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Tracking Dynamic Networks under Sampling Constraints: Supporting Materials

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I. Introduction

This report contains supporting materials such as proofs, discussions, and additional numerical results, for [1]. See the original paper for terms and definitions.

II. Proofs of Selected Theorems

A. Relationship between Whittle's Index and Myopic Index

Proposition 2.1: For the link sampling problem, the Whittle's index (if exists) is no smaller than the myopic index, *i.e.*, $W(x) \geq Y(x)$ $\forall x \in [0,\ 1]$. Moreover, $W(x) \to Y(x)$ as $|p_{11} - p_{01}| \to 0$.

Proof: Due to the convexity of $V_{\beta,m}(x)$ [2], $V_{\beta,m}(\mathcal{T}(x)) \leq xV_{\beta,m}(p_{11}) + (1-x)V_{\beta,m}(p_{01})$. Since Whittle's index must satisfy $V_{\beta,W(x)}(x;\ U=0) = V_{\beta,W(x)}(x;\ U=1)$, plugging in the Bellman equations for $V_{\beta,m}(x;\ U=0)$ and $V_{\beta,m}(x;\ U=1)$ gives $W(x) + \max(x,\ 1-x) \geq 1$ and hence $W(x) \geq Y(x)$. As $p_{11} \to p_{01}$, equality will be achieved. ■

B. Threshold Structure of the Optimal Policy for Single-Armed Bandit with Subsidy

Lemma 2.2: The optimal policy for the single-armed bandit with subsidy m is a threshold policy: $\pi^*_m(x) = 1$ if and only if $\tau^-(m) < x < \tau^+(m)$, *i.e.*, $\mathcal{P}(m) = [0,\ \tau^-(m)] \cup [\tau^+(m),\ 1]$.

Proof: Note that $V_{\beta,m}(x;\ U=1)$ is linear in x. By the convexity of the value function [2], we have that $V_{\beta,m}(x; U=0)$ is also convex in x. At $x=0$ or 1, we have

$$\begin{cases} V_{\beta,m}(0;\ U=0) = m + 1 + \beta V_{\beta,m}(p_{01}), \\ V_{\beta,m}(1;\ U=0) = m + 1 + \beta V_{\beta,m}(p_{11}); \end{cases}$$

$$\begin{cases} V_{\beta,m}(0;\ U=1) = 1 + \beta V_{\beta,m}(p_{01}), \\ V_{\beta,m}(1;\ U=1) = 1 + \beta V_{\beta,m}(p_{11}), \end{cases}$$

which implies that the endpoints of $V_{\beta,m}(x; U=0)$ are above, equal to, or below those of $V_{\beta,m}(x; U=1)$ for $m>0, m=0$, or $m<0$, respectively, as illustrated in Fig. 1. Due to the convexity of $V_{\beta,m}(x; U=0)$, it must have at most two intersections with $V_{\beta,m}(x; U=1)$, denoted by $\tau^-(m)$ and $\tau^+(m)$; for cases without intersection, define $\tau^-(m) = \tau^+(m) \triangleq \tau^*$, where τ^* is the tangent point under a certain m_{max}. Then as in Fig. 1, $V_{\beta,m}(x;\ U=1) > V_{\beta,m}(x;\ U=0)$ if and only if $x \in (\tau^-(m),\ \tau^+(m))$. ■

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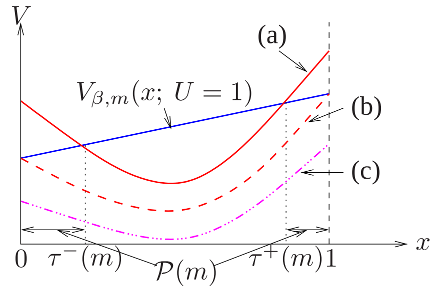


Fig. 1. Threshold structure of the optimal policy: Value function $V_{\beta,m}(x;\ U=0)$ for (a) $m>0$, (b) $m=0$, and (c) $m<0$.

C. Pseudo-Linear Form of Value Function $V_{\beta,m}(x)$

Lemma 2.3: Given thresholds $\tau^-(m)$ and $\tau^+(m)$, define coefficients $a_i,\ b_i$ $(i=1,\ 2)$ as in (3–6) and define functions

$$\begin{aligned} a(x) &\triangleq \frac{1-\beta^{L(x)}}{1-\beta} + \beta^{L(x)+1}\mathcal{T}^{L(x)}(x)a_2 \\ &\quad + \beta^{L(x)+1}(1-\mathcal{T}^{L(x)}(x))a_1, \quad (1) \\ b(x) &\triangleq f(x;\ L(x)) + \beta^{L(x)} + \beta^{L(x)+1}\mathcal{T}^{L(x)}(x)b_2 \\ &\quad + \beta^{L(x)+1}(1-\mathcal{T}^{L(x)}(x))b_1, \quad (2) \end{aligned}$$

where $L(x) \triangleq \mathcal{L}(x;\ \tau^-(m),\ \tau^+(m))$. Then the value function is equal to $V_{\beta,m}(x) = a(x)m + b(x)$, with end values $V_{\beta,m}(p_{01}) = a_1 m + b_1$, and $V_{\beta,m}(p_{11}) = a_2 m + b_2$.

Proof: The linear forms of $V_{\beta,m}(p_{01})$ and $V_{\beta,m}(p_{11})$ are obtained by simply rewriting their expressions in (29, 30). Substituting them into (28) gives the linear form of $V_{\beta,m}(x)$. ■

D. Piecewise-Linear Property of $a(x),\ b(x)$

Proposition 2.4: The functions $a(x),\ b(x)$ defined in Lemma 3 of [1] are both piecewise-linear functions of x.

Proof: The proof is based on the piecewise-constant property of $L(x)$. For fixed $L(x) \equiv l$, it is easy to see that $\mathcal{T}^{L(x)}(x)$ and $f(x;\ L(x))$ are both linear in x, which implies the linearity of $a(x)$ and $b(x)$. Thus, each constant piece of $L(x)$ corresponds to a linear piece of $a(x)$ and $b(x)$, respectively. ■

E. Monotonicity of $\tau^-(m),\ \tau^+(m)$

Lemma 2.5: The thresholds $\tau^-(m),\ \tau^+(m)$ are monotone increasing and decreasing, respectively, with m for $\beta \leq 0.5$.

Proof: It suffices to show ([3]) that for any given thresholds $(\tau^-(m'),\ \tau^+(m'))$ corresponding to some $m' \in [0,\ m_{\text{max}}]$,

$$\frac{\partial}{\partial m}V_{\beta,m}(x;\ U=0) \geq \frac{\partial}{\partial m}V_{\beta,m}(x;\ U=1) \quad (7)$$

$$a_1 \triangleq \frac{(1-\beta^{L_1+1}\mathcal{T}^{L_1}(p_{11}))\left(\frac{1-\beta^{L_2}}{1-\beta}\right)+\beta^{L_2+1}\mathcal{T}^{L_2}(p_{01})\left(\frac{1-\beta^{L_1}}{1-\beta}\right)}{\eta}, \tag{3}$$

$$b_1 \triangleq \frac{(1-\beta^{L_1+1}\mathcal{T}^{L_1}(p_{11}))(f(p_{01};\ L_2)+\beta^{L_2})+\beta^{L_2+1}\mathcal{T}^{L_2}(p_{01})(f(p_{11};\ L_1)+\beta^{L_1})}{\eta}, \tag{4}$$

$$a_2 \triangleq \frac{(1-\beta^{L_2+1}(1-\mathcal{T}^{L_2}(p_{01})))\left(\frac{1-\beta^{L_1}}{1-\beta}\right)+\beta^{L_1+1}(1-\mathcal{T}^{L_1}(p_{11}))\left(\frac{1-\beta^{L_2}}{1-\beta}\right)}{\eta}, \tag{5}$$

$$b_2 \triangleq \frac{(1-\beta^{L_2+1}(1-\mathcal{T}^{L_2}(p_{01})))(f(p_{11};\ L_1)+\beta^{L_1})+\beta^{L_1+1}(1-\mathcal{T}^{L_1}(p_{11}))(f(p_{01};\ L_2)+\beta^{L_2})}{\eta}. \tag{6}$$

for $x=\tau^-(m'),\ \tau^+(m')$, because this condition guarantees that for all $m>m'$, $V_{\beta,m}(x;\ U=0)\geq V_{\beta,m}(x;\ U=1)$ at $x=\tau^-(m')$ and $\tau^+(m')$, implying $\tau^-(m)\geq\tau^-(m')$, $\tau^+(m)\leq\tau^+(m')$. Next, by Lemma 2.3, we have

$$V_{\beta,m}(x;\ U=0)=m+\max(x,\ 1-x)+\beta a(\mathcal{T}(x))m +\beta b(\mathcal{T}(x)),$$
$$V_{\beta,m}(x;\ U=1)=1+\beta x(a_2m+b_2)+\beta(1-x)(a_1m+b_1).$$

Substituting these into (7) and noting that a_i, b_i and $a(\cdot)$, $b(\cdot)$ are constants for fixed thresholds reduce (7) into

$$1+\beta a(\mathcal{T}(x))\geq\beta x a_2+\beta(1-x)a_1,\quad x=\tau^-(m'),\ \tau^+(m'). \tag{8}$$

For $\beta\leq 1/2$, $1+\beta a(\mathcal{T}(x))\geq 1\geq\beta/(1-\beta)$. Meanwhile, $a_1,\ a_2\leq 1/(1-\beta)$ (since they are the discounted total passive time) implies $\beta/(1-\beta)\geq\beta x a_2+\beta(1-x)a_1$, proving (8). ■

III. Supporting Steps in Computing Whittle's Index

A. *Computing Hitting Time* $\mathcal{L}(x;\ c_1,\ c_2)$

For the ease of presentation, we introduce the following auxiliary functions:

$$g_1(y;\ x) \triangleq \frac{\log\left(\max(y-x_0,\ 0)\right)-\log|x-x_0|}{\log|p_{11}-p_{01}|}, \tag{9}$$

$$g_2(y;\ x) \triangleq \frac{\log(\max(x_0-y,\ 0))-\log|x-x_0|}{\log|p_{11}-p_{01}|}. \tag{10}$$

Then some calculation will show that for $p_{11}>p_{01}$,

$$\mathcal{L}(x;c_1,c_2)=\begin{cases}\min\mathbb{N}\cap(g_1(c_2;\ x),\ g_1(c_1;\ x)) & \text{if } x\geq x_0,\\ \min\mathbb{N}\cap(g_2(c_1;\ x),\ g_2(c_2;\ x)) & \text{if } x<x_0,\end{cases} \tag{11}$$

where $\mathbb{N}$ denotes the set of nonnegative integers. For $p_{11}<p_{01}$, if $x\geq x_0$,

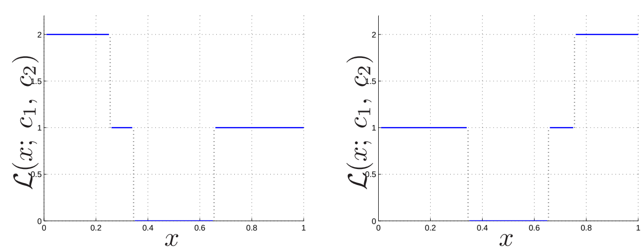
$$\mathcal{L}(x;\ c_1,\ c_2)=\min\big(\min\mathbb{N}_{\mathrm{o}}\cap(g_2(c_1;\ x),\ g_2(c_2;\ x)), \min\mathbb{N}_{\mathrm{e}}\cap(g_1(c_2;\ x),\ g_1(c_1;\ x))\big), \tag{12}$$

where $\mathbb{N}_{\mathrm{e}}$ is the set of nonnegative even numbers $(0,2,\ldots)$ and $\mathbb{N}_{\mathrm{o}}$ the set of nonnegative odd numbers $(1,\ 3,\ldots)$. Similarly, if $x<x_0$,

$$\mathcal{L}(x;\ c_1,\ c_2)=\min\big(\min\mathbb{N}_{\mathrm{o}}\cap(g_1(c_2;\ x),\ g_1(c_1;\ x)), \min\mathbb{N}_{\mathrm{e}}\cap(g_2(c_1;\ x),\ g_2(c_2;\ x))\big). \tag{13}$$

Sanity check: Consider the case $p_{11}>p_{01}$ (positively correlated arm). If $c_1<x<c_2$, it is easy to see that $\mathcal{L}(x;\ c_1,\ c_2)=0$. Indeed, in (11), either $g_1(c_2;\ x)<0$ and $g_1(c_1;\ x)>0$ (if $x\geq x_0$), or $g_2(c_1;\ x)<0$ and $g_2(c_2;\ x)>0$ (if $x<x_0$), yielding $\mathcal{L}(x;\ c_1,\ c_2)=0$. If $x,\ x_0\geq c_2$ or $x,\ x_0\leq c_1$, it is easy to see that $\mathcal{L}(x;\ c_1,\ c_2)=\infty$. Indeed, for $x,\ x_0\geq c_2$, either $g_1(c_2;\ x)=\infty$ and $g_1(c_1;\ x)=\infty$ (if $x\geq x_0$), or $g_2(c_1;\ x)<0$ and $g_2(c_2;\ x)<0$ (if $x<x_0$); for $x,x_0\leq c_1$, either $g_1(c_2;x)<0$ and $g_1(c_1;x)<0$ (if $x\geq x_0$), or $g_2(c_1;\ x)=\infty$ and $g_2(c_2;\ x)=\infty$ (if $x<x_0$), both yielding $\mathcal{L}(x;c_1,c_2)=\infty$ (define $(\infty,\infty)\triangleq\emptyset$ and $\min\emptyset\triangleq\infty$). Similar sanity check holds for $p_{11}<p_{01}$.

Due to the integral requirement, $\mathcal{L}(x;\ c_1,\ c_2)$ is always a piecewise-constant function of x for any c_1, c_2 and p_{01}, p_{11}, as illustrated in Fig. 2.



(a) (b)

Fig. 2. Piece-wise constant property of $\mathcal{L}(x;\ c_1,\ c_2)$ ($c_1=0.35$, $c_2=0.65$): (a) $p_{01}=0.25$, $p_{11}=0.65$ (positively correlated); (b) $p_{01}=0.65$, $p_{11}=0.25$ (negatively correlated).

B. *Computing Auxiliary Function* $f(x;\ L)$

First of all, note that since L may be infinity, we cannot always compute $f(x;\ L)$ by the definition. Fortunately, due to its special structure, we can provide a closed-form solution as follows. The computation is based on the observation that $f(x;\ L)$ is a piecewise power series. We will treat positively-correlated and negatively-correlated arms separately.

For positively-correlated arms (*i.e.*, $p_{11}>p_{01}$), let $L_{1/2}$ denote the hitting time (*i.e.*, smallest l) for $\mathcal{T}^l(x)$ to cross $1/2$, if the crossing occurs within L steps. That is, $L_{1/2}\triangleq\min(L,\ \mathcal{L}(x;\ 1/2,\ 1))$ if $x\leq 1/2$, and $\min(L,\ \mathcal{L}(x;\ 0,\ 1/2))$ if $x>1/2$. It is easy to see that

$$f(x;L)=\begin{cases}\displaystyle\sum_{i=0}^{L_{1/2}-1}\beta^i(1-\mathcal{T}^i(x))+\sum_{i=L_{1/2}}^{L-1}\beta^i\mathcal{T}^i(x) & \text{if } x\leq\frac{1}{2},\\ \displaystyle\sum_{i=0}^{L_{1/2}-1}\beta^i\mathcal{T}^i(x)+\sum_{i=L_{1/2}}^{L-1}\beta^i(1-\mathcal{T}^i(x)) & \text{o.w.}\end{cases}$$

By plugging in the expression of $\mathcal{T}^l(x)$, it can be shown that for $x \leq 1/2$,

$$f(x;\ L) = \frac{1-\beta^{L_{1/2}}}{1-\beta} - \frac{x_0(1-2\beta^{L_{1/2}}+\beta^L)}{1-\beta}$$
$$+\frac{(x_0-x)(1-2(\beta(p_{11}-p_{01}))^{L_{1/2}}+(\beta(p_{11}-p_{01}))^L)}{1-\beta(p_{11}-p_{01})}, \quad (14)$$

and for $x > 1/2$,

$$f(x;\ L) = \frac{\beta^{L_{1/2}}-\beta^L}{1-\beta} + \frac{x_0(1-2\beta^{L_{1/2}}+\beta^L)}{1-\beta}$$
$$-\frac{(x_0-x)(1-2(\beta(p_{11}-p_{01}))^{L_{1/2}}+(\beta(p_{11}-p_{01}))^L)}{1-\beta(p_{11}-p_{01})}. \quad (15)$$

For negatively-correlated arms (*i.e.*, $p_{11} < p_{01}$), the even steps $\mathcal{T}^{2k}(x)$ and the odd steps $\mathcal{T}^{2k+1}(x)$ will converge toward x_0 from opposite directions. Let $\mathcal{K}(x; c_1, c_2) \triangleq \min\{k : \mathcal{T}^{2k} \in (c_1,\ c_2)\}$ denote the hitting time of $(c_1,\ c_2)$ from x by taking two steps at a time, and $\mathcal{K}_{1/2}(x)$ the number of step pairs needed to first cross $1/2$ starting from x, *i.e.*, $\mathcal{K}_{1/2}(x) \triangleq \mathcal{K}(x; 1/2, 1)$ if $x \leq 1/2$, and $\mathcal{K}_{1/2}(x) \triangleq \mathcal{K}(x; 0, 1/2)$ if $x > 1/2$. Define $K_{1/2} \triangleq \min(\mathcal{K}_{1/2}(x),\ \lfloor (L-1)/2 \rfloor + 1)$, and $K'_{1/2} \triangleq \min(\mathcal{K}_{1/2}(\mathcal{T}(x)),\ \lfloor (L-2)/2 \rfloor + 1)$. Note that $\mathcal{K}(x;\ c_1,\ c_2)$ (thus $K_{1/2},\ K'_{1/2}$) can be computed similarly as $\mathcal{L}(x;\ c_1,\ c_2)$ (see Section III-A). We can write $f(x;\ L)$ as

$$f(x;\ L) = \sum_{k=0}^{K_{1/2}-1} \beta^{2k} \max(\mathcal{T}^{2k}(x),\ 1-\mathcal{T}^{2k}(x)) \quad (16)$$
$$+ \sum_{k=K_{1/2}}^{\lfloor \frac{L-1}{2} \rfloor} \beta^{2k} \max(\mathcal{T}^{2k}(x),\ 1-\mathcal{T}^{2k}(x)) \quad (17)$$
$$+ \sum_{k=0}^{K'_{1/2}-1} \beta^{2k+1} \max(\mathcal{T}^{2k+1}(x),\ 1-\mathcal{T}^{2k+1}(x)) \quad (18)$$
$$+ \sum_{k=K'_{1/2}}^{\lfloor \frac{L-2}{2} \rfloor} \beta^{2k+1} \max(\mathcal{T}^{2k+1}(x),\ 1-\mathcal{T}^{2k+1}(x)). \quad (19)$$

This decomposition guarantees that (16) is on the same side of $1/2$ as x, (17) on the other side, (18) on the same side as $\mathcal{T}(x)$, and (19) on the other side.

We now calculate (16–19) by cases. If $x \leq 1/2$, then (16) is equal to $\sum_{k=0}^{K_{1/2}-1} \beta^{2k}(1-\mathcal{T}^{2k}(x))$ and (17) to $\sum_{k=K_{1/2}}^{\lfloor \frac{L-1}{2} \rfloor} \beta^{2k}\mathcal{T}^{2k}(x)$. Calculation will yield the closed-form results as in (20–21). Otherwise (*i.e.*, $x > 1/2$), (16) becomes $\sum_{k=0}^{K_{1/2}-1} \beta^{2k}\mathcal{T}^{2k}(x)$ and (17) becomes $\sum_{k=K_{1/2}}^{\lfloor \frac{L-1}{2} \rfloor} \beta^{2k}(1-\mathcal{T}^{2k}(x))$, which yield (22–23). Similarly, if $\mathcal{T}(x) \leq 1/2$, then (18) becomes $\sum_{k=0}^{K'_{1/2}-1} \beta^{2k+1}(1-\mathcal{T}^{2k+1}(x))$ and (19) becomes $\sum_{k=K'_{1/2}}^{\lfloor \frac{L-2}{2} \rfloor} \beta^{2k+1}\mathcal{T}^{2k+1}(x)$, which gives the results in (24–25). Otherwise (*i.e.*, $\mathcal{T}(x) > 1/2$), (18) is $\sum_{k=0}^{K'_{1/2}-1} \beta^{2k+1}\mathcal{T}^{2k+1}(x)$ and (19) is $\sum_{k=K'_{1/2}}^{\lfloor \frac{L-2}{2} \rfloor} \beta^{2k+1}(1-\mathcal{T}^{2k+1}(x))$, yielding (26–27).

C. *Computing Value Function* $V_{\beta,m}(x)$

It is shown in [1] that given m, $\tau^-(m)$, and $\tau^+(m)$, we can compute the value function of the single-armed bandit with subsidy m by

$$V_{\beta,m}(x) = \frac{(1-\beta^L)m}{1-\beta} + f(x;\ L) + \beta^L$$
$$+\beta^{L+1}\mathcal{T}^L(x)V_{\beta,m}(p_{11}) + \beta^{L+1}(1-\mathcal{T}^L(x))V_{\beta,m}(p_{01}). \quad (28)$$

The only unknowns left are $V_{\beta,m}(p_{11})$, $V_{\beta,m}(p_{01})$. Note that $x = p_{11}$ or p_{01} should also satisfy (28), giving us two equations with two unknowns. Solving these equations yields the results in (29, 30).

IV. Additional Numerical Results

We first verify the properties of $a(x)$, $b(x)$ given in Proposition 2.4. As shown in Fig. 3, $a(x)$ and $b(x)$ are indeed piecewise-linear functions of x.

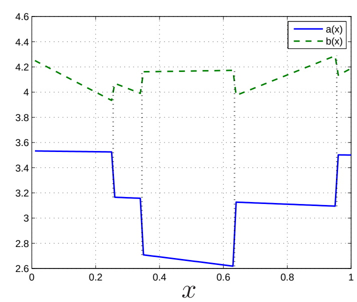


Fig. 3. Coefficients $a(x)$, $b(x)$ vs. x ($\beta = 0.8$, $p_{01} = 0.25$, $p_{11} = 0.65$, $m = 0.4039$, $\tau^-(m) = 0.35$, $\tau^+(m) = 0.6329$).

We then verify the convexity of $V_{\beta,m}(x)$ with respect to m, which is needed to ensure that the performance upper bounds derived in [1] are well-defined and the associated subsidies are unique. We plot the value function $V_{\beta,m}(x_0)$ (with the steady state as the initial state) for the single-armed bandit as a function of subsidy m under positive and negative correlation, respectively, as shown in Fig. 4. In both cases, $V_{\beta,m}(x)$ is a monotone increasing, convex function of m. This observation holds even if we vary the parameters (not shown). Therefore, the expressions within the minimization of the bounds are convex in m (or $\mathbf{m}$), and hence the bounds are well-defined and the dual variables (subsidies) achieving them are unique.

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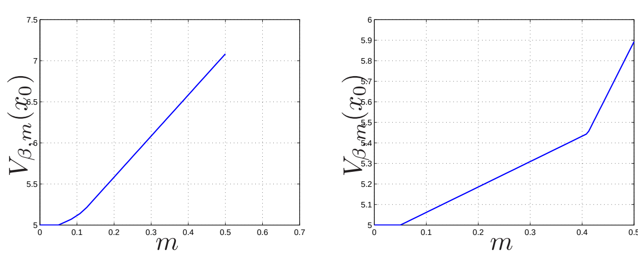
[3] K. Liu and Q. Zhao, "Indexability of Restless Bandit Poblems and Optimality of Whittle's Index for Dynamic Multichannel Access." Submitted to IEEE Trans. on Information Theory, Jan 2010.

$$
\begin{aligned}
x \le \frac{1}{2}:\ (16) &= \frac{(1-x_0)(1-\beta^{2K_{1/2}})}{1-\beta^2} + \frac{(x_0-x)[1-(\beta^2(p_{11}-p_{01})^2)^{K_{1/2}}]}{1-\beta^2(p_{11}-p_{01})^2}, && (20)\\
(17) &= \frac{x_0(\beta^{2K_{1/2}}-\beta^{2(\lfloor\frac{L-1}{2}\rfloor+1)})}{1-\beta^2} - \frac{(x_0-x)[(\beta^2(p_{11}-p_{01})^2)^{K_{1/2}}-(\beta^2(p_{11}-p_{01})^2)^{\lfloor\frac{L-1}{2}\rfloor+1}]}{1-\beta^2(p_{11}-p_{01})^2}; && (21)\\
x > \frac{1}{2}:\ (16) &= \frac{x_0(1-\beta^{2K_{1/2}})}{1-\beta^2} - \frac{(x_0-x)[1-(\beta^2(p_{11}-p_{01})^2)^{K_{1/2}}]}{1-\beta^2(p_{11}-p_{01})^2}, && (22)\\
(17) &= \frac{(1-x_0)(\beta^{2K_{1/2}}-\beta^{2(\lfloor\frac{L-1}{2}\rfloor+1)})}{1-\beta^2} + \frac{(x_0-x)[(\beta^2(p_{11}-p_{01})^2)^{K_{1/2}}-(\beta^2(p_{11}-p_{01})^2)^{\lfloor\frac{L-1}{2}\rfloor+1}]}{1-\beta^2(p_{11}-p_{01})^2}. && (23)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}(x) \le \frac{1}{2}:\ (18) &= \frac{(1-x_0)\beta(1-\beta^{2K'_{1/2}})}{1-\beta^2} + \frac{(x_0-x)\beta(p_{11}-p_{01})[1-(\beta^2(p_{11}-p_{01})^2)^{K'_{1/2}}]}{1-\beta^2(p_{11}-p_{01})^2}, && (24)\\
(19) &= \frac{x_0\beta(\beta^{2K'_{1/2}}-\beta^{2(\lfloor\frac{L-2}{2}\rfloor+1)})}{1-\beta^2} - \frac{(x_0-x)\beta(p_{11}-p_{01})[(\beta^2(p_{11}-p_{01})^2)^{K'_{1/2}}-(\beta^2(p_{11}-p_{01})^2)^{\lfloor\frac{L-2}{2}\rfloor+1}]}{1-\beta^2(p_{11}-p_{01})^2}; && (25)\\
\mathcal{T}(x) > \frac{1}{2}:\ (18) &= \frac{x_0\beta(1-\beta^{2K'_{1/2}})}{1-\beta^2} - \frac{(x_0-x)\beta(p_{11}-p_{01})[1-(\beta^2(p_{11}-p_{01})^2)^{K'_{1/2}}]}{1-\beta^2(p_{11}-p_{01})^2}, && (26)\\
(19) &= \frac{(1-x_0)\beta(\beta^{2K'_{1/2}}-\beta^{2(\lfloor\frac{L-2}{2}\rfloor+1)})}{1-\beta^2} + \frac{(x_0-x)\beta(p_{11}-p_{01})[(\beta^2(p_{11}-p_{01})^2)^{K'_{1/2}}-(\beta^2(p_{11}-p_{01})^2)^{\lfloor\frac{L-2}{2}\rfloor+1}]}{1-\beta^2(p_{11}-p_{01})^2}. && (27)
\end{aligned}
$$

$$
\begin{aligned}
V_{\beta,m}(p_{01}) &= \frac{(1-\beta^{L_1+1}\mathcal{T}^{L_1}(p_{11}))v_2+\beta^{L_2+1}\mathcal{T}^{L_2}(p_{01})v_1}{\eta}, && (29)\\
V_{\beta,m}(p_{11}) &= \frac{(1-\beta^{L_2+1}(1-\mathcal{T}^{L_2}(p_{01})))v_1+\beta^{L_1+1}(1-\mathcal{T}^{L_1}(p_{11}))v_2}{\eta}, && (30)
\end{aligned}
$$

where $L_1 \stackrel{\Delta}{=} \mathcal{L}(p_{11};\tau^-(m),\tau^+(m))$, $L_2 \stackrel{\Delta}{=} \mathcal{L}(p_{01};\tau^-(m),\tau^+(m))$, $v_1 \stackrel{\Delta}{=} \frac{(1-\beta^{L_1})m}{1-\beta}+f(p_{11};L_1)+\beta^{L_1}$, $v_2 \stackrel{\Delta}{=} \frac{(1-\beta^{L_2})m}{1-\beta}+f(p_{01};L_2)+\beta^{L_2}$, and $\eta \stackrel{\Delta}{=} (1-\beta^{L_1+1}\mathcal{T}^{L_1}(p_{11}))(1-\beta^{L_2+1}(1-\mathcal{T}^{L_2}(p_{01})))-\beta^{L_1+L_2+2}(1-\mathcal{T}^{L_1}(p_{11}))\mathcal{T}^{L_2}(p_{01})$.



(a) positively-correlated arm (b) negatively-correlated arm

Fig. 4. $V_{\beta,m}(x)$ vs. m ($\beta = 0.8$): (a) $p_{01} = 0.05$, $p_{11} = 0.45$; (b) $p_{01} = 0.45$, $p_{11} = 0.05$.