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Lattice-Free Sets, Branching Disjunctions, and Mixed-Integer Programming

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Abstract

In this paper we study the relationship between valid inequalities for mixed-integer sets, lattice-free sets associated with these inequalities and structured disjunctive cuts, especially the t-branch split cuts introduced by Li and Richard (2008). By analyzing n-dimensional lattice-free sets, we prove that every facet-defining inequality of the convex hull of a mixed-integer polyhedral set with n integer variables is a t-branch split cut for some positive integer t which is a function of n, and not of the data defining the polyhedral set. We use this result to give a finitely convergent, pure cutting-plane algorithm to solve mixed-integer programs. We also show that the minimum value t, for which all facets of n-dimensional polyhedral mixed-integer sets can be expressed as t-branch split cuts, grows exponentially with n. In particular, when $n = 3$, we observe that not all facet-defining inequalities are 6-branch split cuts. We analyze the cases when $n = 2$ and $n = 3$ in detail, and show that an explicit classification of maximal lattice-free sets is not necessary to express facet-defining inequalities as branching disjunctions with a small number of atoms.

1 Introduction

There has been much recent work on obtaining valid inequalities for mixed-integer sets from specific families of lattice-free sets, and also on explaining such valid inequalities as disjunctive cuts. In this paper, we study the connection between valid inequalities for mixed-integer sets, lattice-free sets, structured disjunctive cuts, and cutting-plane algorithms based on such cuts.

In the 1960s, Gomory [15] presented a cutting-plane algorithm which solves any *pure* integer program (without continuous variables) by generating a finite sequence of *cutting planes*, or inequalities satisfied by all integral solutions of the initial linear inequalities defining the integer program. Gomory [16] later introduced another cutting-plane algorithm based on the *Gomory mixed-integer (GMI) cut* for solving mixed-integer programs (MIP). However, this algorithm does not always terminate; Gomory proved finite termination only when the optimal objective value is known a priori to be integral.

Cook, Kannan and Schrijver [9] introduced split cuts and gave a very simple MIP, involving only three variables (with two of them being integer variables), which cannot be solved with split cuts alone. Their result, along with the result of Nemhauser and Wolsey [25] that GMI cuts are equivalent to split cuts (see also [10]) implies that in fact Gomory's algorithm will fail to terminate on this three variable MIP. The question of finite termination has received some attention over the last few years. Andersen, Louveaux, Weismantel and Wolsey [2] show how the three variable MIP of Cook, Kannan and Schrijver can be solved with *2D lattice-free* cuts, which generalize split cuts. They also implicitly show how to use 2D lattice-free cuts to solve two-row continous group relaxations. Another generalization of split cuts is the t-branch split cuts studied by Li and Richard [23]. They give a family of examples with n integer variables for $n \geq 3$ such that each example in their family has a valid inequality with infinite rank with respect to 2-branch split cuts, and conjecture that $(n-1)$-branch split cuts are also not enough to obtain finite rank. Yet another generalization of the split cut is the crooked cross cut, a type of structured disjunctive cut introduced by Dash, Dey and Günlük [11], and shown to be equivalent to 2D lattice-free cuts in some cases.

In other work on this topic, Owen and Mehrotra [26] presented an algorithm which only generates cutting planes based on simple variable disjunctions which converges to the optimal solution of any MIP, but does not terminate in finite time. When the linear constraints defining an MIP form a polytope, Adams and Sherali [1] presented a hierarchy of relaxations which yield the convex hull of solutions in finitely many steps. Under the same assumptions, Markus Jörg presented an algorithm which generates disjunctive cuts and solves an MIP in finite time in his PhD thesis [18] and in an unpublished manuscript [19]. Chen, Küçükyavuz, and Sen [8] also gave a disjunctive cutting-plane algorithm to solve such MIPs in finite time. Recently, Del Pia and Weismantel [13] present a relaxation of mixed-integer programs based on lattice-free cuts which can be iterated finitely many times to obtain the convex hull of integer solutions, thus yielding a finite cutting-plane algorithm for general mixed-integer programs.

It is clear from the preceding discussion that there is a connection between lattice-free sets, disjunctive cuts, and finite cutting-plane algorithms for MIPs, and our goal in this paper is to enhance knowledge of these relationships. We are primarily interested in studying structured disjunctive cuts. In [11], the authors show that a cut derived from a maximal lattice-free convex set B in $\mathbb{R}^2$ can be expressed as a *crooked-cross* cut, and therefore as a 3-branch split cut (both of these are types of disjunctive cuts). They show further in [12] that that result implies that every facet-defining inequality for a mixed-integer program with two integer variables is a crooked-cross cut (and is also a 3-branch split cut). In this paper, we are interested in extending the latter result to higher dimensions, namely we want to express facet-defining inequalities for MIPs with n integer variables as t-branch split cuts for some t, and thus extend the above results from $\mathbb{R}^2$ to $\mathbb{R}^n$. Our interest in obtaining such an extension is also motivated by the current research on *classifying* all maximal lattice-free sets in $\mathbb{R}^3$ and in higher dimensions, i.e., enumerating them by number of facets, and number of integer points in the interior of each facet, up to unimodular transformations. The search for such a classification result is motivated by the belief that knowing the list of maximal lattice-free sets can lead to useful families of cutting planes. However, this classification project seems very difficult in $\mathbb{R}^3$ and in higher dimensions; further using the classes of lattice-free sets effectively is not trivial. Even in $\mathbb{R}^2$, there are uncountably many maximal lattice-free quadrilaterals, and it is not clear how to choose from this list in order to generate cuts. On the other hand, there are only countably many 2-branch split cuts, and they imply all cuts obtained from maximal lattice-free quadrilaterals. In other words, we are interested in obtaining classes of cutting planes which are not hard to describe, but yet contain fairly complicated classes such as all cuts based on lattice-free sets in $\mathbb{R}^n$, without *explicit classification of maximal lattice-free sets*.

We show that ideas used in Lenstra's algorithm [22] on integer programming in fixed dimensions easily yield a number t which is a function of n alone. Further, we give a strong lower bound on t by presenting a family of MIPs, one for each n, such that t-branch split cuts yield the integer hull only when $t \geq 3 \cdot 2^{n-2}$ if the MIP has n integer variables. When $n = 3$, we show that 6-branch split cuts do not suffice to yield the integer hull, but 21-branch split cuts do. In order to obtain this result, we show that the lattice-width of lattice-free convex sets in $\mathbb{R}^3$ is at most 4.2439. We then observe that our characterization of facets of an MIP as t-branch split cuts leads to a trivial, pure cutting-plane algorithm for mixed-integer programs which terminates in finitely many steps, and is different from the algorithm of Del Pia and Weismantel. We also study some types of disjunctive cuts different from t-branch split cuts when $n = 2$. We also show that any lattice-free set in $\mathbb{R}^2$ is contained in a set $\{x \in \mathbb{R}^2 : b \le a^T x \le b+3\}$ for some $a \in \mathbb{Z}^2$ and an integer b.

2 Preliminaries

In this paper we work with polyhedral mixed-integer sets of the form

$$P = \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^l \; : \; Ax + Gy = b, \; y \ge 0\},$$

where A, G and b have m rows and rational components. Note that the solution set of any mixed-integer linear program can be modeled in this way. For example, the constraint $x_i \ge 0$, where x_i is an integer variable for some i, can be replaced by the constraints $x_i - s = 0, s \ge 0$. We denote the continuous relaxation of P by P^{LP}.

2.1 Disjunctive and lattice-free cuts

Disjunctive programming was introduced by Balas [4] and has proved to be a very important tool for generating valid inequalities for mixed-integer sets. We next review the main ideas that are relevant for this paper. Let $D_k = \{(x, y) \in \mathbb{R}^{n+l} : \; A^k x + G^k y \le b^k\}$ be polyhedral sets indexed by $k \in K$ and let $D = \cup_{k \in K} D_k$. We call D a *disjunction* if it satisfies

$$\mathbb{Z}^n \times \mathbb{R}^l \subseteq D$$

and we call each D_k an *atom* of the disjunction D. Clearly, if the disjunction D satisfies this condition, it has the form $D = D^x \times \mathbb{R}^l$ where $D^x \subseteq \mathbb{R}^n$ is the projection of D in the space of the integer variables. Notice that the above condition is same as requiring that $\mathbb{Z}^n \setminus D^x = \emptyset$, which, in general, is not trivial to verify. In the next section, we will discuss simple disjunctions for which validity of the disjunction can be verified trivially or is obvious from the definition.

A linear inequality is called a *disjunctive cut* for P obtained from the disjunction D if it is valid for $P^{LP} \cap D_k$ for all $k \in K$. Disjunctive cuts are valid for all points in P and multiple disjunctive cuts can be derived from the same disjunction. More precisely, given a disjunction D,

$$P \subseteq P_D = \operatorname{conv}\left(P^{LP} \cap D\right) = \operatorname{conv}\left(\bigcup_{k \in K} (P^{LP} \cap D_k)\right),$$

where P_D is called the *disjunctive hull of P with respect to D*. In other words, P_D is the set of points in P^{LP} satisfying all disjunctive cuts obtained from this disjunction. Also note that, by definition, this set equals the convex hull of points in P^{LP} not contained in $\mathbb{R}^{n+l} \setminus D$.

Disjunctive cuts can also be seen as *lattice-free cuts*. Given a set $B \subseteq \mathbb{R}^n$, we call B *strictly lattice-free* if $B \cap \mathbb{Z}^n = \emptyset$, and we say that B is *lattice-free* if $\operatorname{int}(B) \cap \mathbb{Z}^n = \emptyset$, where $\operatorname{int}(B)$ stands for the points in the interior of B. Thus a lattice-free set may have integral points on its boundary. If B is strictly lattice-free, we define $P'(B)$ as

$$P'(B) = \operatorname{conv}\left(P^{LP} \setminus (B \times \mathbb{R}^l)\right) \Rightarrow P \subseteq P'(B),$$

and any inequality valid for $P'(B)$ is called a *lattice-free* cut derived from the set B. Note that the definition of a lattice-free cut above is seemingly different from most recent work starting with [2] where convex sets which have strictly lattice-free interior are called lattice-free sets and cuts derived from these sets are called lattice-free cuts. We will observe shortly that there is no distinction between these definitions.

It is easy to see that a disjunctive cut derived from the disjunction $D^x \times \mathbb{R}^l$ is a lattice-free cut derived from the set $\mathbb{R}^n \setminus D^x$. Consequently, all disjunctive cuts are lattice-free cuts. As we discuss below, it is also possible to show that valid inequalities obtainable as lattice-free cuts from convex, strictly lattice-free sets are disjunctive cuts. Therefore all lattice-free cuts are disjunctive cuts. Before establishing the equivalence between lattice-free and disjunctive cuts we first make an important observation which we use throughout the paper:

Observation 2.1. *Let $D = D^x \times \mathbb{R}^l$ be a disjunction and let B be a strictly lattice-free set. If $D^x \cap B = \emptyset$, then any lattice-free cut derived from B is a disjunctive cut obtained from D, i.e. $P_D \subseteq P'(B)$.*

Let $c^T x + d^T y \geq f$ be a given valid inequality for P and let $V \subseteq \mathbb{R}^n$ be the points in P^{LP} that violate this inequality. In other words,

$$V = \{(x, y) \in P^{LP} \;:\; c^T x + d^T y < f\}. \tag{1}$$

Furthermore, let $V^x \subset \mathbb{R}^n$ denote the projection of the set V in the space of the integer variables and note that $V^x \cap \mathbb{Z}^n = \emptyset$. It is known that V^x is defined by a finite collection of strict and non-strict rational inequalities, see [12]. Jörg [19] observes that the set V^x is contained in the interior of a polyhedral lattice-free set. In other words, there is a rational polyhedral set $B = \{x \in \mathbb{R}^n \;:\; \pi_i^T x \geq \gamma_i,\ i \in K\}$, where $\pi_i \in \mathbb{Z}^n$ and $\gamma_i \in \mathbb{Z}$ for all $i \in K$, such that $\mathrm{int}(B) \cap \mathbb{Z}^n = \emptyset$ and

$$V^x \subseteq \mathrm{int}(B) = \{x \in \mathbb{R}^n \;:\; \pi_i^T x > \gamma_i,\ i \in K\}.$$

Therefore $c^T x + d^T y \geq f$ is valid for $P'(\mathrm{int}(B)) \subseteq P'(V^x)$. Based on this observation, Jörg then argues that

$$D \;=\; \bigcup_{i \in K} \{(x, y) \in \mathbb{R}^{n+l} \;:\pi_i^T x \leq \gamma_i\} \tag{2}$$

defines a valid disjunction and the cut $c^T x + d^T y \geq f$ is a disjunctive cut derived from the disjunction D. Therefore, any valid inequality for P, and in particular, any facet-defining inequality for P is a disjunctive cut for *some* disjunction D (and therefore a lattice-free cut derived from some convex lattice-free set). We emphasize that this approach is not prescriptive in the sense that the disjunction D is defined using the valid inequality $c^T x + d^T y \geq f$ and not the other way around.

2.2 Multi-branch split disjunctions

We next discuss simple disjunctions D for which it is easy to verify that $\mathbb{Z}^n \times \mathbb{R}^l \subseteq D$. The building block of these disjunctions is a *split disjunction* which is a disjunction with two atoms D_1, D_2, where

$$D_1 = \{(x, y) \in \mathbb{R}^{n+l} \;:\; \pi^T x \leq \gamma\} \text{ and } D_2 = \{(x, y) \in \mathbb{R}^{n+l} \;:\; \pi^T x \geq \gamma + 1\}$$

for some $\pi \in \mathbb{Z}^n, \gamma \in \mathbb{Z}$. We denote this disjunction as $D(\pi, \gamma)$, and define the associated *split set* as

$$S(\pi, \gamma) = \mathbb{R}^n \setminus D(\pi, \gamma) = \{(x, y) \in \mathbb{R}^{n+l} \;:\; \gamma < \pi^T x < \gamma + 1\}.$$

Clearly if $x \in \mathbb{Z}^n$ then $\pi^T x \in \mathbb{Z}$ implying that $\pi^T x$ either satisfies $\pi^T x \leq \gamma$ or $\pi^T x \geq \gamma + 1$ and therefore $D(\pi, \gamma)$ is a valid disjunction. Split disjunctions were introduced by Cook, Kannan, and Schrijver [9]. We will denote the closure of the set $S(\pi, \gamma)$ by $\bar{S}(\pi, \gamma)$.

Li and Richard [23] defined a generalization of split disjunctions called *t-branch split disjunctions*. Let $\pi_i \in \mathbb{Z}^n$ and $\gamma_i \in \mathbb{Z}$ for $i = 1, \ldots, t$. Then,

$$D(\pi_1, \ldots, \pi_t, \gamma_1, \ldots, \gamma_t) = \bigcup_{S \subseteq \{1,\ldots,t\}} \{(x, y) \in \mathbb{R}^{n+l} \ : \pi_i^T x \leq \gamma_i \text{ if } i \in S,\ \pi_i^T x \geq \gamma_i + 1 \text{ if } i \notin S\} \quad (3)$$

is called a t-branch split disjunction. Clearly a split disjunction is simply a 1-branch split disjunction. Further,

$$\mathbb{R}^{n+l} \setminus D(\pi_1, \ldots, \pi_t, \gamma_1, \ldots, \gamma_t) = \bigcup_{i=1,\ldots,t} S(\pi_i, \gamma_i).$$

In other words, the complement of $D(\pi_1, \ldots, \pi_t, \gamma_1, \ldots, \gamma_t)$ can be expressed as the union of t split sets and is thus trivially lattice-free, and therefore $D(\pi_1, \ldots, \pi_t, \gamma_1, \ldots, \gamma_t)$ defines a valid disjunction. On the other hand, verifying that a disjunction of the type (2) used by Jörg is valid requires solving an integer program.

A t-branch split disjunction has at most 2^t atoms. If the vectors $\pi_1, \ldots, \pi_t$ defining a t-branch split disjunction are linearly independent, then each atom of the disjunction is guaranteed to be full-dimensional.We refer to disjunctive cuts obtained from t-branch split disjunctions as *t-branch split cuts*.

2.3 Asymmetric multi-branch split disjunctions

It is possible to generalize multi-branch split disjunctions by using the "*cutting-plane tree*" approach described in Chen, Küçükyavuz, and Sen [8]. This gives a more general family of disjunctions compared with the *asymmetric disjunctions* described in [11]. We say that D is an *asymmetric multi-branch split disjunction* (or, *branching disjunction* for short) if there exists an associated rooted binary tree such that (i) the leaf nodes of the binary tree correspond to the atoms of the disjunction, (ii) the root node corresponds to the space of integer variables, and, (iii) each set associated with a non-leaf node is subdivided into two sets using a split disjunction and the new sets are associated with the offsprings of the current node. In other words, a branching disjunction is a disjunction which can be constructed by recursively applying split disjunctions. We will briefly consider such disjunctions only in Section 5.

By this definition, a t-branch split disjunction is a special kind of branching disjunction where the same split disjunction is used to subdivide the sets associated with nodes at the same level of the binary tree.

3 Expessing valid inequalities as t-branch split cuts

In this section we show that any valid inequality for P can be expressed as a t-branch split cut for some t. Such a result is implied by the work of Chen, Küçükyavuz, and Sen [8] when P is a bounded set; in this case the vectors $\pi_1, \ldots, \pi_t$ defining the disjunction can be assumed to be unit vectors. However t in this case depends on the data defining P. We will next remove the boundedness condition on P and also remove the dependence of t on the data defining P.

To show that a given valid inequality is a t-branch split cut for some t, we will consider the strictly lattice-free set V^x defined in (1) on the space of integer variables, and cover it by the union of t split sets.

Given a closed, bounded, convex set (or convex *body*) $B \subseteq \mathbb{R}^n$ and a vector $c \in \mathbb{Z}^n$, let the *lattice width of B along the direction c*, denoted by $w(B, c)$, be defined as

$$w(B, c) = \max\{c^T x : x \in B\} - \min\{c^T x : x \in B\}. \quad (4)$$

The *lattice width* of B, denoted here as $w(B)$, is defined as

$$w(B) = \min_{c \in \mathbb{Z}^n \setminus \{\mathbf{0}\}} w(B, c).$$

If the set is not closed, we define its lattice width to be the lattice width of its closure.

Lenstra [22] gave a polynomial-time algorithm to solve the feasibility version of integer programs in fixed dimension. Given a polyhedron, Lenstra's algorithm either finds an integral point contained in the polyhedron or certifies that no such point exists. A central component of this algorithm is the use of algorithmic versions of Khinchine's *flatness* theorem. The flatness theorem shows that there exists a function $f(\cdot) : \mathbb{Z}_+ \to \mathbb{R}_+$ such that for any strictly lattice-free bounded convex set $B \subseteq \mathbb{R}^n$,

$$w(B) \leq f(n).$$

Notice that the function $f(\cdot)$ only depends on the dimension of B and not on the complexity of the body B. In [22], Lenstra uses this result to construct a finite enumeration tree to solve the integer feasibility problem. The number of nodes in the tree is bounded above by a function of n which again is independent of the complexity of the body B. Modifying Lenstra's idea slightly, we later show that every lattice-free convex body in $\mathbb{R}^n$ can be covered by the union of t split sets, where t is essentially the maximum number of enumeration nodes used in Lenstra's algorithm.

Lenstra showed that $f(n) \leq 2^{n^2}$, which was later improved to $f(n) \leq c_0(n+1)n/2$ by Kannan and Lovász [20] for some constant c_0. This bound was subsequently improved by Banaszczyk, Litvak, Pajor, and Szarek to $O(n^{3/2})$ and by Rudelson [27] to $O(n^{4/3} \log^c n)$ for some constant c. The constant c_0 used by Kannan and Lovász [20] is $c_0 = \max\{1, 4/c_1\}$ where c_1 is another constant defined by Bourgain and Milman [7]. Independent of the value of c_1, the constant $c_0 \geq 1$ and therefore the upper bound defined by Kannan and Lovász on the lattice width is at least 3 for $\mathbb{R}^2$ and at least 6 for $\mathbb{R}^3$. For $\mathbb{R}^2$, Hurkens [17] proved that $f(2) \leq 1 + 2/\sqrt{3} \approx 2.1547$, and showed that this bound is tight. More precisely he showed the following result.

Theorem 3.1. *[17] If $B \in \mathbb{R}^2$ is lattice-free, then $w(B) \leq 1 + \frac{2}{\sqrt{3}}$. Furthermore there exists B for which the bound is tight. In addition, if $w(B) = 1 + \frac{2}{\sqrt{3}}$ then B is a triangle with vertices q_1, q_2, q_3 such that (let $q_4 := q_1$)*

$$\frac{1}{\sqrt{3}}\, q_i + \left(1 - \frac{1}{\sqrt{3}}\right) q_{i+1} = b_i, \text{ for } i = 1, 2, 3.$$

where $b_i \in \mathbb{Z}^2$ for $i = 1, 2, 3$.

By taking $b_1 = (0,0)^T$, $b_2 = (0,1)^T$, and $b_3 = (1,0)^T$ in the above description, one obtains a triangle $T \in \mathbb{R}^2$ with $w(T) = 1 + \frac{2}{\sqrt{3}}$. It is easy to check that the three vertices q_1, q_2, q_3 of this triangle are given by the columns of the following matrix:

$$\frac{1}{3} \begin{pmatrix} 2 & -1-\sqrt{3} & 2+\sqrt{3} \\ -1-\sqrt{3} & 2+\sqrt{3} & 2 \end{pmatrix}.$$

Furthermore, T is a so-called type 3 maximal lattice-free triangle that contains the lattice points b_1, b_2 and b_3 in the relative interior of its sides. As shown in Figure 1, T does not contain any other lattice points.

Recently, Averkov, Wagner and Weismantel [3] enumerated all maximal lattice-free bodies in $\mathbb{R}^3$ with integer vertices up to unimodular transformations. From their catalogue, one can easily see that the lattice

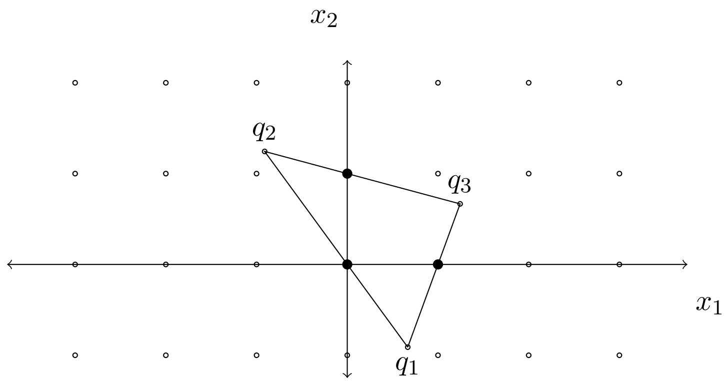


Figure 1: The lattice-free triangle T

width of such bodies does not exceed three in $\mathbb{R}^3$. However, in $\mathbb{R}^3$, it is not hard to construct lattice-free bodies with lattice width slightly greater than 3. Recall the vectors $q_1, \ldots, q_3 \in \mathbb{R}^2$ which define the vertices of the triangle T. Consider the tetrahedron H with vertices $s_1, \ldots, s_4$, where $s_4 = (0, 0, 2 + 2/\sqrt{3})$, and s_1, s_2, s_3 are points on the plane $\{x : x_3 = 0\}$ such that the points $(1, q_i) \in \mathbb{R}^3$ lie on the line segment from s_i to s_4. See Figure 2. By definition, $H \cap \{x : x_3 = 1\}$ is congruent to T. It is easy to verify that H has lattice width $2 + 2/\sqrt{3} \approx 3.1547$. We do not know of any result analogous to Hurkens' result which gives the best possible value of the lattice width in $\mathbb{R}^3$.

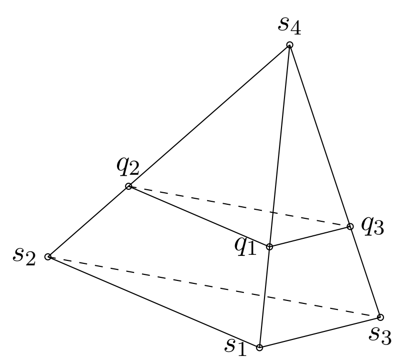


Figure 2: A lattice-free tetrahedron in R^3 with lattice width $2 + 2/\sqrt{3}$

Using the best known value for c_1, and refining the result of Kannan and Lovász slightly, we next give an upper bound in $\mathbb{R}^3$ on the lattice width of strictly lattice-free convex bodies. We need a few definitions to give the result and its proof. In [7], Bourgain and Milman show that if $K \in \mathbb{R}^n$ is a convex body symmetric about the origin and K^* is its polar body, i.e., $K^* = \{y \in \mathbb{R}^n : y^T x \leq 1 \; \forall x \in K\}$, then

$$\mathrm{vol}(K)\mathrm{vol}(K^*) \geq \left(\frac{c_1}{n}\right)^n \tag{5}$$

where $\mathrm{vol}(K)$ stands for the volume of K and $c_1 > 0$ is a universal constant that does not depend on n.

If S and T are subsets of $\mathbb{R}^n$, and δ is a real number, then let $S + T = \{s + t : s \in S, t \in T\}$, and let

$\delta S = \{\delta s : s \in S\}$. $S - T$ is similarly defined. For a convex body B in $\mathbb{R}^n$, let $\mu_j(B)$ be defined as

$$\mu_j(B) = \inf\{t \in \mathbb{R} : tB + \mathbb{Z}^n \text{ intersects every } (n-j)\text{-dimensional affine subspace of } \mathbb{R}^n\},$$

where inf is short for infimum. Therefore, $\mu_n(B)$ is the essentially the smallest t such that $tB + \mathbb{Z}^n = \mathbb{R}^n$, and is called the *covering radius* of B. Clearly $\mu_n(B) \geq 1$ if B is lattice-free and convex and $\mu_n(B) > 1$ if B is a strictly lattice-free convex body. Let

$$\lambda_1(B) = \inf\{t \in \mathbb{R} : t(B - B) \text{ contains a nonzero integer vector}\}.$$

Theorem 3.2. *If $B \in \mathbb{R}^3$ is lattice-free, then $w(B) \leq 1 + 2/\sqrt{3} + (90/\pi^2)^{\frac{1}{3}} \approx 4.2439$.*

Proof. We first define functions $\phi_0, \phi_1 : \mathbb{Z}_+ \to \mathbb{R}_+$ that we will use instead of the universal constants c_0 and c_1. Let B_n stand for the unit ball in $\mathbb{R}^n$ and define

$$\phi_1(n) = n\Big(\frac{2^n(n!)^2}{(2n)!}\mathrm{vol}(B_n)^2\Big)^{\frac{1}{n}}$$

and let $\phi_0(n) = 4/\phi_1(n)$. Subsequently, we will refer to c_0 as the least upper bound on $\phi_0(n)$ for all n, and c_1 as the largest lower bound on $\phi_1(n)$ for all n.

In [21], Kuperberg gave the best-known value for c_1 and showed that if K is a convex body symmetric about the origin, then

$$\mathrm{vol}(K)\mathrm{vol}(K^*) \geq \frac{2^n(n!)^2}{(2n)!}\mathrm{vol}(B_n)^2.$$

Using our notation, this can be rewritten as

$$\mathrm{vol}(K)\mathrm{vol}(K^*) \geq \left(\frac{\phi_1(n)}{n}\right)^n \tag{6}$$

which is identical to (5) except the universal constant c_1 is now replaced with the function $\phi_1(n)$.

In [20], Kannan and Lovász show that $\lambda_1(B)w(B) \leq 4/(\mathrm{vol}(B-B)\mathrm{vol}((B-B)^*)^{1/n}$ which implies that

$$w(B) \leq \frac{4n}{\lambda_1(B)\phi_1(n)} = \frac{n\phi_0(n)}{\lambda_1(B)}$$

by (6). In addition (in Lemma 2.3) they also show that $\mu_1(B) = 1/w(B)$ and therefore, substituting out $w(B)$ from the inequality above, we obtain $\lambda_1(B) \leq n\phi_0(n)\mu_1(B)$.

Now combining $\mu_n(B) \leq \mu_{n-1}(B) + \lambda_1(B)$ [20, Lemma 2.5] with the fact that $\mu_2(B) \leq (1 + 2/\sqrt{3})\mu_1(B)$ (see [20, p 587]) we obtain

$$\mu_3(B) \leq \mu_2(B) + \lambda_1(B) \leq (1 + 2/\sqrt{3})\mu_1(B) + \lambda_1(B) \leq (1 + 2/\sqrt{3} + 3\phi_0(3))\mu_1(B).$$

As $1 \leq \mu_3(B)$ for a lattice-free body in $\mathbb{R}^3$, we have

$$\frac{1}{\mu_1(B)} = w(B) \leq 1 + 2/\sqrt{3} + 3\phi_0(3)$$

Substituting $\phi_0(3) = (10/3\pi^2)^{\frac{1}{3}} \approx 0.6964$ we obtain the desired value. ∎

We can similarly refine the lattice-width bound in [20] in higher dimensions. Lemma 2.6 in [20] asserts that $\mu_{j+1}(B) \leq \mu_j(B) + (j+1)c_0\mu_1(B)$ for $j = 1, \ldots, n-1$. In [20, Theorem 2.7], the authors use the previous inequality to show that if $B \subseteq \mathbb{R}^n$, then

$$\mu_n(B) \leq (1 + c_0 \sum_{i=2}^{n} i)\mu_1(B) \Rightarrow \mu_n(B) \leq c_0 n(n+1)/(2w(B)) \text{ as } \mu_1(B) = 1/w(B), \text{ if } c_0 \geq 1.$$

Looking at the proofs of Lemma 2.6 and Lemma 2.5, and the fact that $\lambda_1(B) \leq \phi_0(n)n\mu_1$, one can replace Lemma 2.6 by

$$\mu_{j+1}(B) \leq \mu_j(B) + (j+1)\phi_0(j+1)\mu_1(B) \text{ for } j = 1, \ldots, n-1 \Rightarrow$$

$$\mu_n(B) \leq (1 + 2/\sqrt{3} + \sum_{i=3}^{n} i\phi_0(i))/w(B),$$

and therefore $w(B) \leq 1 + 2/\sqrt{3} + \sum_{i=3}^{n} i\phi_0(i)$ for lattice-free bodies in $\mathbb{R}^n$. As $\phi_0(4) \approx 0.6510$, we can conclude that if $B \subseteq \mathbb{R}^4$ and B is convex and lattice-free, then $w(B) \leq 6.8481$.

In the remainder of the section, we do not use any specfic bound on the lattice width, but just use $f(n)$ to stand for a function which gives an upper bound on the lattice width of strictly lattice-free bodies in $\mathbb{R}^n$. We use the precise values for $\mathbb{R}^2$ and $\mathbb{R}^3$ in Section 5.

Lemma 3.3. *Let B be a bounded, strictly lattice-free convex set in $\mathbb{R}^n$. Then B is contained in the union of at most $\Pi_{k=1}^{n}(2 + \lceil f(k) \rceil)$ split sets.*

Proof. Let $g(n)$ stand for the minimum number of split sets required to cover any bounded, strictly lattice-free convex set in $\mathbb{R}^n$. It is easy to see that $g(n)$ is an non-decreasing function of n and $g(1) = f(1) = 1$. Thus the result is trivially true when $n = 1$. Assume it is true for all dimensions up to $n-1$ and consider a bounded, strictly lattice-free convex set $B \subseteq \mathbb{R}^n$. By Khinchine's flatness result, there is an integer vector $a \in \mathbb{Z}^n$ such that $f(n) \geq u - l$ where $u = \max\{a^T x : x \in B\}$ and $l = \min\{a^T x : x \in B\}$. Clearly, $B \subseteq \{x \in \mathbb{R}^n : l \leq a^T x \leq u\} \subseteq \{x \in \mathbb{R}^n : \lfloor l \rfloor \leq a^T x \leq \lceil u \rceil\}$.

Let U be the collection of the split sets $S(a,b)$ for $b \in V = \{\lfloor l \rfloor, \ldots, \lceil u \rceil - 1\}$ and notice that

$$B \setminus \bigcup_{b \in V} S(a,b) = \bigcup_{b \in \bar{V}} \{x \in B : a^T x = b\}$$

where $\bar{V} = \{\lceil l \rceil, \ldots, \lfloor u \rfloor\}$. Each one of the $|\bar{V}|$ sets in the right hand side of this expression is strictly lattice-free, has dimension at most $n-1$ and, by induction, can be covered by a collection of at most $g(n-1)$ split sets in $\mathbb{R}^{n-1}$. Each of these lower dimensional split sets can trivially be extended to a split set in $\mathbb{R}^n$ and added to U. Therefore the resulting collection U has size at most $|V| + |\bar{V}|g(n-1)$. Notice that

$$u - l \leq f(n) \implies \lceil u \rceil - \lceil l \rceil \leq \lceil f(n) \rceil,$$

and $|V|, |\bar{V}| \leq \lceil u \rceil - \lceil l \rceil + 1$. Consequently, $|V|, |\bar{V}| \leq \lceil f(n) \rceil + 1$ and the set U (and therefore $g(n)$) has size at most $(\lceil f(n) \rceil + 1)(1 + g(n-1))$. Let $\bar{f}_n$ stand for $1 + \lceil f(n) \rceil$. Expressing $g(i)$ in terms of $f(i)$ for $i = n-1, \ldots, 1$, and using the fact that $g(1) = f(1) = 1$, we obtain

$$g(n) \leq \bar{f}_n + \bar{f}_n \bar{f}_{n-1} + \bar{f}_n \bar{f}_{n-1} \bar{f}_{n-2} + \ldots + \Pi_{i=1}^{n} \bar{f}_i \leq \Pi_{i=1}^{n}(1 + \bar{f}_i),$$

and the desired bound follows. ∎

The previous result obviously also holds for the interior of any bounded (maximal) lattice-free convex in $\mathbb{R}^n$. If the strictly lattice-free convex set is unbounded, additional conditions are needed for Lemma 3.3 to hold; the conditions we choose may not be the least restrictive but suffice for our purpose. Lovász [24] showed that any maximal lattice-free convex set is a polyhedron. Furthermore, if such a set is unbounded, then it is either an irrational hyperplane or it is full-dimensional and it can be expressed as $Q + L$ where Q is a polytope and L a rational linear space. In the latter case, $Q + L$ is called a *cylinder over the polytope* Q. Also see Basu et. al. [6] for a more recent and complete proof of Lovász's result.

Lemma 3.4. *Let B be a strictly lattice-free, convex, unbounded set in $\mathbb{R}^n$ which is contained in the interior of a maximal lattice-free convex set. Then B can be covered by $\Pi_{k=1}^n(2 + \lceil f(k)\rceil)$ split sets.*

Proof. Let B' be a maximal lattice-free convex set containing B in its interior; then B' is full-dimensional. By Lovász's result, $B' = Q + L$, where L is rational, $dim(L) = r > 0$, and Q is a lattice-free polytope contained in the orthogonal complement of L and has dimension $n - r$. Furthermore, B is contained in $\text{int}(Q) + L$. As L is rational, we can define a $n \times n$ unimodular matrix such that every point in the set $UQ = \{Ux : x \in Q\}$ has its last r components zero. Further UQ is lattice-free, and the previous theorem gives $t = \Pi_{k=1}^{n-r}(2 + \lceil f(k)\rceil)$ split sets in $\mathbb{R}^{n-r}$ whose union covers the projection of $\text{int}(UQ)$ on the first $n - r$ components. Let $S(\pi_i, \gamma_i)$ be the ith split set in the above union. Let $S(\pi'_i, \gamma_i)$ be the corresponding split set in $\mathbb{R}^n$ which is defined as follows: π'_i is obtained by appending r zeros to π_i and then multiplying by U^{-1}. It is easy to see that $\text{int}(Q) + L$ and therefore B is covered by $\cup_{i=1}^t S(\pi'_i, \gamma_i)$. ∎

Using the bound of Rudelson cited above, we get an exponential upper bound on the number of split sets needed to cover a strictly lattice-free convex set (which is at least $O(n!^{4/3})$). We will give a (smaller) exponential lower bound in the next section.

Now let $c^T x + d^T y \geq f$ be a non-trivial valid inequality for $\text{conv}\,(P)$, i.e., $c^T x + d^T y \geq f$ is not valid for P^{LP}, but is valid for $\text{conv}\,(P)$. Let $V \subseteq \mathbb{R}^{n+l}$ be defined as in (1), and let V^x be defined as the projection of V on the space of the integer variables. V^x is strictly lattice-free, and is non-empty as $c^T x + d^T y \geq f$ is not valid for P^{LP}. As we discussed earlier, Jörg [19] showed that V^x is contained in the interior of a lattice-free rational polyhedron $B \subseteq \mathbb{R}^n$, and thus in the interior of a maximal lattice-free convex set. Depending on whether V^x is bounded or unbounded, we can use either of the previous two lemmas to obtain the following result.

Theorem 3.5. *Every facet-defining inequality for P is a t-branch split cut for $t = \Pi_{k=1}^n(2 + \lceil f(k)\rceil)$.*

We observed earlier that Jörg's results already express every facet-defining inequality as a disjunctive cut. The previous theorem gives an alternative expression of every facet-defining inequality as a disjunctive cut. A nice aspect of this alternative expression is that the validity of the disjunction is obvious and need not be verified.

We note that one can easily obtain finite disjunctive cutting-plane algorithms for arbitrary MIPs based on Theorem 3.5. We will describe such an algorithm based on Theorem 3.5, which is however of purely theoretical interest, and is not likely to be practical.

Theorem 3.6. *The mixed-integer program*

$$\min\{c^T x + d^T y \;:\; (x, y) \in P\}, \textit{ where } P = \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^l \;:\; Ax + Gy = b,\ y \geq 0\},$$

and A, G and b, c, and d have rational components, can be solved in finite time via a pure cutting-plane algorithm which generates only t-branch split cuts.

Proof. Let $t = \Pi_{i=1}^{n}(2+\lceil f(i)\rceil)$. We will represent any t-branch split disjunction $D(\pi_1,\dots,\pi_t,\gamma_1,\dots,\gamma_t)$ by a vector v in $\mathbb{Z}^{(n+1)t}$; the components of $\pi_1,\dots,\pi_t$ are arranged as the first nt components of v, and $\gamma_1,\dots,\gamma_t$ form the last n components. Let $\Omega = \mathbb{Z}^{(n+1)t}$. As Ω is a countable set, by definition the vectors in Ω can be arranged in a sequence $\{\Omega_i\}$, say by increasing norm. Further let D_i be the t-branch split disjunction defined by Ω_i. For any facet-defining inequality of $\mathrm{conv}\,(P)$, there exists a (finite) integer k such that the inequality is a t-branch split cut defined by the disjunction D_k. Let k^* be the largest index of a disjunction associated with facet-defining inequalities. Now consider the following algorithm which does not compute or use the value of k^*. Let P_0 denote the continuous relaxation of P.

Repeat the following two steps for $i = 1, 2, \dots$

1. Compute $P_i = P_{i-1} \cap \mathrm{conv}\,(P_0 \cap D_i)$.
2. If the basic optimal solution of $\min\{c^Tx + d^Ty \;:\; (x,y) \in P_i\}$ is integral, terminate.

As P_i is a relaxation of P, an integral optimal solution over P_i is also an optimal solution over P. Further, as $P_{k^*} = \mathrm{conv}\,(P)$, the algorithm must terminate for some $i \le k^*$.

∎

If one wants to check validity of a given inequality, the termination criterion in the above algorithm can be modified to check it. Finally, if one wants to compute $\mathrm{conv}\,(P)$, then the termination criterion can be changed to verifying that all vertices of P_i are integral.

We also note that it is possible to produce a similar algorithm using Jörg's characterization, except one needs an additional step to verify the validity of each potential disjunction.

4 Results on covering lattice-free sets with split sets

In this section, we construct a lattice-free set in $\mathbb{R}^n$ that cannot be covered by fewer than $\Omega(2^n)$ split sets. Clearly our lower bound is significantly smaller than the upper bound presented in the previous section and we believe that the best upper bound is likely to be considerably closer to our lower bound.

Recall that $S(a,b) = \{x \in \mathbb{R}^n : b < a^Tx < b+1\}$ is an open set. Given an integer vector $a \in \mathbb{Z}^n$, we will refer to the collection of split sets $\{S(a,b) : b \in \mathbb{Z}\}$ as the family of split sets defined by a. We also refer to a as the defining vector of these split sets, and denote this fact using a function $d.v.(\cdot)$ where $d.v.(S(a,b)) = a$. We denote the Euclidean norm of a by $||a||$, and the set of all split sets in $\mathbb{R}^n$ by $\mathcal{S}^n$.

Definition 4.1. *Let K be a compact set in $\mathbb{R}^n$ and let $\varepsilon > 0$ be given. We define $\mathcal{L}(K,\varepsilon)$ as the collection of vectors $a \in \mathbb{Z}^n$ such that, for some $b \in \mathbb{Z}$, the volume of $K \cap S(a,b)$ is at least ε.*

Note that $\mathcal{L}(K,\varepsilon)$ can be empty, for example if ε is greater than the volume of K.

Lemma 4.2. *For any compact set $K \subset \mathbb{R}^n$ and any number $\varepsilon > 0$, the list $\mathcal{L}(K,\varepsilon)$ is finite.*

Proof. Let $l \in \mathbb{R}$ be an upper bound on the $(n-1)$-dimensional volume of the intersection of a hyperplane of dimension $n-1$ with B. Clearly, for any vector $a \in \mathbb{Z}^n$, the distance between two parallel hyperplanes of the form $\{x : a^Tx = b\}$ and $\{x : a^Tx = b+1\}$ is $1/||a||$. Therefore, if $l/\varepsilon < ||a||$, the volume of the intersection of a split set $S(a,b)$ (for some $b \in \mathbb{Z}$) with K is at most $l/||a|| < \varepsilon$. Therefore $\mathcal{L}(K,\varepsilon)$ is a subset of $\{a \in Z^n : ||a|| \le l/\varepsilon\}$ and is a finite set. ∎

For a triangle in $\mathbb{R}^2$, we can assume the number l in the proof of the previous lemma equals the length of the longest side.

Lemma 4.3. *Let T be the triangle defined in the previous section with lattice width $1 + 2/\sqrt{3}$. Then there exists an $\varepsilon > 0$ such that $T \setminus (S(a_1, b_1) \cup S(a_2, b_2))$ has area at least ε for every $a_1, a_2 \in \mathbb{Z}^2$ and every $b_1, b_2 \in \mathbb{Z}$.*

Proof. As discussed in the previous section, T is a maximal lattice-free triangle of type 3. It is known from results in [11] that the interior of T is not contained in the union of two split sets defined by linearly independent vectors. On the other hand, as $w(T) > 2$, the interior of T is not contained in the union of two split sets defined by linearly dependent vectors.

Consider the list $\mathcal{L}(T, 1/2)$ and note that the list is not empty as the intersection of the split set $\{x \in \mathbb{R}^n : 0 < x_1 < 1\}$ with T has area at least 1/2, and thus its defining vector belongs to $\mathcal{L}(T, 1/2)$. As this list is finite, one can find a split set that has the largest intersection (in terms of area) with T. Let $\varepsilon_1 > 0$ be the area left uncovered by this split set. Remember that the split sets not contained in the list $\mathcal{L}(T, 1/2)$ can cover an area of T less than the ones contained in the list and consequently, the minimum area of T that is left uncovered by any split set (including the ones not contained in the list) is ε_1.

Now consider the list of split sets $\mathcal{L}(T, \varepsilon_1/2)$. Let ε_2 be the area of T left uncovered by any two split sets from this list. As the list is finite, ε_2 can be computed and as T cannot be covered by the union of two split sets, $\varepsilon_2 > 0$. Now consider any two split sets, not necessarily from the list $\mathcal{L}(T, \varepsilon_1/2)$. If they both belong to the list, then they leave at least $\varepsilon_2 > 0$ of the area of T uncovered. If, on the other hand, at least one of them, say the second one, does not belong to the list, notice that the first split set cannot cover at least ε_1 of the area, and the second split set can cover at most $\varepsilon_1/2$ of the area. Consequently, any two split sets have to leave $\varepsilon_3 = \min\{\varepsilon_1/2, \varepsilon_2\} > 0$ of the area of T uncovered. ∎

Lemma 4.4. *There exists a rational, strictly lattice-free triangle in $\mathbb{R}^2$ such that for some $\varepsilon > 0$, the area left uncovered by any two split sets is at least ε.*

Proof. Note that that T contains lattice points on its boundary but shrinking it slightly, and tilting the sides by a small amount, one can obtain a strictly lattice-free triangle T' whose sides are defined by rational equations, and the $area(T' \cap T)$ is at least $(1 - \varepsilon_3/2)area(T)$. The proof can be completed by setting $\varepsilon = \varepsilon_3/2$. ∎

Definition 4.5. *We say that a set $A \subset \mathbb{R}^n$* can be weakly covered *by j split sets if there exist split sets $S_1, \ldots, S_j \in \mathcal{S}^n$ such that the volume of $A \setminus (S_1 \cup \ldots \cup S_j)$ is zero.*

Recall that any lattice-free set in $\mathbb{R}^2$ can be weakly covered by three split sets, but the triangle in the previous lemma cannot be weakly covered by two split sets.

Lemma 4.6. *Let K be a compact set in $\mathbb{R}^n$ that cannot be weakly covered by any collection of $m - 1$ split sets in $\mathcal{S}^n$ and let l be some fixed constant. Then there exists a finite set such that for any l split sets which cover K, the defining vectors of at least m of them are contained in this set.*

Proof. We use induction with respect to m and construct a family of sets $\Sigma(K, l, m)$ satisfying the desired property.

(i) If $m = 1$ then at least one split set in a weak covering of K by l split sets must cover a volume of $\frac{\mathrm{vol}K}{l}$ of K and therefore we choose

$$\Sigma(K, l, 1) = \mathcal{L}(K, \frac{\mathrm{vol}K}{l}),$$

which is finite by Lemma 4.2.

(ii) $m \Rightarrow m+1$.

For some $m \geq 1$, assume the result has been proved for all compact sets that cannot be weakly covered by $m-1$ split sets. Let K be a compact set that cannot be weakly covered by m split sets. Let σ be a collection of l split sets weakly covering K. Let $S_0 \in \sigma$ be a split set whose intersection with K has the greatest volume of all split sets in σ. Then $\mathrm{vol}(K \cap S_0) \geq \frac{\mathrm{vol}K}{l}$ and therefore $d.v.(S_0) \in \mathcal{L}(K, \mathrm{vol}K/l)$. The set $K \setminus S_0$ is a compact set which is weakly covered by $\sigma \setminus \{S_0\}$.

Further, $K \setminus S_0$ cannot be weakly covered by $m-1$ split sets, otherwise K can be weakly covered by m split sets. By induction, there exists a finite set $\Sigma(K \setminus S_0, l-1, m)$ such that at least m of the split sets in $\sigma \setminus \{S_0\}$ have their defining vectors in $\Sigma(K \setminus S_0, l-1, m)$. We take

$$\Sigma(K, l, m+1) = \mathcal{L}(K, \frac{\mathrm{vol}K}{l}) \bigcup_{S \in \mathcal{S}^n : \mathrm{vol}(K \cap S) \geq \frac{\mathrm{vol}K}{l}} \Sigma(K \setminus S, l-1, m)$$

which is a finite union of finite families of sets.

∎

Given an $n \times n$ matrix M and a set $S \subseteq \mathbb{R}^n$, we define $MS = \{Ms : s \in S\}$.

Lemma 4.7. *Given any two finite sets of vectors $V = \{v^1, \ldots, v^k\} \subseteq \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, $W = \{w^1, \ldots, w^m\} \subseteq \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, there exists an unimodular matrix M such that $MV \subseteq \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ and $MV \cap W = \emptyset$.*

Proof. Let q be the largest absolute value of the coefficients of the vectors $v^1, \ldots, v^k, w^1, \ldots, w^m$. Then let

$$M = \begin{pmatrix} 1 & \mu \\ \mu & \mu^2 + 1 \end{pmatrix} \text{ where } \mu = 3q.$$

Clearly M is an unimodular matrix. To see that $Mv \notin W$ for $v \in V$ let $v = (v_1, v_2)^T \in V$. Denote the first row of Mv by α which equals $v_1 + \mu v_2$. If $|v_2| \geq 1$, then α is a nonzero integer as $|v_1| < \mu$. If $v_2 = 0$, then α equals v_1 which is nonzero as every vector in S is nonzero.

The second row of Mv equals $\mu(v_1 + \mu v_2) + v_2 = \mu\alpha + v_2$. As $|\alpha| \geq 1$, $|v_2| \leq q$ and $\mu = 3q$, we have $|\mu\alpha + v_2| \geq 3q - q > q$. Thus $||Mv||_\infty > q$ and therefore Mv can not belong to W. ∎

In the next Lemma we use some elementary properties of unimodular transformations. First note that any split set remains a split set under a unimodular transformation. Also, remember that the volume of a bounded set stays the same after a unimodular transformation. In particular, given a bounded set A and a split set $S \in \mathcal{S}^n$, the volume of $A \cap S$ is the same as that of $MA \cap MS$ provided that M is a unimodular matrix of appropriate dimension.

Lemma 4.8. *Given any integers $l, k > 0$, there exist rational, strictly lattice-free triangles $T_1, \ldots, T_k \subseteq \mathbb{R}^2$ which cannot be weakly covered by two split sets with the property that the sets $\Sigma(T_1, l, 3), \ldots, \Sigma(T_k, l, 3)$ are pairwise-disjoint.*

Proof. Let T_1 be the strictly lattice-free triangle constructed in Lemma 4.4. Given any $\varepsilon > 0$ and any 2×2 unimodular matrix N, clearly NT_1 is strictly lattice-free. Further, one can easily verify that $\mathcal{L}(NT_1, \varepsilon) = N\mathcal{L}(T_1, \varepsilon)$. From the proof of Lemma 4.6, and by induction, it follows that $\Sigma(T_1, l, 3)$ can be expressed as the union of finitely many lists of the form $\mathcal{L}(K', \varepsilon')$, where K' is obtained from T_1 by subtracting up to two split sets in $\mathcal{S}^3$ from T_1, and ε' is some number. Therefore, we can conclude that $\Sigma(NT_1, l, 3) = N\Sigma(T_1, l, 3)$ for any integer $l > 0$.

We will now construct unimodular matrices $N_1, \ldots, N_k$, where N_1 is the identity matrix, such that the triangles $T_i = N_i T_1$ for $i = 1, \ldots, k$, have the desired property. Consider any $k > 1$, and assume we have constructed $N_1, \ldots, N_{k-1}$. Let $V = \Sigma(T_1, l, 3)$, and let $W = \cup_{i=1}^{k-1} \Sigma(N_i T_1, l, 3)$. By Lemma 4.7, we can construct a unimodular matrix N_k such that $N_k V = \Sigma(N_k T_1, l, 3)$ has no elements in common with W. The result follows by induction with respect to k. ∎

For any $n \geq 2$, setting $l = 3 \times 2^{n-2} - 1$ and $k = 2^{n-2}$ in Lemma 4.8, we get triangles $T_1, T_2, \ldots, T_{2^{n-2}}$ such that no triangle T_i can be weakly covered by two split sets and such that for $i \neq j$ the intersection $\Sigma(T_i, 3 \times 2^{n-2} - 1, 3) \cap \Sigma(T_j, 3 \times 2^{n-2} - 1, 3) = \emptyset$. For an integer $\Delta \in \{0, \ldots, 2^{n-2} - 1\}$, let δ_l stand for the lth bit in the binary expansion of Δ in $n - 2$ bits. In other words, $\Delta = \sum_{l=0}^{n-3} \delta_l 2^l$ with each $\delta_l \in \{0, 1\}$. For each $\Delta \in \{0, \ldots, 2^{n-2} - 1\}$, we define corresponding (2-dimensional) planes

$$V_\Delta := \{(\delta_0, \ldots, \delta_{n-3}, x, y) | x, y \in \mathbb{R}\}$$

and a triangle in each plane

$$\mathbf{T}_\Delta = \{(\delta_0, \ldots, \delta_{n-3}, x, y) | (x, y) \in T_{1+\Delta}\}.$$

Let $\mathbf{T} = \operatorname{conv}\left(\bigcup_{\Delta=0}^{2^{n-2}-1} \mathbf{T}_\Delta \right)$.

Lemma 4.9. $\mathbf{T}$ *is a strictly lattice-free rational polytope.*

Proof. By construction, $\mathbf{T}$ is a rational polytope as it is the convex hull of finitely many rational polytopes. If $v \in \mathbb{Z}^n$ is an integer point in $\mathbf{T}$, the first $n - 2$ components of v must be 0-1, and thus $v \in V_\Delta$ for some $0 \leq \Delta < 2^{n-2}$. But, by construction, $\mathbf{T} \cap V_\Delta = \mathbf{T}_\Delta$, which is strictly lattice-free. ∎

Let $\Pi : \mathbb{R}^n \to \mathbb{R}^2$ denote projection to the first two coordinates.

Lemma 4.10. *Let* $S = S(a, b)$ *be a split set in* $\mathbb{R}^n$ *where* $a = (\alpha_1, \ldots \alpha_n) \in \mathbb{Z}^n$ *and* $b \in \mathbb{Z}$. *Let* $\Delta \in \{0, \ldots, 2^{n-2} - 1\}$. *If* $\alpha_{n-1} = \alpha_n = 0$ *then* $S \cap V_\Delta = \emptyset$. *Otherwise* $\Pi(S \cap V_\Delta)$ *is a split set in* $\mathbb{R}^2$ *with defining vector* $(\alpha_{n-1}, \alpha_n)^T$.

Proof.

$$S \cap V_\Delta = \{(\delta_0, \ldots, \delta_{n-3}, x, y) : b - \sum_{l=0}^{n-3} \delta_l \alpha_l < \alpha_{n-1} x + \alpha_n y < b + 1 - \sum_{l=0}^{n-3} \delta_l \alpha_l\}.$$

If $\alpha_{n-1} = \alpha_n = 0$, then $\alpha_{n-1} x + \alpha_n y = 0$, but 0 is not strictly contained between two consecutive integers, and thus $S \cap V_\Delta = \emptyset$. If $(\alpha_{n-1}, \alpha_n) \neq (0, 0)$, then $S \cap V_\Delta$ is a nonempty split set and its defining vector is $(\alpha_{n-1}, \alpha_n)^T$. ∎

For a split set S in $\mathbb{R}^n$, we define $\hat{\Pi}(S \cap V_\Delta)$ to be $d.v.(\Pi(S \cap V_\Delta))$; if $S \cap V_\Delta = \emptyset$, then we set $\hat{\Pi}(S \cap V_\Delta) = (0, 0)^T$.

Theorem 4.11 (Lower bound result). $\mathbf{T}$ *is a strictly lattice-free rational polytope in* $\mathbb{R}^n$ *which cannot be covered by fewer than* $3 \times 2^{n-2}$ *split sets.*

Proof. By contradiction. Let us suppose that $\mathbf{T}$ is covered by a collection σ of fewer than $3 \times 2^{n-2}$ split sets. Let $\Delta \in \{0, \ldots, 2^{n-2} - 1\}$. Then $\mathbf{T} \cap V_\Delta = \mathbf{T}_\Delta$. Since $\mathbf{T}_\Delta$ is covered, by construction at least three split sets $S_1, S_2, S_3 \in \sigma$ satisfy $\hat{\Pi}(S_i \cap V_\Delta) \in \Sigma(T_{1+\Delta}, 3 \times 2^{n-2} - 1, 3)$ for $i = 1, 2, 3$.

By Lemma 4.10, the collection of vectors $\{\hat{\Pi}(S \cap V_\Delta) : S \in \sigma\}$ does not depend on Δ. The sets $\Sigma(T_{1+\Delta}, 3 \times 2^{n-2} - 1, 3)$, for $\Delta = 0, \ldots, 2^{n-2} - 1$, are pairwise disjoint. Therefore, there are at least $3 \times 2^{n-2}$ sets in σ, a contradiction. ∎

For a convex body A and a point x, let $\operatorname{dist}_\infty(x, A)$ stand for the l_∞ distance of the point x from A.

Definition 4.12. *Let $A \in \mathbb{R}^n$ be a convex set. Then the l_∞ ε-neighborhood of A is the set $D_\varepsilon(A) = \{x : \operatorname{dist}_\infty(x, A) \leq \varepsilon\}$.*

The l_∞ ε-neighborhood of a convex set is always convex, and the if ε is a rational number then the ε-neighborhood of a rational polytope is again a rational polytope. In the latter case, the l_∞ ε-neighborhood is the convex hull of the l_∞ ε-neighborhoods of the vertices of the polytope.

Remark 4.13. *As the polytope $\mathbf{T}$ is at a positive distance, say ε' from every lattice point, for any rational $\varepsilon < \varepsilon'$, the l_∞ ε-neighborhood of $\mathbf{T}$ is a rational polytope which is strictly lattice-free. Therefore, the interior of a maximal lattice-free set containing $D_\varepsilon(\mathbf{T})$ cannot be covered by fewer than $3 \times 2^{n-2}$ split sets.*

In $\mathbb{R}^3$, Theorem 4.11 yields a strictly lattice-free rational polytope which needs 6 split sets to cover it. We can improve this number by one (in general, we can improve the bound in Theorem 4.11 by one too).

Theorem 4.14. *In $\mathbb{R}^3$, there is a strictly lattice-free rational polytope which cannot be covered by fewer than 7 split sets.*

Proof. Let T_0 and T_1 be triangles in $\mathbb{R}^2$, constructed using Lemma 4.8, such that each triangle cannot be weakly covered by two split sets and $\Sigma(T_0, 6, 3) \cap \Sigma(T_1, 6, 3) = \emptyset$. Let

$$V_i = \{(x, y, i) | x, y \in \mathbb{R}\} \text{ and } \mathbf{T}_i = \{(x, y, i) | (x, y) \in T_i\} \text{ for } i = 0, 1.$$

In addition let $Q = \{(x, y) : 0 \leq x, y \leq 7\}$. Q has an area of 49, but the area of Q covered by any (translated) split set in $\mathbb{R}^2$ is at most 7, and hence at least 7 such (translated) split sets are needed to cover Q. Finally, let $\mathbf{T}' = \operatorname{conv}\left(\mathbf{T}_0, \mathbf{T}_1, Q^{1/2}\right)$, where $Q^{1/2} = \{(x, y, z) : 0 \leq x, y \leq 7,\ z = 1/2\}$. We will show that $\mathbf{T}'$ has the desired property.

Assume that σ is a collection of six split sets in $\mathbb{R}^3$ which covers $\mathbf{T}'$. It is clear from the proof of Theorem 4.11, that there must be three distinct split sets S_1, S_2, S_3 in σ such that $\hat{\Pi}(S_i \cap V_0) \in \Sigma(T_0, 6, 3)$ for $i = 1, 2, 3$. Similarly, there must be three distinct split sets S'_1, S'_2, S'_3 such that $\hat{\Pi}(S'_i \cap V_1) \in \Sigma(T_1, 6, 3)$ for $i = 1, 2, 3$. As $\Sigma(T_0, 6, 3) \cap \Sigma(T_1, 6, 3) = \emptyset$, the above six split sets comprise all of σ. Therefore all split sets in σ have the property that their defining vectors $(\alpha_1, \alpha_2, \alpha_3)$ satisfy $(\alpha_1, \alpha_2) \neq (0, 0)$, and therefore the intersection of these split sets with $\{(x, y, z) : z = 1/2\}$ are nonempty, two-dimensional translated split sets which cover Q. But Q cannot be covered by any six translated split sets in $\mathbb{R}^2$, a contradiction. ∎

The next theorem connects the previous results with the inexpressibility of facet-defining inequalities of polyhedral sets as t-branch split cuts.

Theorem 4.15. *Let $B \in \mathbb{R}^n$ be a strictly lattice-free rational polytope that cannot be covered by t split sets for some positive integer t. Then there is a rational mixed-integer polyhedral set in $\mathbb{Z}^n \times \mathbb{R}$ with a facet-defining inequality that cannot be expressed as a t-branch split cut.*

Proof. Let B and t satisfy the conditions of the theorem. By Remark 4.13, there exists a rational $\varepsilon > 0$ such that $D_\varepsilon(B)$ is strictly lattice-free. Let $\bar{B} = D_\varepsilon(B)$. Then $\bar{B}$ is full-dimensional as it contains B in its interior, and the interior of $\bar{B}$ cannot be covered by t split sets. Let $\bar{x}$ be a point in the interior of $\bar{B}$. Let B' be the polyhedron in $\mathbb{R}^{n+1}$ defined as $\operatorname{conv}\big((\bar{B} \times 0) \cup (\bar{x} \times \{1/2\})\big)$. We define a mixed-integer polyhedral set P_B as follows:

$$P_B = \{(x, y) \in \mathbb{Z}^n \times \mathbb{R} : (x, y) \in B'\}.$$

All mixed-integer solutions of P_B satisfy $y \le 0$ (in fact, the convex hull of solutions equals $P_B \cap \{y = 0\}$).

Let $S_1, \ldots, S_t$ be t arbitrary split sets in $\mathbb{R}^{n+1}$ defined on the x variables, i.e., they are of the form $\{(x, y) \in \mathbb{R}^{n+1} : b_i < a_i^T x < b_{i+1}\}$ where $b_1, \ldots, b_t$ are integers and $a_1, \ldots a_t \in \mathbb{Z}^n$. Recall that $y \le 0$ is a t-branch split cut for P_B derived from the disjunction associated with the split sets $S_1, \ldots, S_t$ if and only it is valid for $B' \setminus \cup_{i=1}^t S_i$. The sets $\bar{S}_i = S_i \cap \{(x, y) \in \mathbb{R}^{n+1} : y = 0\}$ are split sets in $\mathbb{R}^n$ and do not cover the interior of $\bar{B}$. Let $\hat{x} \in \operatorname{int}(\bar{B}) \setminus \cup_{i=1}^t \bar{S}_i$. Then $B' \setminus \cup_{i=1}^t S_i$ contains a point of the form $(\hat{x}, \varepsilon')$ for some $\varepsilon' > 0$. This point violates the inequality $y \le 0$, and thus $y \le 0$ cannot be expressed as a t-branch split cut.
∎

5 Two and three dimensions

In this section, one of our goals is to express lattice-free cuts as t-branch split cuts, in dimensions two and three. As mentioned earlier, it is shown in [11] that every cut based on a maximal lattice-free set in $\mathbb{R}^2$ is implied by a crooked cross cut and by a 3-branch split cut. Such results are not well-studied in $\mathbb{R}^3$. This is partially because the result in [11] is derived by using the classification of maximal lattice-free sets in $\mathbb{R}^2$ of Dey and Wolsey [14]. An analogous classification result is not yet known in $\mathbb{R}^3$, and seems unattainable in $\mathbb{R}^n$ with current tools.

A second goal in this section is to express lattice-free cuts as cuts based on branching disjunctions with few atoms. As an example, consider the fact that a cut based on a maximal lattice-free triangle of type 3 (*triangle cuts of type 3*) is also a 3-branch split cut. Based on this result, one can separate over a family of cuts containing all triangle cuts of type 3, by separating over 3-branch split cuts with 6 atoms. On the other hand, the result in [11] showing that a triangle cut of type 3 is a crooked cross cut implies that such a cut can be obtained from a disjunction with only 4 atoms instead of 6. For a fixed disjunction, the complexity of optimizing over the set of all disjunctive cuts obtainable from the disjunction is directly related to the number of atoms in the disjunction.

Therefore, the number of parameters required to specify a disjunctive cut (e.g., the t vectors in a t-branch split cut), and the number of atoms in the disjunction are both interesting quantities. We are interested in obtaining the best-possible bounds on these quantities in $\mathbb{R}^2$ and $\mathbb{R}^3$ without any knowledge of the classification of maximal lattice-free sets in these dimensions.

In $\mathbb{R}^2$ the lattice width of any lattice-free convex body is strictly less than 3; this is implied both by the result of Hurkens (Theorem 3.1) and by the result of Kannan and Lovász. If a stands for the direction of minimum lattice width, B may intersect up to 4 parallel split sets of the form $\{x \in \mathbb{R}^3 : b < a^T x < b + 1\}$. Then B minus the union of these sets consists of one-dimensional sets of the form $\{x \in B : a^T x = b\}$ for at most 3 consecutive values of b; each such set needs at most one split set to cover it. We thus get the following result.

Lemma 5.1. *Any strictly lattice-free convex set B in $\mathbb{R}^2$ is contained in the union of 7 split sets. Further, these split sets yield a branching disjunction with 8 atoms.*

Note that the number of atoms is only two more than the number of atoms in a 3-branch split disjunction which implies a triangle cut of type 3. To improve Lemma 5.1, we will next show that in fact any strictly lattice-free convex set in $\mathbb{R}^2$ intersects at most three parallel split sets. We start by making the following basic observations

Lemma 5.2. *Two intersecting lines divide the plane into four open sectors. If a convex body B intersects a sector and B minus the closure of this sector intersects the closure of each of the other sectors, then q is in the interior of B.*

Note that the above lemma only requires three points. For example, if P contains $(-1, 0), (1, -1)$ and any point in the interior of the first quadrant $\{x \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$ then it contains the intersection of the axes, the point $(0, 0)$.

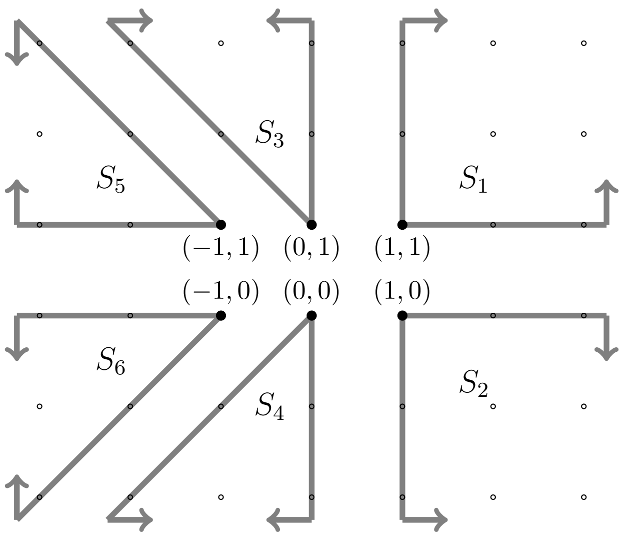
We are now ready to show that any strictly lattice-free convex set in $\mathbb{R}^2$ intersects at most three parallel split sets. In the proof, we use that any maximal lattice-free set contains at least 2 integer points on its boundary. Note that this fact does not require the classification of maximal lattice-free sets.

Theorem 5.3. *Let B be a maximal lattice-free set in $\mathbb{R}^2$. Then there is a vector $a \in \mathbb{Z}^2$ and an integer b such that B is contained in $\{x \in \mathbb{R}^2 : b \leq a^T x \leq b + 3\}$.*

Proof. B contains two lattice points in its boundary and after a unimodular transformation, we can assume these points are $(0, 0)$ and $(0, 1)$. Assuming B is not contained in three vertical strips, it has (without loss of generality) a point $x = (x_1, x_2)$ with $1 < x_1 < 2$, and by convexity a point $y = (1, y_2)$ with $k < y_2 < k+1$, for some $k \in \mathbb{Z}$. Left-multiplying B by the unimodular matrix

$$M = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$$

which leaves points $(0, 0), (0, 1)$ unchanged, we can assume $0 < y_2 < 1$.



$$S_1 := \{x \in \mathbb{R}^2 : x_1 > 1, x_2 > 1\}$$

$$S_2 := \{x \in \mathbb{R}^2 : x_1 > 1, x_2 < 0\}$$

$$S_3 := \{x \in \mathbb{R}^2 : x_1 < 0, x_2 + x_1 > 1\}$$

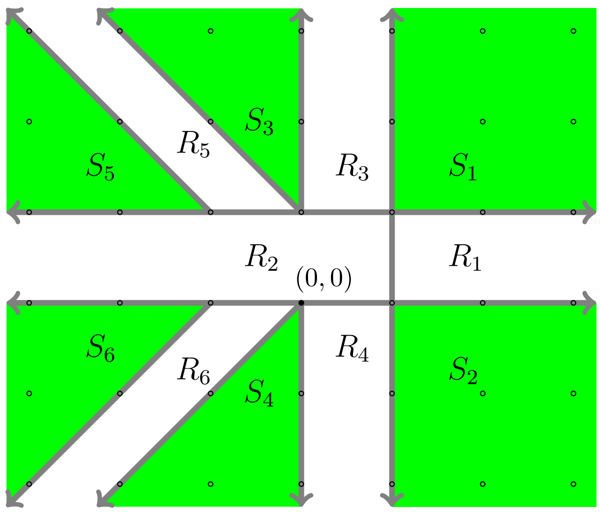
$$S_4 := \{x \in \mathbb{R}^2 : x_1 < 0, x_2 - x_1 < 0\}$$

$$S_5 := \{x \in \mathbb{R}^2 : x_2 > 1, x_2 + x_1 < 0\}$$

$$S_6 := \{x \in \mathbb{R}^2 : x_2 < 0, x_2 - x_1 > 1\}$$

Figure 3: Regions that have empty intersection with B

Consider the 6 sets $S_1, S_2, \ldots, S_6 \in \mathbb{R}^2$ shown in Figure 3. Applying Lemma 5.2 with the two bounding lines of the set and two of the points $(0, 0), (0, 1), y$, it is easy to argue that none of these sets can intersect B. Therefore $B \subset \bigcup_{i=1}^{6} R_i$, see Figure 4 for the sets $R_1, R_2, \ldots, R_6 \in \mathbb{R}^2$.



$R_1 := \{x \in R^2 : x_1 \geq 1, 0 \leq x_2 \leq 1\}$

$R_2 := \{x \in R^2 : x_1 \leq 1, 0 \leq x_2 \leq 1\}$

$R_3 := \{x \in R^2 : x_2 \geq 1, 0 \leq x_1 \leq 1\}$

$R_4 := \{x \in R^2 : x_2 \leq 0, 0 \leq x_1 \leq 1\}$

$R_5 := \{x \in R^2 : x_2 \geq 1, 0 \leq x_2 + x_1 \leq 1\}$

$R_6 := \{x \in R^2 : x_2 \leq 0, 0 \leq x_2 - x_1 \leq 1\}$

Figure 4: Regions that can have non-empty intersection with B

Suppose there exists $z \in B \cap R_1$ satisfying $z_1 \geq 2$. Then we claim that B lies in $\{x \in R^2 : -1 \leq x_2 \leq 2\}$. Indeed, if $u \in B$ with $u_2 > 2$ then the line from u to z crosses either S_1 or S_3, depending on whether u is in R_3 or R_5, contradiction. We can similarly rule out points $u \in B$ with $u_2 < -1$, showing the claim. If there is $z \in B \cap R_2$ with $z_1 \leq -1$ we can also show that B lies in $\{x \in \mathbb{R}^2 : -1 \leq x_2 \leq 2\}$.

These cases being treated, we can henceforth assume that if $u \in B$ satisfies $0 \leq u_2 \leq 1$ (so $u \in R_1 \cup R_2$), then $-1 < u_1 < 2$.

If $B \not\subset \{x \in \mathbb{R}^2 : -1 \leq x_1 \leq 2\}$ (three vertical strips), then the set $P := B \cap \{x \in \mathbb{R}^2 : x_1 < -1\}$ is non-empty and is contained in either R_5 or R_6, but not both. The picture is still symmetric (in the line $x_2 = 1/2$), so it suffices to treat the case $P \subset R_5$.

Note that in this case, $B \cap R_3 = \emptyset$, for otherwise B would contain a point in S_3. Also, $B \cap R_4 \subset \{x \in \mathbb{R}^2 : x_2 + 2x_1 > 0\}$, otherwise B would contain a point in S_4.

Suppose $B \cap \{x \in R^2 : x_2 > 2\} \neq \emptyset$, then B and in particular $B \cap R_1$ lie in $\{x \in \mathbb{R}^2 : x_2 + x_1 < 2\}$, since $B \cap S_3 = \emptyset$. Then B contains a point in $\{x \in \mathbb{R}^2 : x_2 + x_1 < -1\}$, necessarily in R_6, otherwise B is contained in three diagonal strips. Since $B \cap S_4 = \emptyset$, we deduce that $B \cap R_4 = \emptyset$. To finish this case, we need a slightly more complicated forcing argument with three points, where the first point restricts the location of the second, the second restricts the third and the third restricts the first, arising in a contradiction. Let $y = (1, y_2) \in B$ be as before, with $0 < y_2 < 1$. Let $u \in B$ satisfy $u_2 = 2$. Since $B \cap S_3 = \emptyset$ (so the line joining u and y does not intersect S_3) we have $u_1 < -1 - y_2$. Let $z \in B$ satisfy $z_2 + z_1 = -1$. Since the line joining u and z does not cross S_5, we deduce (that $z_1 + 1 \geq -1 - u_1 > y_2$ and thus) $z_2 < -y_2$. But then the line joining z and y crosses S_4, contradiction, see Figure 5.

Next, suppose $B \cap \{x \in \mathbb{R}^2 : x_2 > 2\} = \emptyset$. We can also suppose B contains a point $u \in R_4$ with $u_2 = -1$ and $0 < u_1 < 1$, otherwise B is contained in three horizontal strips. Then $B \cap R_6 = \emptyset$. Assume B is not contained in the three diagonal strips $\{x \in \mathbb{R}^2 : -1 \leq x_2 + x_1 \leq 2\}$. Then there is a point $z \in B$, necessarily in R_1, with $z_2 + z_1 = 2$ and $0 < z_2 < 1$. We do the three-point forcing argument again. Let $v \in B$ satisfy $v_1 = -1$, $1 < v_2 < 2$, which exists since P is non-empty. The line joining u to v does not intersect S_4, so $v_2 - 1 > 1 - u_1$. Joining v to z, we do not cross S_3, so $v_2 - 1 < 1 - z_2$, so $1 - z_2 > 1 - u_1$. Then the line joining u to z crosses S_2, contradiction. ∎

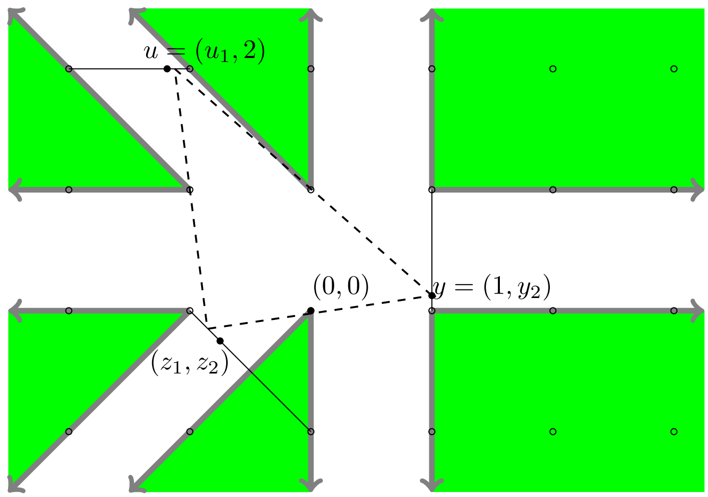


Figure 5: When B intersects with $\{x \in \mathbb{R}^2 : x_2 > 2\}$

Using Theorem 5.3, it is possible to improve Lemma 5.1 by reducing the number of atoms to 6, and the number of split sets to 6. We will next show that if we use branching disjunctions, we can reduce the number of atoms to 4 without explicitly using the classification of maximal lattice-free sets in $\mathbb{R}^2$, as used in [11].

We will say that a disjunction D in $\mathbb{R}^2$ *excludes* a lattice-free convex set B if D does not interset the interior of B.

Theorem 5.4. *Let B be a maximal lattice-free set in $\mathbb{R}^2$. Then there is a branching disjunction D with 4 atoms which excludes B.*

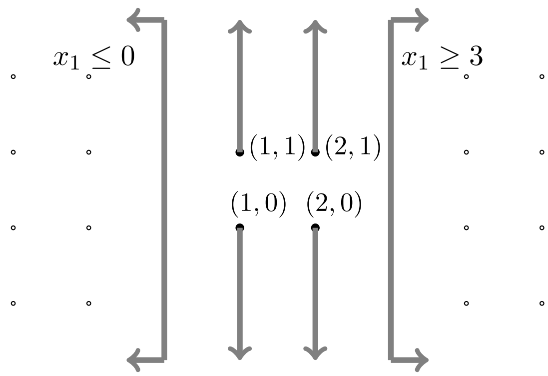
Proof. By Theorem 5.3, $B \subseteq \{x \in \mathbb{R}^2 : b \leq a^T x \leq b + 3\}$ for some $a \in \mathbb{Z}^2$ and $b \in Z$. Without loss of generality (and after a unimodular transformation and shifting, if necessary) we can assume that $a = [1, 0]^T$ and $b = 0$. Clearly,

$$B \cap \{x \in \mathbb{R}^2 : a^T x = b + 1\} = B \cap \{x \in \mathbb{R}^2 : x_1 = 1\} \subseteq \{x \in \mathbb{R}^2 : c \leq x_2 \leq c + 1\}$$

for some $c \in Z$ and similarly, $B \cap \{x \in \mathbb{R}^2 : x_1 = 2\} \subseteq \{x \in \mathbb{R}^2 : d \leq x_2 \leq d+1\}$ for some $d \in Z$. Again without loss of generality we can assume that $c = d = 0$. Therefore, after some shifting and unimodular transformations, the interior of B is contained in $\mathbb{R}^2 \setminus D'$ for the branching disjunction $D' = \cup_{i=1,\ldots,6} D_i$ with 6 atoms as shown in Figure 6. In the rest of the proof we will consider different cases to reduce the number of atoms.

Let $S_1 = \{x \in \mathbb{R}^2 : 0 \leq x_1 < 1\}$ and $S_3 = \{x \in \mathbb{R}^2 : 2 < x_1 \leq 3\}$. First assume that $B \cap S_3 = \emptyset$. In this case the 4 atom disjunction consisting of D_1, D_2, D_3 of Figure 6 and $\{x \in \mathbb{R}^2 : x_1 \geq 2\}$ proves the claim. Similarly $B \cap S_1 = \emptyset$ is easily handled and therefore in the rest of the proof we will assume that $B \cap S_i \neq \emptyset$ for $i = 1, 3$.

Now assume that B has exactly one vertex v_1 in S_1. In this case, let q be the x_2 coordinate of the point that lies on the intersection of the boundary of D_1 and the line that passes through the points $(1, 1)$ and v_1. In other words, the following three points $(0, q)$, v_1 and $(1, 1)$ lie on the same line. In this case, the disjunction defined by the atoms D_4, D_5, D_6 together with $D_2' = \{x \in \mathbb{R}^2 : x_1 \leq 1, \lfloor q \rfloor x_1 + x_2 \geq \lfloor q \rfloor + 1\}$ and $D_3' = \{x \in \mathbb{R}^2 : x_1 \leq 1, \lfloor q \rfloor x_1 + x_2 \leq \lfloor q \rfloor\}$ excludes B, see Figure 7. If in addition to having exactly



$D_1 = \{x \in \mathbb{R}^2 : x_1 \le 0\}$

$D_2 = \{x \in \mathbb{R}^2 : x_1 = 1, x_2 \ge 1\}$

$D_3 = \{x \in \mathbb{R}^2 : x_1 = 1, x_2 \le 0\}$

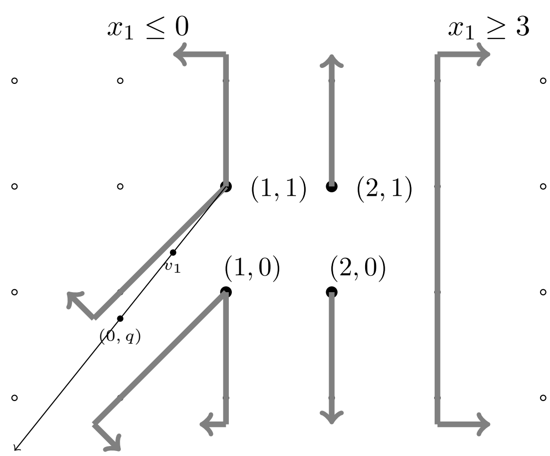
$D_4 = \{x \in \mathbb{R}^2 : x_1 = 2, x_2 \ge 1\}$

$D_5 = \{x \in \mathbb{R}^2 : x_1 = 2, x_2 \le 0\}$

$D_6 = \{x \in \mathbb{R}^2 : x_1 \ge 3\}$

Figure 6: A branching disjunction with 6 atoms

one vertex in S_1, the set B also has exactly one vertex v_3 in S_3, then using a symmetric argument, it is easy to replace D_4, D_5, D_6 with two new atoms D'_4 and D'_5 to show that B is in fact excluded by a branching 4 atom disjunction. We therefore conclude that one of S_1 or S_3 must contain at least 2 vertices of B and in the rest of the proof, without loss of generality, we assume that S_3 contains at least 2 vertices.



$D'_2 = \{x \in \mathbb{R}^2 : x_1 \le 1, \lfloor q \rfloor x_1 + x_2 \ge \lfloor q \rfloor + 1\}$

$D'_3 = \{x \in \mathbb{R}^2 : x_1 \le 1, \lfloor q \rfloor x_1 + x_2 \le \lfloor q \rfloor\}$

$D_4 = \{x \in \mathbb{R}^2 : x_1 = 2, x_2 \ge 1\}$

$D_5 = \{x \in \mathbb{R}^2 : x_1 = 2, x_2 \le 0\}$

$D_6 = \{x \in \mathbb{R}^2 : x_1 \ge 3\}$

Figure 7: A branching disjunction with 5 atoms when B has exactly one vertex in S_1

Furthermore, if neither one of the sets $D_{2,4} = conv(D_2, D_4)$ and $D_{3,5} = conv(D_3, D_5)$ contain a vertex of B in their interior, then the disjunction obtained by $D_1, D_{2,4}, D_{3,5}$ and D_6 proves the claim, see Figure 8. Consequently, B must have at least one vertex in one of $D_{2,4}$ or $D_{3,5}$. Without loss of generality, assume that $D_{2,4}$ contains a vertex in its interior. Now remember that B is a maximal lattice-free set and as such has at most 4 vertices. Therefore, B must have exactly one vertex in S_1, one vertex in the interior of $D_{2,4}$ and two vertices in S_3.

As B has a vertex in the interior of $D_{2,4}$ (and therefore with an x_2 coordinate greater than 1), the vertex in S_1 must have its x_2 coordinate less than 1, otherwise B would contain the point $(1, 1)$ in its interior. Similarly the two vertices in S_3 have their x_2 coordinate less than 1. Further, the line segment joining these two vertices contains an integer point in its relative interior as B is a maximal lattice-free set. As $B \subseteq \{x : x_1 \le 3\}$, this integer point must lie on the line $x_1 = 3$ and therfore the two vertices in S_3 also lie

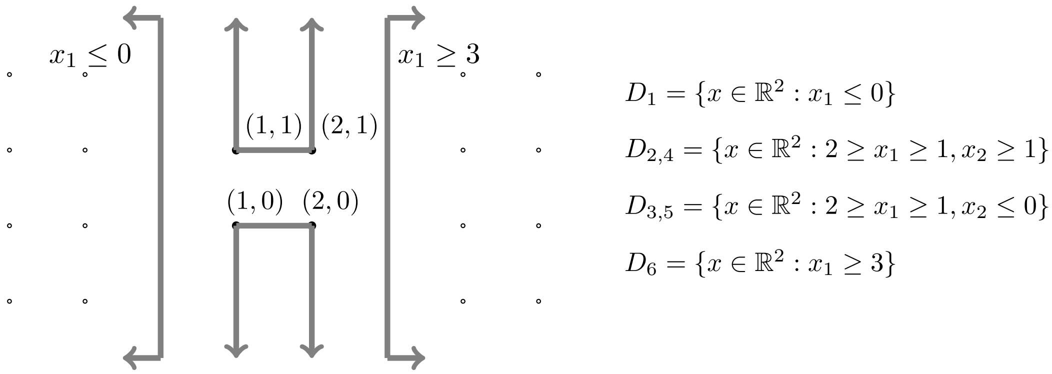


Figure 8: A branching disjunction with 4 atoms

on the line $x_1 = 3$. Further, given the position of the other two vertices, the integer point must be $(3, 0)$, and the vertices in S_3 must have their x_2 coordinates in the intervals $(0, 1)$ and $(-1, 0)$, respectively. Therefore the vertex in S_1 has its x_2 coordinate greater than 0, and thus the disjunction defined by D_4 and D_6 together with $D'_{35} = \{x \in \mathbb{R}^2 : x_1 \leq 2, x_2 \leq 0\}$ and $D'_2 = \{x \in \mathbb{R}^2 : x_1 \leq 0, x_2 \geq 1\}$ excludes B.

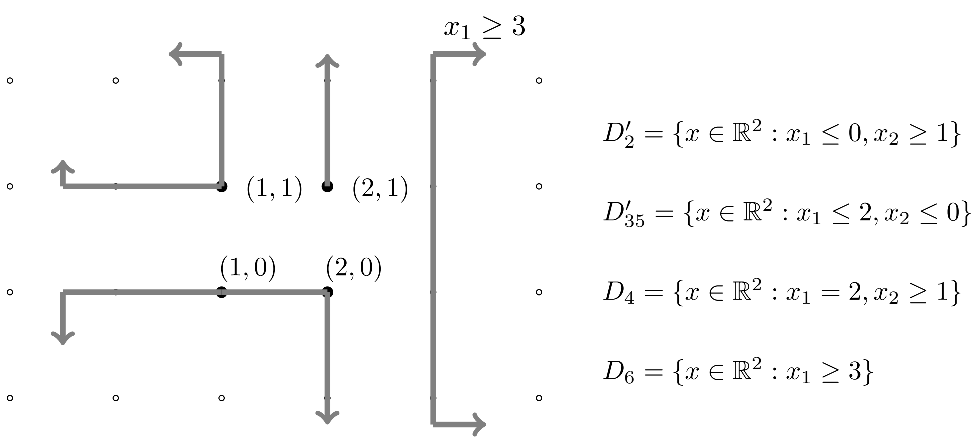


Figure 9: Another branching disjunction with 4 atoms

∎

The previous result and Theorem 3.2 imply the following result.

Theorem 5.5. *Any strictly lattice-free convex set B in $\mathbb{R}^3$ is contained in the union of 21 split sets. Further, there is a branching disjunction with 22 atoms which exculdes B.*

Proof. If a stands for the direction of minimum lattice width, we need split sets of the form $\{x \in \mathbb{R}^3 : b < a^T x < b + 1\}$ for up to 6 consecutive values of b. Then B minus the union of these sets consists of two-dimensional lattice-free sets of the form $\{x \in B : a^T x = b\}$ for at most 5 consecutive values of b. Each such lattice-free set needs 3 split sets for a total of 21 split sets. If $l = \min\{a^T x : x \in B\}$ and $u = \max\{a^T x : x \in B\}$, then there is a branching disjunction with 22 atoms excluding B; it consists of the atoms $\{x \in \mathbb{R}^3 : a^T x \leq \lfloor l \rfloor\}, \{x \in \mathbb{R}^3 : a^T x \geq \lceil u \rceil\}$ and 4 atoms for each nonempty set of the form $\{x \in B : a^T x = b\}$. ∎

The above upper bound of 21 split sets is quite a bit higher than the lower bound of 7 split sets we obtained earlier. We believe the best possible value is close to 7.

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