv(P).*

Proof. Let $cx + dy \geq f$ be a valid inequality for conv(P) and let $S \in \mathbb{R}^{2+n_2}$ be the points in P^{LP} that violate this inequality. That is,

$$S = \{(x, y) \in \mathbb{R}^{2+n_2} : cx + dy < f,\ Ax + Gy = b,\ y \geq 0\}.$$

If S is empty, then the inequality $cx + dy \geq f$ is valid for P^{LP} and therefore it is a CC cut. We therefore assume that $S \neq \emptyset$ and let $S^x = \text{proj}_x(S)$ denote the projection of S in the space of x variables. As $cx + dy \geq f$ is valid for P, S does not contain any integral points, that is, $S^x \cap \mathbb{Z}^2 = \emptyset$, and therefore S^x is a convex lattice-free set in $\mathbb{R}^2$. As all maximal convex lattice-free sets in $\mathbb{R}^2$ are contained in CC sets (see [3]), S^x is contained in some CC set $C = \mathbb{R}^2 \setminus D(\pi_1, \pi_2, \gamma_1, \gamma_2)$.

Consider the CC set $\bar{C}$ obtained by extending C. As $S^x \subseteq C$, we have $S \subseteq \bar{C}$ and therefore $cx + dy \geq f$ is valid for $P^{LP} \setminus \bar{C}$. In other words, $cx + dy \geq f$ is a CC cut for P. ∎

When $n_1 = 2$ and $m = 2$, the convex hull of P is given by the two-dimensional (2D) lattice-free cuts [2], and also by CC cuts. Lemma 2.1, therefore, generalizes the latter result to arbitrary m. It is, however, still an open question if CC cuts strictly dominate 2D lattice-free cuts or not. A possible way to answer this question is to study facet-defining inequalities for P when $n_1 = 2$ and $m > 2$ and investigate if they can always be derived as lattice-free cuts for a two-row relaxation of P. We refer the reader to [3] for a discussion on 2D lattice-free cuts for general mixed-integer sets.

3 An alternative characterization of the CC closure

Cook, Kannan, and Schrijver [4] gave an alternative definition of split cuts: they observe that the class of split cuts for P is equivalent to the class of inequalities valid for $P^{LP} \cap \{(x, y) \; : \; \pi x \in \mathbb{Z}\}$ for all integral vectors π. We next present an alternative characterization of the CC closure of P similar to the result on split cuts above.

Let

$$P_{\Pi} = \bigcap_{\pi_1, \pi_2 \in \mathbb{Z}^{n_1}} \text{conv}\left(\{(x, y) \in P^{LP} \; : \; \pi_1 x \in \mathbb{Z}, \pi_2 x \in \mathbb{Z}\}\right).$$

Theorem 3.1. *For a polyhedral mixed-integer set P, $P_{CC} = P_{\Pi}$.*

Proof. For fixed $\pi_1, \pi_2 \in \mathbb{Z}^{n_1}$, let

$$P_{\pi_1,\pi_2} = \text{conv}\left(\{(x, y) \in P^{LP} \; : \; \pi_1 x \in \mathbb{Z}, \pi_2 x \in \mathbb{Z}\}\right) \tag{5}$$

and consider a point $p \in P_{\pi_1,\pi_2}$. Clearly, p is a convex combination of points $p^k = (x^k, y^k), \; k \in K$, such that $p^k \in P^{LP}$ and $\pi_1 x^k, \pi_2 x^k \in \mathbb{Z}$ for all k. Then for any choice of $\gamma_1, \gamma_2 \in \mathbb{Z}$, it is clear that p^k does not belong to the CC set associated with the CC disjunction $\bar{D}(\pi_1, \pi_2, \gamma_1, \gamma_2)$, as it belongs to one of the atoms. In other words

$$p^k \in \bigcap_{\gamma_1,\gamma_2 \in \mathbb{Z}} P_{CC}(\pi_1, \pi_2, \gamma_1, \gamma_2)$$

for all $k \in K$ and therefore, $p \in \bigcap_{\gamma_1,\gamma_2 \in \mathbb{Z}} P_{CC}(\pi_1, \pi_2, \gamma_1, \gamma_2)$. Consequently,

$$P_{\pi_1,\pi_2} \subseteq \bigcap_{\gamma_1,\gamma_2 \in \mathbb{Z}} P_{CC}(\pi_1, \pi_2, \gamma_1, \gamma_2) \tag{6}$$

for any fixed $\pi_1, \pi_2 \in \mathbb{Z}^{n_1}$. Therefore

$$P_{\Pi} \; = \bigcap_{\pi_1,\pi_2 \in \mathbb{Z}^{n_1}} P_{\pi_1,\pi_2} \subseteq \bigcap_{\pi_1,\pi_2 \in \mathbb{Z}^{n_1}} \; \bigcap_{\gamma_1,\gamma_2 \in \mathbb{Z}} P_{CC}(\pi_1, \pi_2, \gamma_1, \gamma_2) \; = \; P_{CC}.$$

We will now prove the reverse inclusion, $P_{CC} \subseteq P_{\Pi}$, by showing that $P_{CC} \subseteq P_{\pi_1,\pi_2}$, for every choice of $\pi_1, \pi_2 \in \mathbb{Z}^{n_1}$. To prove this, note that P_{π_1,π_2} is the projection of the set conv (S) on the x, y variables where S is defined as

$$S = \{(x, y, z) \; : \; (x, y) \in P^{LP}, z \in \mathbb{Z}^2, z_1 = \pi_1 x, z_2 = \pi_2 x\}.$$

As S is a mixed-integer polyhedral set with only two integer variables, by Lemma 2.1, conv (S) equals S_{CC}. We will next show that $P_{CC} \subseteq \text{proj}_{x,y}(S_{CC}) = \text{proj}_{x,y}(\text{conv}\,(S))$.

Consider a CC cut for S, say

$$\alpha_1 z_1 + \alpha_2 z_2 + cx + dy \geq f, \tag{7}$$

derived from a CC disjunction $\bar{D}(\mu_1, \mu_2, \gamma_1, \gamma_2)$ on the z variables. Substituting out the z variables, we obtain the inequality

$$(\alpha_1 \pi_1 + \alpha_2 \pi_2 + c)x + dy \geq f \tag{8}$$

which is valid for $\text{proj}_{x,y}(S_{CC})$. This inequality is a CC cut for P obtained from the CC disjunction $\bar{D}(\pi'_1, \pi'_2, \gamma_1, \gamma_2)$ in the x, y space, where

$$\pi'_1 = \mu_1 \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \pi'_2 = \mu_2 \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}.$$

To see this, consider an atom of the disjunction $\bar{D}(\mu_1, \mu_2, \gamma_1, \gamma_2)$, say

$$\bar{D}_4(\mu_1, \mu_2, \gamma_1, \gamma_2) = \{(x, y, z) : \mu_1 z \geq \gamma_1 + 1, \mu_2 z \geq \gamma_2 + 1\}.$$

By definition, inequality (7) is valid for $S^{LP} \cap \bar{D}_4(\mu_1, \mu_2, \gamma_1, \gamma_2)$. Suppose inequality (8) is not valid for the atom $\bar{D}_4(\pi'_1, \pi'_2, \gamma_1, \gamma_2)$. By definition, there is a point $(\hat{x}, \hat{y})$ such that

$$(\hat{x}, \hat{y}) \in P^{LP},\ \pi'_1\hat{x} \geq \gamma_1 + 1,\ \pi'_2\hat{x} \geq \gamma_2 + 1 \text{ and } (\alpha_1\pi_1 + \alpha_2\pi_2 + c)\hat{x} + d\hat{y} < f.$$

Consider the point $(\hat{x}, \hat{y}, \hat{z})$ defined by setting $\hat{z}_1 = \pi_1\hat{x}$ and $\hat{z}_2 = \pi_2\hat{x}$. Clearly, this point satisfies

$$(\hat{x}, \hat{y}, \hat{z}) \in S^{LP},\ \mu_1\hat{z} \geq \gamma_1 + 1,\ \mu_2\hat{z} \geq \gamma_2 + 1 \text{ and } \alpha_1\hat{z}_1 + \alpha_2\hat{z}_2 + c\hat{x} + d\hat{y} < f,$$

which is a contradiction. Thus inequality (8) is a CC cut for P.

As every valid inequality for $\text{proj}_{x,y}(\text{conv}\,(S))$ is implied by a nonnegative linear combination of CC cuts (7) for S, we can conclude that a valid inequality for $\text{proj}_{x,y}(\text{conv}\,(S))$ is implied by a nonnegative linear combination of CC cuts (8) for P. ∎

Let t be a fixed integer, and consider a disjunctive cut obtained by modifying the sets in (1) and (2) as follows:

$$D_1^t(\pi_1, \pi_2, \gamma_1, \gamma_2) = \{x \in \mathbb{R}^{n_1} \ :\ \pi_1 x \leq \gamma_1,\ (\pi_2 - t\pi_1)x \leq \gamma_2 - t\gamma_1\}, \quad (9)$$
$$D_2^t(\pi_1, \pi_2, \gamma_1, \gamma_2) = \{x \in \mathbb{R}^{n_1} \ :\ \pi_1 x \leq \gamma_1,\ (\pi_2 - t\pi_1)x \geq \gamma_2 - t\gamma_1 + 1\}. \quad (10)$$

Such cuts are referred to as parametric cross cuts in [3]. Note that when $t = 1$, parametric cross cuts are just CC cuts, and when $t = 0$, they reduce to the 2-branch split cuts of Li and Richard [6] (also referred to as *cross cuts* in [3]). Let $P^t(\pi_1, \pi_2, \gamma_1, \gamma_2)$ be the set of points in P^{LP} that satisfy all parametric cross cuts derived from the above disjunction. Observe that

$$P_{\pi_1,\pi_2} \subseteq \bigcap_{\gamma_1,\gamma_2\in\mathbb{Z}} P^t(\pi_1, \pi_2, \gamma_1, \gamma_2),$$

where P_{π_1,π_2} is defined in (5). Therefore, arguing as in the proof of Theorem 3.1, one obtains that

$$P_{\Pi} \subseteq \bigcap_{\pi_1,\pi_2\in\mathbb{Z}^{n_1}} \bigcap_{\gamma_1,\gamma_2\in\mathbb{Z}} P^t(\pi_1, \pi_2, \gamma_1, \gamma_2).$$

Therefore, we have the following corollary of Theorem 3.1.

Corollary 3.2. *P_{CC} equals the set of points in P^{LP} that satisfy all (i.e., for any t) parametric cross cuts for P. In particular, P_{CC} is contained in the 2-branch split closure of P.*

4 Mixed-integer sets with simple structure

We next extend Lemma 2.1 to show that CC cuts are sufficient to define the convex hull of the mixed-integer set P for $n_1 > 2$ provided that the coefficients of the integer variables form a matrix of rank 2.

Theorem 4.1. *If $rank(A) = 2$, then any facet-defining inequality for conv (P) is a CC cut and consequently $P_{CC} =$ conv (P).*

Proof. We will show that conv $(P) = P_\Pi$, and by Theorem 3.1 the result will follow. Clearly conv $(P) \subseteq P_\Pi$. We will next show the reverse inclusion.

As A is rational, we can assume, without loss of generality, that A, G are scaled such that A is an integral matrix. As $rank(A) = 2$, there exists a unimodular matrix $U \in \mathbb{Z}^{n_1 \times n_1}$ with the property that $AU = \left[\begin{array}{cc} T & 0 \end{array}\right]$ where $T \in \mathbb{Z}^{m\times 2}$ and has rank 2; see [7]. Let

$$Q = \{(z, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \; : \; AUz + Gy = b, \; y \geq 0\}.$$

As only the first two columns of AU are nonzero, it follows that the variables $z_3, \ldots, z_{n_1}$ are not restricted in any way. Rewriting $Ax+Gy = b$ as $AUU^{-1}x+Gy = b$, it follows that there is a one-to-one correspondence between the points in P^{LP} and Q^{LP} via the mapping $(x, y) \to (U^{-1}x, y)$, whose inverse mapping is $(z, y) \to (Uz, y)$. We denote the latter mapping by h. Furthermore, as U is unimodular, the same one-to-one correspondence holds between the integral points in P and Q as well.

Consider a point $(\bar{x}, \bar{y}) \in P_\Pi$. Let π_1 and π_2 stand for the first and second rows of U^{-1}, respectively. By the definition of P_Π, $(\bar{x}, \bar{y})$ is in the convex hull of points in P^{LP} satisfying $\pi_1 x \in \mathbb{Z}$, and $\pi_2 x \in \mathbb{Z}$. In other words,

$$(\bar{x}, \bar{y}) = \sum_{i=1}^{t} \lambda_i (x^i, y^i) \text{ where } \sum_{i=1}^{t} \lambda_i = 1, \text{ and } \lambda_i \geq 0 \text{ for } i = 1, \ldots, t,$$
$$(x^i, y^i) \in P^{LP} \text{ and } \pi_1 x^i \in \mathbb{Z}, \pi_2 x^i \in \mathbb{Z} \text{ for } i = 1, \ldots, t.$$

Therefore,

$$(U^{-1}\bar{x}, \bar{y}) = \sum_{i=1}^{t} \lambda_i (U^{-1}x^i, y^i) \in Q^{LP}, \text{ and } (U^{-1}x^i, y^i) \in Q^{LP} \text{ for } i = 1, \ldots, t. \tag{11}$$

Now let $(z^i, y^i) = (U^{-1}x^i, y^i)$ for any $i \in \{1, \ldots, t\}$. As $\pi_1 x^i, \pi_2 x^i \in \mathbb{Z}$ for all i, the first two components of z^i are integral, but the remaining components may not be integral. But the vector consisting of all but the first two components of z^i can be expressed as a convex combination of integral vectors in $\mathbb{Z}^{n_1-2}$. In other words,

$$(z^i, y^i) = \sum_{j=1}^{s_i} \mu_i^j (w^{ij}, y^i) \text{ where } \sum_{j=1}^{s_i} \mu_i^j = 1 \text{ and } w^{ij} \in \mathbb{Z}^{n_1}, \; \mu_i^j \geq 0 \text{ for } j = 1 \ldots, s_i, \tag{12}$$

and the first two components of w^{ij} equal the first two components of z^i. Now each vector (w^{ij}, y_i) is a point in Q. Combining equations (11) and (12), we conclude that $(U^{-1}\bar{x}, \bar{y})$ is a convex combination of some (integral) points $q_1, \ldots, q_l \in Q$. Therefore $h(U^{-1}\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$ is a convex combination of $h(q_1), \ldots, h(q_l)$; but the latter collection of points is contained in P, and thus $(\bar{x}, \bar{y}) \in$ conv (P). ∎

5 Some extensions

We next consider implications of our results in three different settings. First consider a mixed-integer set $Q = \{x : Ax \leq b, x \in \mathbb{Z}^n\}$ such that $A = [v \; T]$, where T is a totally unimodular $m \times (n-1)$ matrix, and v is an arbitrary integer vector, and A and v have m rows. Further, let b be integral. Eisenbrand, Oriolo, Stauffer, and Ventura [5] observe that conv (Q) equals the split closure of Q. Now assume that A has two columns with integral components (assume they are the first two columns of A) such that the remaining columns of A form a totally unimodular matrix. Rewriting $Ax \leq b$ as $Ax + Iy = b, y \geq 0$ where I is an $m \times m$ identity matrix, we can conclude from Theorem 3.1 that CC cuts give the facet-defining inequalities of

$$Q' = \text{conv}\left(\{x \in \mathbb{R}^n : Ax \leq b, x_1, x_2 \in \mathbb{Z}\}\right).$$

On the other hand, for arbitrary integers t_1 and t_2, the points x satisfying $Ax \leq b, x_1 = t_1, x_2 = t_2$ form a polyhedron with integral vertices (in all components), i.e., $Q' = \text{conv}\,(Q)$. Therefore, all facet-defining inequalities of $\text{conv}\,(Q)$ are CC cuts. Consequently, $Q_{CC} = \text{conv}\,(Q)$. More generally, if in the mixed-integer set P, A consists of two columns with integral components, the remaining columns of $[A\ G]$ form a totally unimodular matrix and b is integral, then the above observation implies that $P_{CC} = \text{conv}\,(P)$.

Our results can also be applied to the generalization of the *two-row continuous group relaxation* studied by Andersen, Louveaux, and Weismantel [1], where some of the continous variables have upper bounds in addition to lower bounds of zero. As the number of integer variables in this set is two, all facet-defining inequalities are given by CC cuts.

Finally, consider a set of the form

$$P = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \ : (x, y) \in \mathcal{C}\}, \tag{13}$$

where $\mathcal{C}$ is a general convex set. In this setting P_{CC} and P_{Π} can be defined as before. Now observe that the proofs of Lemma 2.1 and Theorem 3.1 do not use the polyhedrality of the continuous relaxation of P. Therefore we obtain the following result.

Proposition 5.1. *For P given by (13):*

1. *If $n_1 = 2$, then $P_{CC} = \text{conv}\,(P)$;*
2. *$P_{CC} = P_{\Pi}$.*

Finally note that Theorem 4.1 can also be generalized to the mixed integer convex programming setting. Consider P given by

$$P = \{(x, y) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \ : (Ax, y) \in \mathcal{C}\}, \tag{14}$$

where $A \in \mathbb{Z}^{m \times n_1}$ and $\mathcal{C}$ is a general convex set. In this case, again note that the proof of Theorem 2.1 does not use the polyhedrality of the continuous relaxation of P. Thus, we obtain the following result.

Proposition 5.2. *Let P be given by (14). If $\text{rank}(A) = 2$, then $P_{CC} = \text{conv}\,(P)$.*

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