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Dispatch-and-Search: Supporting Proofs

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Dispatch-and-Search: Supporting Proofs

Ting He, Ananthram Swami, and Kang-Won Lee

I. Introduction

This report contains supporting proofs for [1]. See the original paper for problem formulation and notations.

II. Selected Theorems

Lemma 2.1: For any policy π, its reward R^π_∞ is monotone increasing with the conditional contact probability p^π_t for any t.

Lemma 2.2: For any t and any π,

$$p^\pi_t \le \frac{B_{t,k}}{\max(1-\sum_{j=1}^{t-1} B_{j,k},\ B_{t,k})} =: \overline{p}_t. \quad (1)$$

Corollary 2.3: For k ferries and a domain of size N (cells),

$$R^{\text{MY}}_\infty \ge \beta k/[N(1-\beta)+\beta k]. \quad (2)$$

Corollary 2.4: If the initial belief is the steady-state distribution $\mathbf{b}_*$ ($\mathbf{b}_* = \mathbf{P}^T\mathbf{b}_*$), then

$$\overline{R}_\infty = \beta^{T_0}\left[1 - B_{*,k}\left(T_0-1\right)\right] + \frac{B_{*,k}(\beta-\beta^{T_0})}{1-\beta}, \quad (3)$$

where $B_{*,k} \triangleq \max_{|a|=k}\sum_{s\in a} b_*(s)$ and $T_0 = \lceil B^{-1}_{*,k}\rceil$. Moreover, if $\mathbf{P}$ is doubly stochastic[1], then the above reduces to

$$\overline{R}_\infty \le \beta k/[(1-\beta)N]. \quad (4)$$

Lemma 2.5: For each domain with $a^g(d) > 0$ and local control reward $R^{\pi_l}_d$,

$$R^{\pi_l}_d - \beta^\Delta \le \mathbb{E}[\beta^{\upsilon(d)}] \le R^{\pi_l}_d. \quad (5)$$

Lemma 2.6: Under myopic search policies and the asymptotically approximate myopic dispatch policy in (17) of [1], let $d_\tau \triangleq \arg\max_d m_\tau(d)/N_d$ denote the domain served in round τ. We have

$$q_\tau(d_\tau) \ge 1 - \left(1-\frac{K}{N_{d_\tau}}\right)^\Delta =: \underline{q}_\tau, \quad (6)$$

$$\mathbb{E}[\beta^{\upsilon_\tau(d_\tau)}] \ge \frac{\beta K\left(1-\beta^\Delta\left(1-\frac{K}{N_{d_\tau}}\right)^\Delta\right)}{N_{d_\tau}-\beta(N_{d_\tau}-K)}. \quad (7)$$

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[1]That is, each column of $\mathbf{P}$ also sums up to one.

Theorem 2.7: Under myopic search policies and the asymptotically approximate myopic dispatch policy in (17) of [1], the discounted total throughput is lower bounded by

$$R^{\pi_g}_\infty \ge \mathbb{E}[\sum_{\tau=1}^\infty \beta^{(\Delta+1)(\tau-1)} r(\mathbf{m}_\tau)|\mathbf{m}_1=\mathbf{0}] =: \underline{R}^{\pi_g}_\infty, \quad (8)$$

where the expectation is over the Markov chain $\{\mathbf{m}_\tau\}^\infty_{\tau=1}$ specified in (20) of [1].

III. Proofs of Selected Theorems

A. Proof of Lemma 2.1

By equation (8) in [1], we have

$$\begin{aligned}\frac{\partial R^\pi_\infty}{\partial p^\pi_t} &= \beta^t\prod_{j=1}^{t-1}(1-p^\pi_j) - \sum_{s=t+1}^\infty \beta^s p^\pi_s \prod_{j=1, j\neq t}^{s-1}(1-p^\pi_j)\\ &= \left[\prod_{j=1}^{t-1}(1-p^\pi_j)\right]\left(\beta^t - \mathbb{E}[\beta^{\Upsilon^\pi}|\Upsilon^\pi > t]\right) \ge 0.\end{aligned}$$

Therefore, R^π_∞ is monotone increasing with p^π_t. ■

B. Proof of Lemma 2.2

First, we prove by induction that (recall $\mathbf{b}^{(t)} = (\mathbf{P}^T)^t\mathbf{b}_0$)

$$b^\pi_t(i) \le \frac{b^{(t)}(i)}{1-\sum_{j=1}^{t-1}\sum_{s\in a^\pi_j} b^{(j)}(s)}, \quad \forall i\in\mathcal{S},\ t\ge 1. \quad (9)$$

For $t=1$, $\mathbf{b}^\pi_1 = \mathbf{P}^T\mathbf{b}_0 = \mathbf{b}^{(1)}$. For $t>1$,

$$b^\pi_t(i) = \sum_{j\notin a^\pi_{t-1}} \frac{b^\pi_{t-1}(j)P_{j,i}}{1-\sum_{s\in a^\pi_{t-1}} b^\pi_{t-1}(s)} \le \frac{b^{(t)}(i)}{1-\sum_{j=1}^{t-1}\sum_{s\in a^\pi_j} b^{(j)}(s)},$$

obtained by applying (9) for $b^\pi_{t-1}(j)$. This proves (9).

Then, we apply the above to $p^\pi_t = \sum_{s\in a^\pi_t} b^\pi_t(s)$: $p^\pi_t \le [\sum_{s\in a^\pi_t} b^{(t)}(s)]/[1-\sum_{j=1}^{t-1}\sum_{s\in a^\pi_j} b^{(j)}(s)] \le \overline{p}_t$ for $\overline{p}_t$ defined as in (1). ■

C. Proof of Corollary 2.3

The proof is based on the fact that $p^{\text{MY}}_t \ge k/N$, $\forall t$. By Lemma 2.1,

$$R^{\text{MY}}_\infty \ge \sum_{t=1}^\infty \beta^t\frac{k}{N}(1-\frac{k}{N})^{t-1} = \frac{\beta k}{N(1-\beta)+\beta k}.$$

■

D. Proof of Corollary 2.4

If $\mathbf{b}_0 = \mathbf{b}_*$, then $\mathbf{b}^{(t)} \equiv \mathbf{b}_*$, $\forall t$. Accordingly, $B_{t,k} \equiv B_{*,k}$, and $T_0 = \lceil B_{*,k}^{-1} \rceil$. Plugging these into equation (9) of [1] gives (3). If, in addition, $\mathbf{P}$ is doubly stochastic, then it is known that $\mathbf{b}_*$ is uniform, i.e., $B_{*,k} = k/N$, applying which to (3) gives

$$\overline{R}_\infty = \beta^{\lceil N/k \rceil} \left[1 - \frac{k}{N}\left(\lceil \frac{N}{k} \rceil - 1\right)\right] + \frac{k(\beta - \beta^{\lceil N/k \rceil})}{N(1-\beta)}. \tag{10}$$

If we maximize the right-hand side of (10) with respect to $\lceil \frac{N}{k} \rceil$, calculation shows that the maximum is achieved at $\lceil \frac{N}{k} \rceil = \frac{N}{k}$, $\frac{N}{k} + 1$, or $\frac{N}{k} - \frac{1}{1-\beta} - \frac{1}{\log \beta} + 1$. At the first two values, the right-hand side is $\frac{k\beta(1-\beta^{N/k})}{N(1-\beta)} < \frac{\beta k}{(1-\beta)N}$; at the third value, it is $\frac{k}{N}\left(\beta^{\frac{N}{k}+1-\frac{1}{1-\beta}-\frac{1}{\log \beta}} / \log \beta + \frac{\beta}{1-\beta}\right) < \frac{\beta k}{(1-\beta)N}$. Thus, $\overline{R}_\infty \leq \beta k/[(1-\beta)N]$. ■

E. Proof of Lemma 2.5

The upper bound holds because $R_d^{\pi_l} = \mathbb{E}[\beta^{\Upsilon(d)}]$ while $\mathbb{E}[\beta^{\upsilon(d)}]$ is a truncated average. The lower bound is because $R_d^{\pi_l} - \mathbb{E}[\beta^{\upsilon(d)}] = \sum_{t=\Delta+1}^{\infty} \beta^t \Pr\{\Upsilon(d) = t\} \leq \beta^{\Delta} \Pr\{\Upsilon(d) > \Delta\} \leq \beta^{\Delta}$. ■

F. Proof of Lemma 2.6

Let p_t be the conditional contact probability in domain d_τ. By definition, we have $p_t \geq K/N_{d_\tau}$ for the myopic search policy. From the analysis of local control (Section 4.3 in [1]), we have $q_\tau(d_\tau) = 1 - \prod_{t=1}^{\Delta}(1 - p_t)$ and $\mathbb{E}[\beta^{\upsilon_\tau(d_\tau)}] = \sum_{t=1}^{\Delta} \beta^t \Pr\{\upsilon_\tau(d_\tau) = t\} = \sum_{t=1}^{\Delta} \beta^t p_t \prod_{j=1}^{t-1}(1-p_j)$, both increasing with p_t. Plugging in the lower bound of p_t yields the results. ■

G. Proof of Theorem 2.7

Due to discount, the total throughput is an increasing function of the probability of delivery $q_\tau(d_\tau)$ and is thus lower bounded if $q_\tau(d_\tau)$ is replaced by its lower bound via the Markov chain defined in (20) of [1]. For given $(\mathbf{b}_\tau, \mathbf{m}_\tau)$, the expected immediate reward under the asymptotically approximate myopic dispatch policy given by (17) of [1] is $m_\tau(d_\tau)\mathbb{E}[\beta^{\upsilon_\tau(d_\tau)}]$, which is lower bounded by $r(\mathbf{m}_\tau)$ for any $\mathbf{b}_\tau$ due to (7). Combining these two proves the result. ■

REFERENCES

[1] T. He, A. Swami, and K.-W. Lee, "Dispatch-and-Search: Dynamic Multi-Ferry Control in Partitioned Mobile Networks," in *Proc. ACM MobiHoc*, May 2011.