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On the p-Median Polytope of Fork-free Graphs

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Abstract

We study a prize collecting version of the uncapacitated facility location problem and of the p-median problem. We say that the uncapacitated facility location polytope has the intersection property, if adding the extra equation that fixes the number of opened facilities does not create any fractional extreme point. We show that this property holds if and only if the graph has no fork. A fork is a particular subgraph. We give a complete description of the polytope for this class of graphs.

Keywords: uncapacitated facility location, p-median.

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1 Introduction

The *uncapacitated facility location problem* (UFLP) and the *p-median problem* (pMP) are among the most studied problems in combinatorial optimization. Here we deal with a *prize collecting* version of them, that we denote by UFLP$'$ and pMP$'$ respectively. We assume that $G = (U \cup V, A)$ is a bipartite directed graph, not necessarily connected and with no isolated nodes. The arcs are directed from U to V. The nodes in U are called *customers* and the nodes in V are called *locations*. Each location v has a weight $f(v)$ that corresponds to the revenue obtained by opening a facility at that location, minus the cost of building this facility. Each arc (u, v) has a weight $c(u, v)$ that represents the revenue obtained by assigning the customer u to the opened facility at location v, minus the cost originated by this assignment. The difference between the UFLP and the UFLP$'$ is that in the first problem each customer *must* be assigned to an opened facility, whereas in the second problem a customer could be not assigned to any facility. If the number of opened facilities is required to be exactly p, we have the pMP and pMP$'$ respectively.

An integer programming formulation of the UFLP$'$ is

(1) $\max \sum_{(u,v)\in A} c(u, v)x(u, v) + \sum_{v\in V} f(v)y(v)$

(2) $\sum_{v:(u,v)\in A} x(u, v) \le 1 \quad \forall u \in U,$

(3) $x(u, v) \le y(v) \quad \forall (u, v) \in A,$

(4) $y(v) \le 1 \quad \forall v \in V,$

(5) $x(u, v) \ge 0 \quad \forall (u, v) \in A,$

(6) $y(v) \in \{0, 1\} \quad \forall v \in V,$

(7) $x(u, v) \in \{0, 1\} \quad \forall (u, v) \in A.$

If inequalities (2) are set to equations, then we have a formulation of the UFLP. If we add the equation

$$\sum_{v\in V} y(v) = p \tag{8}$$

to (1)-(7), we have a formulation of the pMP$'$ and if inequalities (2) are set to equations, we have the pMP.

For a given bipartite graph $G = (U \cup V, A)$, let $UFLP'(G)$ be the convex

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hull of the solutions of (2)-(7), and $pMP'(G)$ be the convex hull of the solutions of (2)-(8). Analogously we can define the polytopes $UFLP(G)$ and $pMP(G)$. Notice that $UFLP(G)$ is a face of $UFLP'(G)$, and $pMP(G)$ is a face of $pMP'(G)$. Thus a characterization of $pMP'(G)$ and $UFLP'(G)$ yields to a characterization of $pMP(G)$ and $UFLP(G)$. We denote by $P(G)$ the linear relaxation of $UFLP'(G)$ defined by (2)-(5), and by $P_p(G)$ the linear relaxation of $pMP'(G)$ defined by (2)-(5) and (8).

Let us call the graph of Figure 1 a *fork*, and denote it by F. By setting

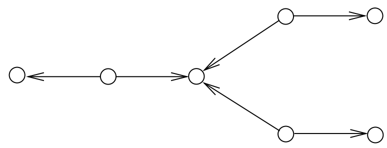

Fig. 1. A fork.

each variable associated with F to $\frac{1}{2}$, we obtain a fractional extreme point of $P_2(F)$. In general assume that a bipartite graph G contains a fork F. We can set to $\frac{1}{2}$ all variables associated with F, and set to zero the remaining variables. This is an extreme point of $P_2(G)$. Such fractional extreme points may be cutoff by using a set of valid inequalities for $pMP'(G)$ introduced in [6].

In this extended abstract we describe some of the results of [2]. Here we will consider a set of valid inequalities for $UFLP'(G)$ introduced in [5]. We call them *$CJPR$-inequalities*, using the initials of the authors' names. These inequalities are also valid for $pMP'(G)$ since $pMP'(G) \subseteq UFLP'(G)$. We will show that the addition of these inequalities to $P(G)$ yields an integral polytope, when G does not contain a fork.

We say that $UFLP'(G)$ has the *intersection property* with respect to (8), if the intersection of $UFLP'(G)$ with the hyperplane defined by (8), is an integral polytope for every nonnegative integer p. We show that $UFLP'(G)$ has this property if and only if G contains no fork. Based on this we show that the addition of the $CJPR$-inequalities to the system defining $P_p(G)$ gives an integral polytope for every nonnegative integer p, if and only if G does not contain a fork. This is our main result. We also give combinatorial polynomial time algorithms to solve the problems pMP', $UFLP'$, pMP and $UFLP$ when the underlying graph does not contain a fork.

This paper is organized as follows. In Section 2, we give some notations and definitions and some preliminary results that will be useful all along the paper. Section 2.4 gives a complete characterization of $UFLP'(G)$ if G has no fork. In Section 3, we discuss the intersection of the polytope $UFLP'(G)$ with the hyperplane defined by (8), we also establish $pMP'(G)$ for this class of graphs.

2 Preliminaries

2.1 Some definitions and notations

Let $G = (U \cup V, A)$ be a bipartite graph. Denote by $\beta(G)$ the *covering number* of G, that is the minimum number of locations $v \in V$ needed to cover all customers $u \in U$. Let $F \subseteq A$ be a subset of arcs in A. Denote by $N^-(F)$ (resp. $N^+(F)$) the set of nodes in U (resp. V) incident to an arc in F. Let $G(F) = (N^-(F) \cup N^+(F), F)$ be the bipartite subgraph of G induced by F. Hence $\beta(G(F))$ is the minimum number of nodes in $N^+(F)$ necessary to cover all the nodes in $N^-(F)$ using only arcs in F.

We denote by $\delta^+(S)$ the set of arcs $(u, v) \in A$ with $u \in S$ and by $\delta^-(W)$ the set of arcs $(u, v) \in A$ with $v \in W$. For a node $u \in U$ (resp. $v \in V$), we write $\delta^+(u)$ (resp. $\delta^-(v)$) instead of $\delta^+(\{u\})$ (resp. $\delta^-(\{v\})$). Usually $d(v)$ denotes the *degree* of a node v in a simple graph, that is the number of edges incident to v. We keep this notation in our case, that is $d(u) = |\delta^+(u)|$ for $u \in U$ and $d(v) = |\delta^-(v)|$ for $v \in V$. If there is a risk of confusion we specify by $d_G(v)$ the degree of the node v with respect to a given graph G.

2.2 CPJR-inequalities

Theorem 1 *[5] Let G be a bipartite directed graph, for any subgraph $G(F)$ of G the inequality*

$$(9) \quad \sum_{(u,v)\in F} x(u, v) - \sum_{v\in N^+(F)} y(v) \leq |N^-(F)| - k,$$

is valid for $UFLP'(G)$ if and only if $k \leq \beta(G(F))$.

Let $G(F)$ be a subgraph of G induced by $F \subseteq A$, where each node in $N^+(F)$ has degree two. In this case $\beta(G(F)) \geq \left\lceil \frac{|N^-(F)|}{2} \right\rceil$. It follows that the following inequalities:

$$(10) \quad \sum_{(u,v)\in F} x(u, v) - \sum_{v\in N^+(F)} y(v) \leq \left\lfloor \frac{|N^-(F)|}{2} \right\rfloor,$$

for all $F \subseteq A$, where $d_{G(F)}(v) = 2$ for all $v \in N^+(F)$, $|N^-(F)| \geq 3$ and odd, are of type (9). Thus inequalities (10) are valid for $UFLP'(G)$. We call them

CJPR-inequalities. These inequalities are $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts, cf. [4]. They can be obtained by adding some of the inequalities (2)-(5) multiplied by 1/2 and by rounding down the right hand side. An *odd cycle* in a bipartite graph is a cycle having $2(2k+1)$ nodes, for some integer $k \geq 1$. When the subgraph $G(F)$ of G is an odd cycle, then inequalities (10) are known as the *odd cycle inequalities*, their separation can be done in polynomial time see [1,4].

2.3 Decomposition of graphs with no fork

Let $G = (U \cup V, A)$ be a bipartite graph with no fork. If there are two nodes u and v in U with $\delta^+(u) = \{(u, w_1), (u, w_2)\}$ and $\delta^+(v) = \{(v, w_1), (v, w_2)\}$, we say that u and v are *twins*. This is an equivalence relation. The first step of the decomposition is as follows:

Step 1. For every equivalence class of U by the relation twin, leave only one node (and remove the others). The next step is as follows.

Step 2. Remove every node $u \in U$ with degree equal to one.

Lemma 2.1 *[2] After applying Steps 1 and 2, we are left with a bipartite graph with no fork where each of its connected components either*

(i) contains exactly three locations, or

(ii) the degree of each location is at most two.

2.4 The characterization of $UFLP'(G)$

In [2] we proved the following.

Theorem 2 *[2] Let $G = (U \cup V, A)$ be a bipartite graph. If G has no fork, then $UFLP'(G)$ is described by inequalities (2)-(5) and inequalities (10).*

3 The intersection property for $UFLP'(G)$ and the characterization of $pMP'(G)$

Let P be an integral polytope in $\mathbb{R}^n$. Let q be an integer valued row vector in $\mathbb{R}^n$ such that the g.c.d. of its components is one. For an integer p let $H_p = \{x \in \mathbb{R}^n : qx = p\}$.

We say that P has the *intersection property* with respect to q, if for every integer p the polytope $P \cap H_p$ is integral. For an undirected graph $H = (V, E)$, the *stable set polytope* is the convex hull of all incidence vectors of stable sets of H. The following result has been shown in [3].

Theorem 3 *The stable set polytope of a graph $H = (V, E)$ has the intersection property with respect to $\sum_{v \in V} x(v) = p$ if and only if H is a clawfree graph.*

A *clawfree* graph is a graph that does not contain the bipartite graph $K_{1,3}$ as an induced subgraph. Given a bipartite graph $G = (U \cup V, A)$, we will show, in this section, that the polytope $UFLP'(G)$ has the intersection property with respect to $\sum_{v \in V} y(v) = p$, if and only if G has no fork. To obtain this result we modified the proof given in [3].

Theorem 4 *[2] Let $G = (U \cup V, A)$ a bipartite graph. $UFLP'(G)$ has the intersection property with respect to $\sum_{v \in V} y(v) = p$ if and only if G has no fork.*

From Theorems 2 and 4, we obtain our main result.

Theorem 5 *[2] Let $G = (U \cup V, A)$ be a bipartite graph. The polytope $pMP'(G)$ is described by (2)-(5), (8) and (10) if and only if G does not contain a fork as a subgraph.*

Corollary 6 *Let $G = (U \cup V, A)$ a bipartite graph. If G has no fork, then $pMP(G)$ is described by (3)-(5), inequalities (2) transformed into equations, (8) and (10).*

Using the decomposition procedure of Subsection 2.3 we can solve the separation problem for (10) when the graph has no fork, see [2]. This implies the following.

Theorem 7 *The problems $UFLP'$, pMP', $UFLP$ and pMP can be solved in polynomial time when the underlying graph does not contain a fork as a subgraph.*

Using the decomposition procedure of Subsection 2.3, we can give combinatorial algorithms for these problems. Roughly speaking, these problems reduce to a sequence of matching problems, see [2].

References

[1] M. Baïou and F. Barahona, *On the integrality of some facility location polytopes*, SIAM J. Discrete Math., 23 (2009), pp. 665–679.

[2] M. Baïou, F. Barahona, and J. Correa, *On the p-median polytope and the intersection property*, technical report, IBM Watson Research Center, 2008.

[3] G. Calvillo, *The concavity and intersection properties for integral polyhedra*, Ann. Discrete Math., 8 (1980), pp. 221–228. Combinatorics 79 (Proc. Colloq.,

Univ. Montréal, Montreal, Que., 1979), Part I.

[4] A. Caprara and M. Fischetti, $\{0, \frac{1}{2}\}$-*Chvátal-Gomory cuts*, Math. Programming, 74 (1996), pp. 221–235.

[5] D. C. Cho, E. L. Johnson, M. Padberg, and M. R. Rao, *On the uncapacitated plant location problem. I. Valid inequalities and facets*, Math. Oper. Res., 8 (1983), pp. 579–589.

[6] I. R. de Farias, Jr., *A family of facets for the uncapacitated p-median polytope*, Oper. Res. Lett., 28 (2001), pp. 161–167.