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Mathematics

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Sylvester-Gallai-like Theorems for Polygons in the Plane

Jonathan Lenchner
IBM Research Division
Thomas J. Watson Research Center
P.O. Box 704
Yorktown Heights, NY 10598

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Jonathan Lenchner*

Abstract

Given an arrangement of lines in the plane, an *ordinary point* is a point of intersection of precisely two of the lines. Motivated by a desire to understand the fine structure of ordinary points in line arrangements, we consider the following problem: given a polygon P in the plane and a family of lines passing into the interior of P, how many ordinary intersection points must there be on or inside of P? We answer this question for a variety of different types of polygons. In a similar spirit, we investigate bichromatic arrangements of lines intersecting the interior of polygons and determine when such arrangements must have monochromatic intersection points.

1 Introduction

The Sylvester-Gallai Theorem [10, 15, 20] tells us that a finite collection of lines in the projective plane, not all passing through a single point, must have ordinary points – points of intersection of precisely two lines. Let us call a collection of lines all passing through a common point "trivial," since we shall explicitly exclude such collections from our consideration. Much work has gone into trying to understand, for a given n, what the minimum number of ordinary points can be among all non-trivial arrangements of n lines, and indeed this *quantitative* Sylvester problem[1] has turned out to be quite difficult. In 1993 Csima and Sawyer [5] showed that for $n \neq 7$ there must be at least $\frac{6n}{13}$ ordinary points, while it has been widely conjectured that for $n \neq 7, 13$ there must be at least $\frac{n}{2}$ ordinary points.

Melchior [16] showed that the Sylvester-Gallai Theorem is a consequence of Euler's formula regarding the facial structure of line or pseudoline arrangements in the projective plane (equation (1) of section 3 below). The very same Euler formula also holds within any simple (i.e. non self-intersecting) polygon without holes determined by the lines of an arrangement. However, at least in its most obvious form, there is no Sylvester-Gallai Theorem for the lines intersecting in such polygons. Henceforth we take all of our polygons (which we shall not continue to qualify) to be simple polygons without holes. In this paper we develop variants of the Sylvester-Gallai Theorem suitable for lines intersecting in these polygons. By examining extremal examples of ordinary points in polygons and then studying how these polygons can be pieced together to get line or pseudoline arrangements we hope to develop new methods for attacking the quantitative Sylvester problem.

For concreteness, and simplicity of exposition, we confine our attention to the affine plane. Thus, if we have three lines (alternatively, three points), no two of which are parallel (not all of which lie on a line), they determine a unique triangle.

Following our treatment of questions regarding ordinary points in polygons, we study problems related to the well-known Motzkin-Rabin Theorem regarding the presence of monochromatic intersection points in two-colored collections of lines. Again the context will be affine polygons.

2 Historical Background

In 1893, J. J. Sylvester [20] posed the following celebrated problem: Given a finite collection of points in the affine plane, not all lying on a line, show that there exists a line which passes through precisely two of the points. Sylvester's problem was reposed by Erdős in 1944 [8] and later that year a proof was given by Gallai [10]. The result is therefore most customarily known as the Sylvester-Gallai Theorem. The statement and conclusion of the Sylvester-Gallai Theorem clearly holds in either the affine *or* projective planes. In the projective plane, the dual statement says that given a finite collection of lines, not all passing through a single point, there must be a point of intersection of just two lines. Some time after Gallai's proof was published, it was appreciated that Melchior had already established the same result in 1940 [16].

Shortly after establishment of the Sylvester-Gallai Theorem, work began on trying to understand, for a given n, what the minimum number of ordinary points can be amongst all non-trivial arrangements of n lines in the projective plane [2,9]. It was widely conjectured, beginning with Dirac and Motzkin in 1951 [7, 17], that for sufficiently large n there must be at least $\frac{n}{2}$ ordinary points. Earlier, Böröczky (as cited in [4]) had found examples, for even $n \geq 6$, of n lines and precisely $\frac{n}{2}$ ordinary points. In his 1951 paper, Motzkin [17] showed that there must be at least $O(\sqrt{n})$ ordinary points. In 1958 Kelly and Moser [12] showed that there are always $\frac{3n}{7}$ ordinary points, and then in 1993 Csima and Sawyer [5] showed that for

*IBM T. J. Watson Research Center, Yorktown Heights, NY 10598. lenchner@us.ibm.com .

[1]Phraseology which I attribute to Ricky Pollack [18].

$n \neq 7$ there must be at least $\frac{6n}{13}$ ordinary points. Meanwhile, in their 1958 paper, Kelly and Moser [12] observed that there is an arrangement of 7 lines with just 3 ordinary points, and then in 1968 Crowe and McKee [4] found an arrangement of 13 lines with 6 ordinary points. Thus, the $\frac{n}{2}$-conjecture, as it is sometimes called, now says that if $n \neq 7, 13$, then there must be at least $\frac{n}{2}$ ordinary points.

In a series of papers beginning with [13] and culminating in [14], the current author studied non-trivial arrangements in the affine plane. In the affine plane, arrangements with all lines through a common point or all lines mutually parallel are ruled out as being "trivial." For such non-trivial arrangements, it turns out that there must be at least $\frac{2n-3}{7}$ affine ordinary points, as long as $n \neq 6$.

In the papers of Kelly-Moser [12], Csima-Sawyer [5, 6] and our own [14] use is made of a series of lemmas saying that lines with few ordinary points on them must have a given number of "attached ordinary points." An ordinary point p is *attached* to a line ℓ if the two lines through p, in addition to ℓ, form a cell of the arrangement. These "local" lemmas become the machinery for obtaining the various global bounds on the number of ordinary points. The current paper tries to build up an alternative set of machinery for crafting global incidence theorems based on local results.

The Motzkin-Rabin Theorem says that if a collection of points, not all on a line, are each colored one of two colors, there must be a monochromatic line. In the projective plane the dual theorem states that if each line of a non-trivial collection of lines is colored one of two colors, then there must be a monochromatic point of intersection. A proof of the Motzkin-Rabin Theorem was never published by its authors [11]. A bit more historical context for this theorem is provided in Section 5.

3 Prelude

Let us begin by giving Melchior's famous proof [16] of the Sylvester-Gallai Theorem. Given an arrangement $\mathcal{A}$ of lines in the projective plane with V vertices, E edges, and F faces, Euler's formula says that

$$V - E + F = 1. \tag{1}$$

Following Melchior [16], we put

$$\begin{aligned} t_j &= \text{number of vertices where j lines cross,} \\ p_k &= \text{number of faces surrounded by k edges,} \end{aligned}$$

and write

$$Y = \sum_{j\geq 2}(3-j)t_j + \sum_{k\geq 3}(3-k)p_k. \tag{2}$$

By definition,

$$\sum_{j\geq 2} t_j = V, \quad \sum_{k\geq 3} p_k = F, \tag{3}$$

while,

$$\sum_{j\geq 2} jt_j = E \tag{4}$$

since $\sum_{j\geq 2} 2jt_j$ counts each edge twice, once at each of its defining vertices. and

$$\sum_{k\geq 3} kp_k = 2E, \tag{5}$$

since each edge is incident to two faces.

Thus we have

$$\begin{aligned} Y &= \sum_{j\geq 2}(3-j)t_j + \sum_{k\geq 3}(3-k)p_k \\ &= \sum_{j\geq 2} 3t_j - \sum_{j\geq 2} jt_j + \sum_{k\geq 3} 3p_k - \sum_{k\geq 3} kp_k \\ &= 3V - E + 3F - 2E \\ &= 3(V - E + F) \\ &= 3. \end{aligned}$$

Now, looking at (2) we note that only the coefficient of the t_2 term, which is 1, is positive, so we must have $t_2 \geq 3$, establishing the Sylvester-Gallai Theorem, and in fact a slightly stronger conclusion, namely that there must be at least three ordinary points.

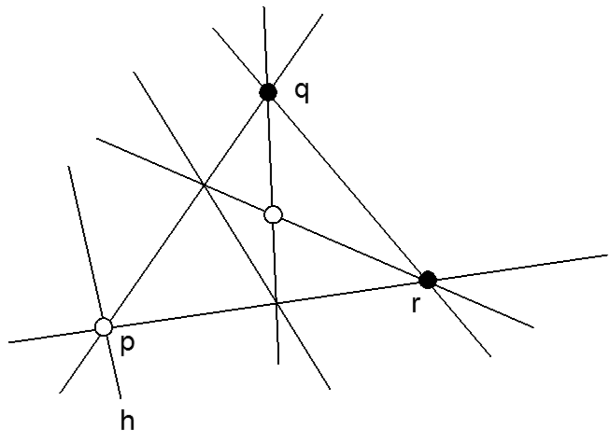

Figure 1. We consider the finite triangle $\triangle(p,q,r)$ in the statement of Theorem 1. Since line h does not intersect the interior of $\triangle(p,q,r)$ it is not considered. Amongst the remaining lines, the Theorem concludes that there must be at least 2 ordinary points in $\triangle(p,q,r)$ – indicated here with hollow circles.

4 Main Results

Theorem 1 *Consider any triangle formed by the lines of an arrangement and the vertices formed inside or on the boundary of the triangle by the defining lines of the triangle together with the lines which intersect the interior of the triangle. Amongst these vertices and lines only, there must be at least* 2 *ordinary points.*

To better understand the statement of the Theorem, see Figure 1.

Proof. Let T be an arbitrary triangle formed by the lines of an arrangement. In any polygon Euler's relation holds and says that

$$V - E + F = 1 \tag{6}$$

where V, E and F, respectively, refer to the vertices, edges and faces of the arrangement lying inside or on the boundary of T. We would like to use (6) to tell us something about the familiar expression

$$Y_T = \sum_{j\geq 2}(3-j)t_j + \sum_{k\geq 3}(3-k)p_k \tag{7}$$

where, in the present context, t_j denotes the number of j-crossings in T and p_k denotes the number of k-gons in T.

We still have

$$\sum_{j\geq 2} t_j = V, \quad \sum_{k\geq 3} p_k = F. \tag{8}$$

However edges along the boundary of T are not shared by two faces so we *don't* have

$$\sum_{k\geq 3} kp_k = 2E$$

and similarly, since edges don't extend outward from the borders of T we *don't* have

$$\sum_{j\geq 2} jt_j = E.$$

However, let us declare that we have a single "external face" wherever there is an edge lying along the border of T and similarly say that every line which is either a defining line of T or intersects the interior of T has a single "external edge" on the outside of T. See Figure 2. If we let

$$F_{ext} = \text{number of External Faces} \tag{9}$$
$$E_{ext} = \text{number of External Edges} \tag{10}$$

then we have

$$\sum_{k\geq 3} kp_k + F_{ext} = 2E, \tag{11}$$

$$\sum_{j\geq 2} jt_j = E + E_{ext}. \tag{12}$$

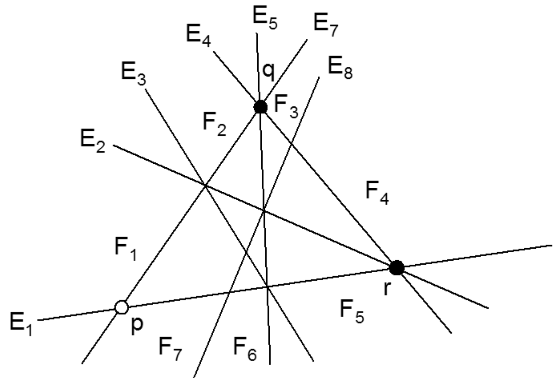

Figure 2. The finite triangle $T = \triangle(p,q,r)$ with external faces $F_1,...,F_7$ and external edges $E_1,...,E_8$ as described in the proof of Theorem 1.

Let us also define

$$\text{Excess Crossings} = \sum_{j\geq 4}(j-3)t_j, \tag{13}$$

and say that a vertex is a "k-crossing" if the vertex comes at the intersection of exactly k lines. The idea behind the naming (13) is that if we are looking for arrangements with few ordinary points, in a certain sense it is "excessive" to have a vertex which is more than a 3-crossing.

We claim that

$$E_{ext} - F_{ext} - \text{Excess Crossings} \leq 1. \tag{14}$$

To see this, start with triangle T consisting just of its three defining lines. We have

$$\begin{aligned} E_{ext} &= 3, \\ F_{ext} &= 3, \\ \text{Excess Crossings} &= 0, \\ t_{2,\partial} &= 3 \end{aligned}$$

where $t_{2,\partial}$ denotes the number of ordinary points on the boundary of T (henceforth denoted ∂T). Thus $E_{ext} - F_{ext} -$ Excess Crossings starts out at 0 and so the claimed inequality (14) starts out strict. Adding a line to T always increments E_{ext} by 1. Adding a line which induces a new crossing on ∂T increments F_{ext} by 1, while if the new line crosses ∂T at an existing vertex which is not an ordinary point, the Excess Crossings term is incremented by 1. Thus the only way to add a line and increase the left hand side of (14) is if the line passes through two previous ordinary points along ∂T, thus decrementing $t_{2,\partial}$ by 2. The effect of adding such a line is that the left hand side of (14) increases by 1. Note that any time a line is added which increases $t_{2,\partial}$, either increasing $t_{2,\partial}$ by 2 – which increases F_{ext} by 2, or increasing $t_{2,\partial}$ by 1 – which increases F_{ext} by 1 and Excess Crossings by 1, the left hand side of (14) decreases by 1. Thus any time lines are placed first to increase $t_{2,\partial}$ by either 1 or 2, and subsequently to decrease $t_{2,\partial}$ by 2, the magnitude of $E_{ext} - F_{ext} -$ Excess Crossings does not increase. Since we start with $E_{ext} - F_{ext} -$ Excess Crossings $= 0$, and we can lower $t_{2,\partial}$ at most once without a corresponding increase in $t_{2,\partial}$, the validity of equation (14) follows.

Substituting (8), (11) and (12) into (7) and applying (6) yields

$$\begin{aligned} Y_T &= 3V - (E + E_{ext}) + 3F - (2E - F_{ext}) \\ &= 3 + F_{ext} - E_{ext}. \end{aligned}$$

And applying (14) gives

$$Y_T \geq 2 - \text{Excess Crossings}. \tag{15}$$

In other words,

$$\sum_{j\geq 2}(3-j)t_j + \sum_{k\geq 3}(3-k)p_k \ \geq \ 2 - \text{Excess Crossings}. \tag{16}$$

Since

$$\sum_{j\geq 2}(3-j)t_j + \text{Excess Crossings} = t_2, \tag{17}$$

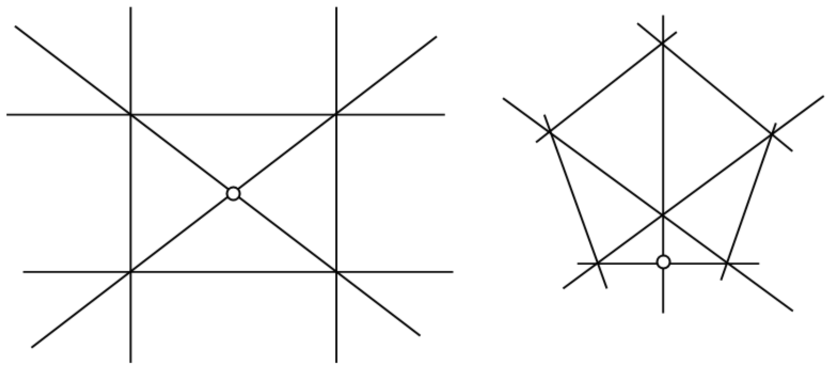

Figure 3. Tight examples for the convex cases in Theorems 3 and 5.

equation (16) becomes

$$t_2 + \sum_{k\geq 3}(3-k)p_k \geq 2, \tag{18}$$

and reasoning as in the punch-line of the Melchior proof, we conclude that $t_2 \geq 2$. □

Lemma 2 *If a triangle T as considered in Theorem 1 has exactly two ordinary points along its boundary then T must contain at least one additional ordinary point in its interior.*

Proof. As in the proof of the previous theorem, we start out with just the defining lines T and the relations

$$\begin{aligned} E_{ext} &= 3, \\ F_{ext} &= 3, \\ \text{Excess Crossings} &= 0, \\ t_{2,\partial} &= 3. \end{aligned}$$

But now, after all lines are added, $t_{2,\partial} = 2$. As we recall, the only time $E_{ext} - F_{ext} -$ Excess Crossings (the left hand side of the former equation (14)) increases is when $t_{2,\partial}$ drops by 2 – in which case $E_{ext} - F_{ext} -$ Excess Crossings increases by 1. Whenever $t_{2,\partial}$ increases, $E_{ext} - F_{ext} -$ Excess Crossings increases. It follows that we cannot lay down additional lines and increase $E_{ext} - F_{ext} -$ Excess Crossings, while still arriving at $t_{2,\partial} = 2$. Thus, under the assumptions of this lemma, the analog of equation (14) is

$$E_{ext} - F_{ext} - \text{Excess Crossings} \leq 0. \tag{19}$$

Therefore

$$\begin{aligned} Y_T &= 3 + F_{ext} - E_{ext} \\ &\geq 3 - \text{Excess Crossings}, \end{aligned}$$

and

$$\sum_{j\geq 2}(3-j)t_j + \sum_{k\geq 3}(3-k)p_k \;\geq\; 3 - \text{Excess Crossings}. \tag{20}$$

Then using (17), we obtain

$$t_2 \geq 3 \tag{21}$$

which is the desired conclusion. □

Henceforth we refer to polygons determined by the lines of an arrangement simply as “polygons in arrangements” and the ordinary points contained in these polygons, formed just by the defining lines of the polygons or lines intersecting the polygon’s interior, as “ordinary points in polygons.”

Theorem 3 *A quadrilateral in an arrangement must contain at least one ordinary point. A non-convex quadrilateral must contain at least two ordinary points.*

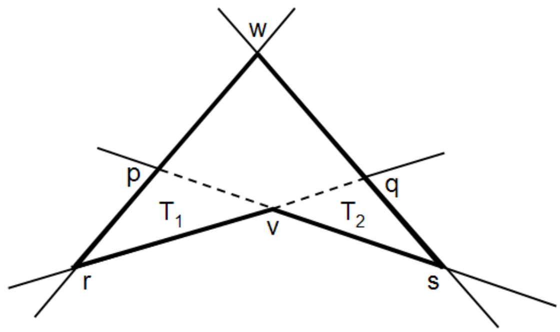

Figure 4. A non-convex quadrilateral with reflex vertex v and triangles $T_1 = \triangle(p,r,v)$ and $T_2 = \triangle(q,s,v)$.

See the left hand example in Figure 3 for a tight example in the convex case.

Proof. Let Q be an arbitrary quadrilateral in an arrangement and first let us suppose that Q is convex. The definitions and relations (9) - (12) are valid in any convex polygon, so we argue as in the proof of the previous theorem and lemma, starting with the defining lines of the quadrilateral and then adding additional lines. The key relation is now

$$E_{ext} - F_{ext} - \text{Excess Crossings} \leq 2, \tag{22}$$

and we start with the relations

$$\begin{aligned} E_{ext} &= 4, \\ F_{ext} &= 4, \\ \text{Excess Crossings} &= 0, \\ t_{2,\partial} &= 4. \end{aligned}$$

Since $E_{ext} - F_{ext} - \text{Excess Crossings}$ starts at 0, and there are just two opportunities to increase $E_{ext} - F_{ext} - \text{Excess Crossings}$ by 1, each time by adding lines and dropping $t_{2,\partial}$ by 2, relation (22) is established.

Then

$$\begin{aligned} Y_Q &= 3 + F_{ext} - E_{ext} \\ &\geq 3 - \text{Excess Crossings} - 2, \end{aligned}$$

so

$$t_2 \geq 1. \tag{23}$$

Now let Q be an arbitrary non-convex quadrilateral, with reflex vertex v as depicted in Figure 4, and consider triangles $T_1 = \triangle(p,r,v)$ and $T_2 = \triangle(q,s,v)$. By Theorem 1 we know that T_1 and T_2 each contain at least two ordinary points. But by Lemma 2, p and v cannot be the only ordinary points in T_1, and v and q cannot be the only ordinary points in T_2. It follows that T_1 contains an ordinary point not contained in T_2 and that T_2 contains an ordinary point not contained in T_1, and, moreover, that both of these ordinary points are ordinary with respect to T. Hence the non-convex quadrilateral Q contains two ordinary points and the theorem follows. $\square$

Lemma 4 *A quadrilateral with one ordinary point along its boundary must contain at least one additional ordinary point.*

Proof. Let Q be a quadrilateral with one ordinary point along its boundary. By Theorem 3 non-convex quadrilaterals always contain at least two ordinary points, hence we may assume that Q is convex. Then, if we start with the four defining lines of Q, we can establish the defining relation

$$E_{ext} - F_{ext} - \text{Excess Crossings} \leq 1, \tag{24}$$

since $E_{ext} - F_{ext} - \text{Excess Crossings}$ starts out at 0 while to begin with $t_{2,\partial} = 4$ and and after all lines are added $t_{2,\partial} = 1$. Hence we obtain

$$\begin{aligned} Y_Q &= 3 + F_{ext} - E_{ext} \\ &\geq 3 - \text{Excess Crossings} - 1, \end{aligned}$$

so

$$t_2 \geq 2, \tag{25}$$

and the lemma is established. $\square$

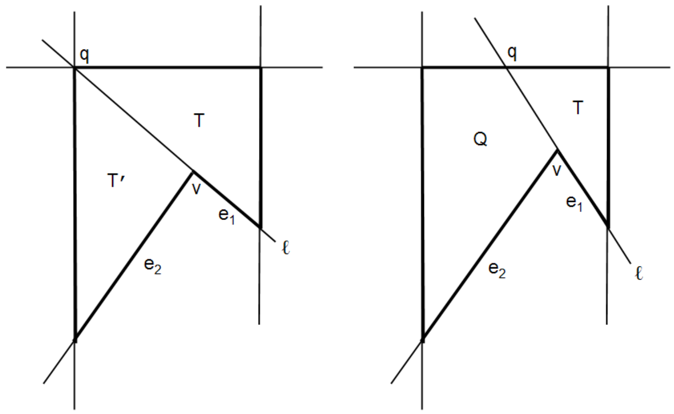

Figure 5. A non-convex pentagon P, with reflex vertex v, and one edge leading to v (the edge e_1), along the line ℓ, meeting P at a point q. Line ℓ may partition P into two triangles, per the left hand side where T and T' are the triangles, or into a triangle and a quadrilaterals, per the right hand side, where T and Q are respectively the triangle and quadrilateral. P may actually contain two reflex vertices, though this fact does not effect the argument.

Theorem 5 *A pentagon in an arrangement must contain at least one ordinary point. A non-convex pentagon must contain at least two ordinary points.*

Proof. Let P be an arbitrary pentagon in an arrangement. If P is convex we may argue precisely as we did in Theorem 3, for the case of a convex quadrilateral, to conclude that P must have at least one ordinary point. Thus assume that P is not convex and let v be one of the (possibly two) reflex vertices of P and let e_1 and e_2 be the two edges of P meeting at v, with ℓ the line in the sub-arrangement of P extending e_1. The line ℓ meets ∂P at a point $q \notin e_1$, and in the process partitions P into either two triangles or a triangle and a quadrilateral. See Figure 5. Regardless of which of these cases holds, the two smaller polygons share two corner points, q and v. Either q or v can be a "phony" ordinary point in the sense of being ordinary in one of the smaller polygons but not in P. First suppose that both small polygons are triangles, i.e., we are in the case depicted in the left-hand drawing in Figure 5. In this case we might as well assume that there are no additional lines through q so that q is ordinary in both triangles T and T'. By Theorem 1, T and T' each contain an additional ordinary point. If this additional ordinary point, call it p, is shared between T and T' then each triangle much contain a third ordinary point by virtue of Lemma 2. Call these, not necessarily distinct, points r and s. If r and s are distinct then two of $\{p,r,s\}$ are ordinary points in P and we are done. If $r = s$ then one of p and r must actually be the point v, lest we again have two ordinary points in P. But then v is necessarily not a phony ordinary point, since it is ordinary in both T and T' and then P has the necessary two ordinary points. Thus the second ordinary point for each small triangle, in addition to q, must not be shared. Call these non-shared ordinary points a and b, with a ordinary in T, and b ordinary in T'. Now either a and b are ordinary in P or one is actually vertex v, with v associated with T say (and so $v = a$). But then, by Lemma 2 again, T must contain a third ordinary point c and both b and c ordinary in P. Thus if ℓ partitions P into two triangles, as per the left-hand drawing in Figure 5, P necessarily has two ordinary points.

So suppose ℓ partitions P into a triangle T and a quadrilateral Q as in the right-hand drawing in Figure 5. In this case, if q is a phony ordinary point it must be ordinary in either T or Q. Also note that if v is phony then it must be ordinary in Q but not T (and not vice versa). Now note that since T has two ordinary points (Theorem 1) if our theorem is to be seen to be false it must be that either q or v is phony, with one or both points ordinary with respect to T, and since we have ruled out v in this respect, it must be that q is ordinary with respect to T. Moreover, T must have one other legitimate ordinary point (in other words, an ordinary point which is also ordinary in P). Let us give the legitimate ordinary point of T the name r. By Theorem 3, Q must have an ordinary point. If this ordinary point is not to constitute a second ordinary point for P it must either be v or r. However, by Lemma 4, it is readily seen that Q must have *both* v *and* r as ordinary points. But then r is necessarily on ∂T and so, by Lemma 2, T must have a third ordinary point, which is then necessarily also ordinary in P. This last ordinary point in addition to r constitute ordinary points in P and the theorem is proven. □

Figure 6 gives examples of a non-convex quadrilateral and a non-convex pentagon with just two ordinary points. There are $2n$-gons for $n \geq 3$ without ordinary points: start with any such regular $2n$-gon and add the n lines through pairs of opposite vertices. It is also not too difficult to see that there are 7-gons and, in general, $(2n+1)$-gons, for $n \geq 3$, which have no ordinary points. Figure 7 describes the construction for the 7-gon case.

Lemma 6 *A non-convex quadrilateral Q with two or three ordinary points on its boundary must contain another ordinary point.*

Proof. Consider the triangles T_1 and T_2 in the generic non-convex quadrilateral of Figure 4 and first suppose Q has just two ordinary points on its boundary. By Theorem 1 and Lemma 2, for T_1 and T_2 to each just have a single ordinary point of Q, requires that T_1 have ordinary points at v and r an T_2 to have ordinary points at v and s. However, if we then consider the quadrilateral Q' above T_1 and T_2 determined by vertices p,v,q and w, then by the first part of Theorem 3, either Q' has

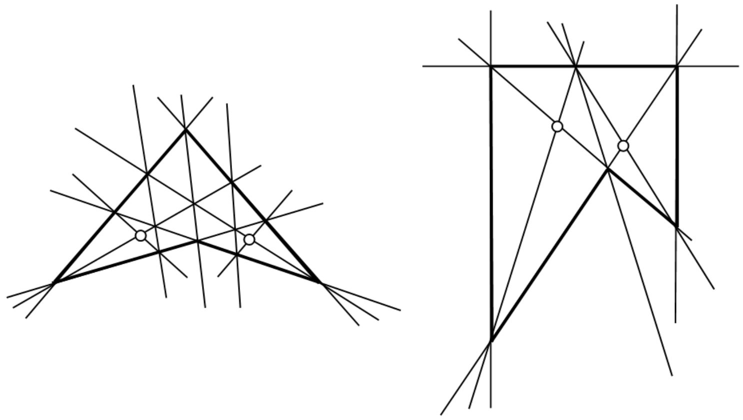

Figure 6. A non-convex quadrilateral (left) and non-convex pentagon (right), each with exactly two ordinary points – showing that the second halves of Theorems 3 and 5 are tight.

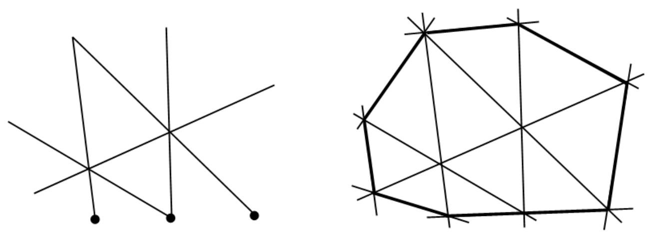

Figure 7. Constructing a 7-gon without ordinary points: Start with 5 line segments as shown on the left with given incidences, such that the bottom three endpoints (those indicated with solid dots) are collinear, there are no other collinearities between triples of endpoints, and the endpoints are in convex position. Now draw lines connecting the endpoints to form a convex 7-gon and add lines determined by the starting line segments.

an ordinary point at p, v or q, in which case these points give rise to third ordinary points in Q, or Q' has an ordinary point elsewhere, in which case, so does Q.

The case where the non-convex quadrilateral, which we again call Q, has three ordinary points on its boundary is similar. Suppose Q has just these three ordinary points and again refer to Figure 4. If T_1 and T_2 each have just one ordinary point of Q on their boundaries, then, as earlier, v and r must be ordinary in T_1 and v and s must be ordinary in T_2. But then, by Lemma 4, given that Q' has an ordinary point along its boundary which is also an ordinary point in Q, it must have an additional ordinary point as well, which, whether that ordinary point is at v, r, s or elsewhere, is ordinary in Q also. The only additional cases are where one of T_1 and T_2 has two ordinary points of Q and the other has one, or that they share an ordinary point at v which is also an ordinary point of Q and each have one additional "proper" ordinary point as well. Let us consider this latter case first. For T_1 and T_2 each not to have a third ordinary point which would be an ordinary point of Q, it must be that p is ordinary in T_1 and q is ordinary in T_2 but that T_1 and T_2 have second ordinary points along their common boundary with Q elsewhere. But then, by Lemma 4, Q' must have an additional ordinary point besides that at v which is ordinary in Q, giving Q four ordinary points in total.

Hence, let us consider the last case, where, without loss of generality, suppose T_1 has two ordinary points of Q, T_2 has one ordinary of Q, and additionally, v is not ordinary in Q. It follows (by Lemma 2) that v and q are ordinary in T_2 but not Q, and either p or v is ordinary in T_1 but not Q. In either case, we again see by Theorem 4 that Q' necessarily has an ordinary point which is also ordinary in Q, giving Q four ordinary points in total. The lemma is thus established. □

Lemma 7 *A non-convex pentagon P with two ordinary points on its boundary must contain another ordinary point.*

Proof. We suppose P has exactly two ordinary points along its boundary and show that it must have an additional ordinary point. We again consider the two cases as depicted in Figure 5 where the extended edge attached to the reflex vertex v either partitions P (i) into two triangles, or (ii) into a triangle and a quadrilateral. In case (i) suppose first that v is ordinary. In addition to a possible ordinary points at q, by Lemma 2, T and T' will necessarily have ordinary points which are ordinary in P, giving three in total. Hence, suppose v is not ordinary in P. Then v can be ordinary in at most one of T and T'. Without loss of generality, suppose, as in the left-hand drawing of Figure 5, that v can only be ordinary in T'[2]. Then, for T to have only boundary ordinary points, in addition to possible ordinary point at q (by Lemma 2), it must have two

[2] Note that had it been that the line extending edge e_2 partitioned P into two triangles, and we had chosen the edge e_2, then v would only be ordinary in T, not T'.

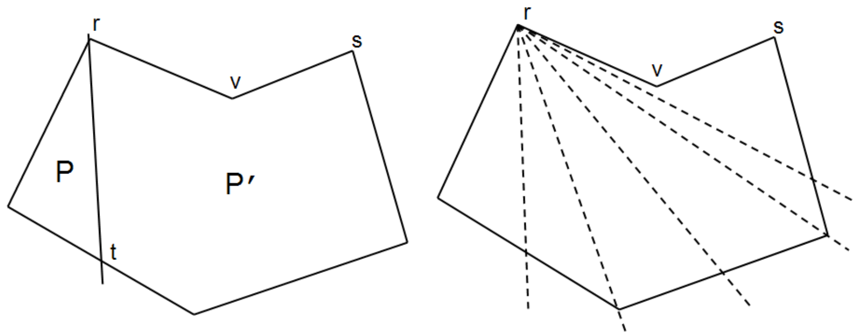

Figure 8. A non-convex hexagon with reflex vertex v and neighboring vertices r and s. We assume r is not ordinary so that a third line through r intersects H at a second a second point t partitioning H into two polygons P and P' with $s \in P'$. If (p, p') denotes the number of edges in P and P' respectively, the possible values for (p, p') are $(3,6), (3,5), (4,5), (4,4)$ and $(5,4)$, as indicated in the right-hand drawing.

ordinary points along its boundary that are ordinary points of P. Analogously, T' must have at least one ordinary point in addition to possible ordinary points at v and q that is ordinary in P – giving P at least three ordinary points in total.

Now consider the case, as depicted in the right hand drawing of Figure 5 where the line ℓ extending edge e_1 partitions P into a triangle and a quadrilateral. If v is ordinary then using Lemma 2 in T and Lemma 4 in Q gives us three ordinary points in P. If v is ordinary in T but not Q then Q must have at least one ordinary point that is also ordinary in P, and the only way for the same to be true for T is if q is also ordinary (for T). But then we inevitably have q ordinary in Q and so in P, and by Lemma 4, Q will have an additional ordinary point which is a proper ordinary point for P, giving P three ordinary points in total. Finally, consider the case where v is ordinary in Q but not T. Then T can only have as few as two proper ordinary points along its boundary if q is also ordinary in T. But then, applying Lemma 4 again, either q or another point along $\partial Q \cap \partial P$ will be a proper ordinary point for P – again giving P three ordinary points in total. The lemma is therefore established. □

Theorem 8 *A non-convex hexagon H must contain at least one ordinary point. Moreover, if H has one ordinary point on its boundary it must contain an additional ordinary point as well.*

Proof. Let v be a reflex vertex of H and let r and s be the two adjacent vertices to v in H. If adjacent vertex r is ordinary then we are done so suppose it is not ordinary. A third line through r intersects H at a second point t partitioning H into two polygons P and P' where, to add concreteness to our labeling, let us say $s \in P'$. If (p, p') denotes the number of edges in P and P' respectively then the possible values for (p, p') are $(3,6), (3,5), (4,5), (4,4)$ and $(5,4)$ as depicted in Figure 8.

We first establish the existence of an ordinary point by considering each of these possibilities in turn. In cases $(3,6)$ and $(3,5)$, in addition to possible phony ordinary points at r and t, P must contain a genuine ordinary point by Lemma 2. In case $(4,5)$ P' must contain a genuine ordinary point in addition to possible phony ordinary points at r and t by Lemma 7, and finally in cases $(4,4)$ and $(5,4)$ P' must contain a genuine ordinary point by Lemma 6. We have thus established the existence of an ordinary point in H. Let us assume this ordinary point is on ∂H and try to establish that there must be another ordinary point in H as well. Let us hold the case $(3,6)$ for last. The case $(3,5)$ follows by applying Lemma 2 in P and Lemma 7 in P'. $(4,5)$ follows by applying Lemma 4 to P and Lemma 7 to P' while cases $(4,4)$ and $(5,4)$ follow by the second half of Lemma 6 applied to polygon P'. It follows that the only troublesome case is $(3,6)$ – the case depicted in the left hand drawing in Figure 8. But in this case $\partial P \cap \partial H$ must contain a ordinary point by Lemma 2, and reasoning in the same way about vertex s, there would be a triangle on the far right, call it P'', with $p \cap P'' = \emptyset$ and $\partial P'' \cap \partial H$ also containing an ordinary point. Hence he second half of the theorem is established and the proof is complete. □

Theorem 9 *A non-convex* 7*-gon S must contain at least one ordinary point.*

Proof. We argue as in the proof of the prior theorem. Let v be a reflex vertex of S with adjacent vertices r and s which we may assume to be non-ordinary. A third line through r partitions S into polygons P and P' with possible corresponding numbers of edges $(3,7), (3,6), (4,6), (4,5), (5,5), (5,4)$ and $(6,4)$. The cases $(3,7)$ an $(3,6)$ are handled by applying Lemma 2 to P. Case $(4,6)$ is handled by applying the first part of Theorem 3 and Lemma 4 to P and Theorem 8 to P'. Cases $(4,5)$ and $(5,5)$ are handled by applying the second half of Theorem 5 and Lemma 7 to P'. Finally, the cases $(5,4)$ an $(6,4)$ are handled by applying the second half of Theorem 3 and Lemma 6 to P'. The theorem follows. □

The example in Figure 9 shows that there are non-convex octagons without ordinary points. The related constructions in Figure 10 shows that there are non-convex 7-gons with a single ordinary point (left-hand drawing) and more interesting examples of hexagons without ordinary points (right-hand drawing) besides the earlier example of the regular hexagon.

A summary of our results is given in the table in Figure 11.

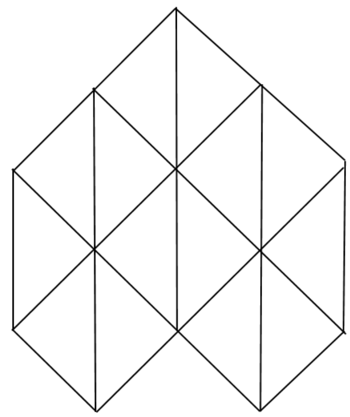

Figure 9. A non-convex octagon without ordinary points.

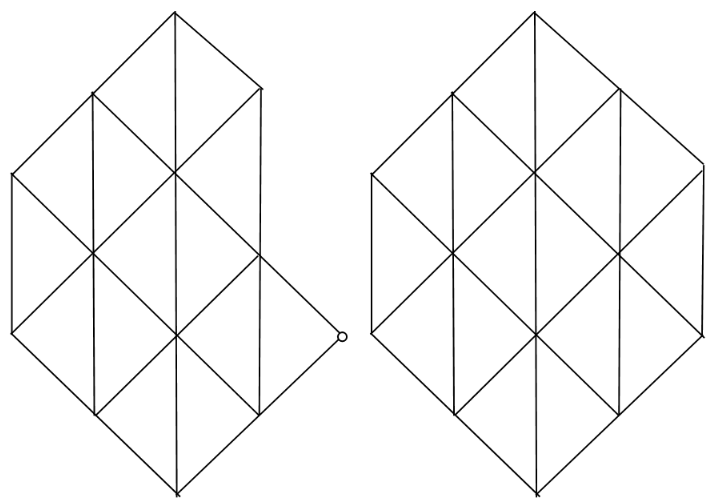

Figure 10. Two related constructions to that in Figure 9, the first (left) showing a non-convex 7-gon with just a single ordinary point, and the second (right) showing an additional case of a hexagon without ordinary points.

5 Colored Variations

In this section we consider the problem where we have an arrangement of lines in a polygon, the lines are colored in one of two colors, and we would like to conclude that there must be one or more monochromatic vertices, in other words, vertices where only lines of a single color meet. A famous unpublished (by the authors) theorem of Motzkin and Rabin (see [11]) states that such a bichromatic arrangement in the plane must have a monochromatic vertex. The Motzkin-Rabin Theorem has a "book proof" based on Euler's formula which is due to Chakerian [1,3]. We adapt the proof of this same result given in [19] and attributed to Motzkin, to prove the following very useful lemma:

Lemma 10 *Suppose we are given a bichromatic arrangement in a polygon P, and a triangle T lying on or within that polygon, whose defining lines are of two colors X and Y. Further, suppose that at the vertex v in which the two lines of one color (X, say) meet, a third line of color Y passes and also passes into the interior of T. Then T must have a monochromatic vertex which is also a monochromatic vertex of P.*

Proof. Figure 12 shows the situation described in the statement of the lemma. The argument is simple: consider the vertex where the two thin lines (i.e. lines of color Y) meet. If the lemma were false and we had a counterexample there would have to be a thick line (of color X) through this vertex which would in turn intersect one of the two existing thick lines. If this vertex were not monochromatic, there would necessarily be a thin line through this next vertex, and so on. At each stage we start at a triangle with two lines of one color and one of the other, conclude that we must have a line of the second color running through the common vertex of the two lines of the first color, and this flips the situation around where we have a triangle with two lines of the second color and one of the first, and must proceed in the same way again. Ultimately this process must end at a monochromatic vertex by virtue of the fact that an arrangement has only a finite

Convex? / # Edges	Yes	No
3	2	--
4	1	2
5	1	2
6	0	1
7	0	1
8	0	0

Figure 11. Guaranteed numbers of ordinary points for arrangements in convex and non-convex polygons with specified numbers of edges. All results have been shown to be best possible, with the exception of the non-convex hexagon.

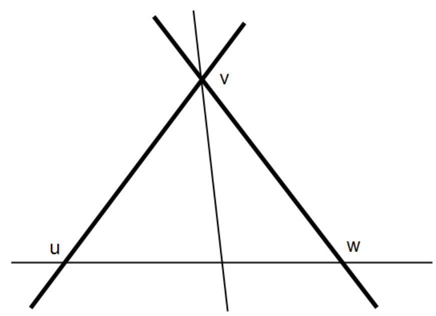

Figure 12. The situation described in Lemma 10, with $T = \triangle(u,v,w)$. Thick lines are of color X and thin lines of color Y.

number of lines. The vertex thus found is not one of the corner vertices from the original triangle T and so is necessarily also monochromatic in P. □

Theorem 11 *A bichromatic arrangement in a triangle T must contain a monochromatic vertex.*

Proof. If the defining lines of a purported counterexample T are two of color X and one of color Y then through the common vertex of the lines of color X must be a line of color Y leading to the exact conditions of Lemma 12, and hence the conclusion that T must contain a monochromatic vertex.

If, on the other hand, the defining lines of the purported counterexample T are all of one color X then consider any two vertices v and w of T. Through each of v and w there must be an additional line of color Y. Now if we consider the two X-colored defining lines of T which pass through w, together with the two new lines of color Y (see Figure 13) we again have the conditions of Lemma 12 and so can conclude that T has a monochromatic vertex. □

There are many easy-to-construct examples of bichromatic arrangements in convex polygons of n sides for any $n \geq 4$. However:

Theorem 12 *A bichromatic arrangement in a non-convex quadrilateral Q must contain a monochromatic vertex.*

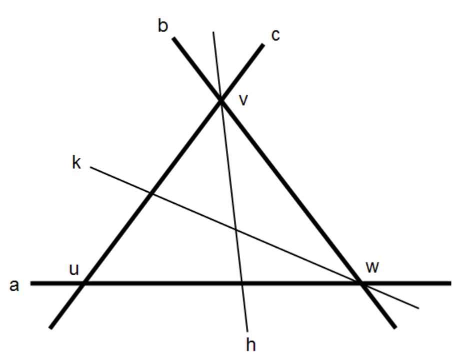

Figure 13. The defining lines of the purported counterexample $T = \triangle(u,v,w)$ are all of one color X, so consider any two vertices v and w of T. Through each of v and w there must be an additional line of color Y - call these lines h and k. The two X-colored defining lines (a and b) of T which pass through w, together with the two new lines, h and k of color Y give rise to the conditions of Lemma 12.

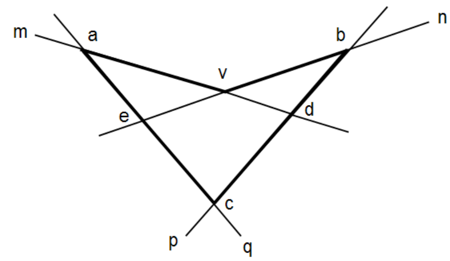

Figure 14. Labeling of the defining lines and vertices of the non-convex quadrilateral Q (the defining edges of which are in bold) in the proof of Theorem 12. The figure makes no assumptions about the colors of lines.

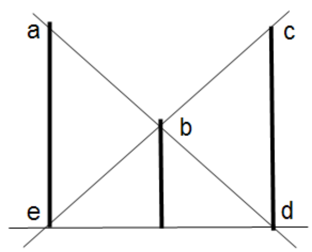

Figure 15. A bichromatic non-convex pentagon $P = P(a,b,c,d,e)$ without a monochrome vertex.

Proof. We adopt the labeling conventions of Figure 14, where in particular, v is the reflex vertex of Q and a and b are the defining vertices of Q adjacent to v. Within Q consider the triangles $T = \triangle(b,e,c)$ and $T' = \triangle(a,e,v)$. By Theorem 11 T must have a monochromatic vertex. If Q is to be a counterexample to the theorem, such a monochrome vertex must not also be monochrome in Q. Notice though that the only vertex that can be monochromatic in T but not Q is the vertex e. In the spirit then of reaching a contradiction, this means that lines n and q are of the same color, say X, and that there is a third line, call it h through e in T' of color Y. Now if the line m is Y-colored then the collection of lines $\{m,n,q,h\}$ satisfy the conditions of Lemma 12, guaranteeing a monochromatic vertex in Q. On the other hand if the line m is X-colored, then to avoid a monochromatic vertex at a there must be a third Y-colored line, call it ℓ through a and T', and then the collection of lines $\{\ell,h,m,q\}$ satisfy the conditions of Lemma 12 again guaranteeing that Q has a monochromatic vertex. The theorem is therefore established. $\square$

Figure 15 provides an example of a bichromatic arrangement in a non-convex pentagon without monochromatic vertices. However, the same would not be possible in a pentagon with two reflex vertices, as the next theorem shows.

Theorem 13 *A bichromatic arrangement in an n-gon, P, with $n-3$ reflex vertices necessarily has a monochromatic vertex.*

Proof. We observe that if an n-gon has $n-3$ reflex vertices, it has just 3 non-reflex vertices which are then also the 3 points giving rise to the convex hull of the complete set of vertices – and P is what is known as a pseudotriangle. We treat the following three cases in sequence:
Case (i): The three non-reflex vertices are all consecutive vertices of the n-gon,
Case (ii): Two of the three of the non-reflex vertices are consecutive but not the remaining one, and
Case (iii): Each non-reflex vertex is separated from the other non-reflex vertices by at least one reflex vertex.

Case (i): The three non-reflex vertices are all consecutive vertices of the n-gon.

This situation is depicted in Figure 16. The lines m,n,p and q give rise to a non-convex quadrilateral Q wholly

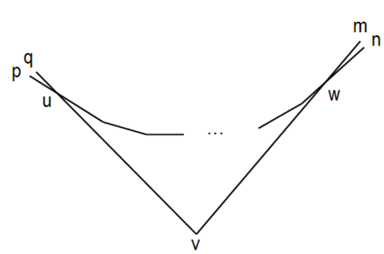

Figure 16. An n-gon with $n-3$ reflex vertices, so 3 non-reflex vertices. Case (i): the three non-reflex vertices, u, v and w, are consecutive.

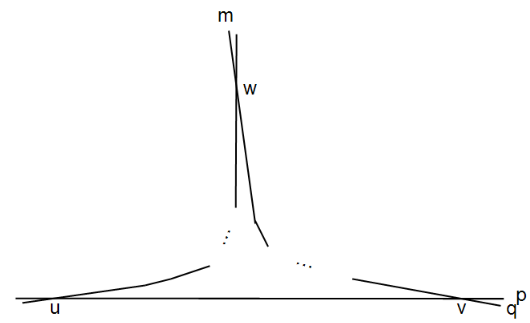

Figure 17. An n-gon with $n-3$ reflex vertices, so 3 non-reflex vertices. Case (ii): two of the three non-reflex vertices, here labeled u and v, are consecutive, but the third, w, is not consecutive with either of the others.

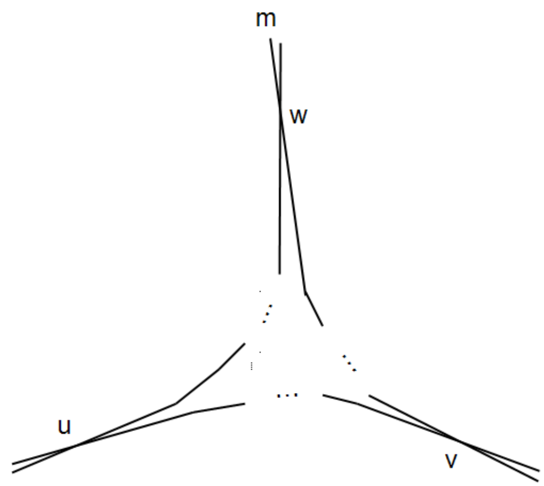

Figure 18. An n-gon with $n-3$ reflex vertices, so 3 non-reflex vertices. Case (iii): each of the three non-reflex vertices, labeled u, v, and w are separated by one or more reflex vertices.

contained in P. By Theorem 12, Q necessarily has a monochromatic vertex and any monochromatic vertex of Q is easily seen to also be monochromatic in P.

Case (ii): Two of the three of the non-reflex vertices are consecutive but not the remaining one.

This second case is depicted in Figure 17. In this case we extend one of the defining edges of P, which we have called m in the figure, that contributes to the non-consecutive, non-reflex, vertex, w, of P. The line containing this edge meets the edge given by $[u, v]$ at a point x. The line m partitions P into two polygons, one to its left and one to its right. Call these polygons P_L and P_R. Note that both P_L and P_R are polygons of the type being considered in the statement of this theorem, and that, moreover, they have three consecutive non-reflex vertices so fall into Case (i) of this proof, and thus necessarily have monochromatic vertices. We actually just exploit this fact for polygon P_L. If P_L is to have a monochromatic vertex that is not monochromatic in P, that vertex must be x. If m and p (the line through u and v) are both X-colored then there must be a line, h, through x entering the interior of P_R, which is Y-colored. Now consider the lines m, q, p and h. If q is Y-colored then Lemma 12 applied to $\{m, q, p, h\}$ guarantees a monochromatic vertex in P. On the other hand, if q is X-colored then if v were not monochromatic there would be a Y-colored line ℓ through v an we can apply Lemma 12 to lines $\{p, q, \ell, h\}$ again yielding a monochromatic vertex in P. Case (ii) is thus settled.

Case (iii): Each non-reflex vertex is separated from the other non-reflex vertices by at least one reflex vertex.

This third case is depicted in Figure 18. Again we consider the line extending edge m and its intersection at a point x along the set of edges separating the pair of non-reflex vertices u and v. The line m once again partitions P into two polygons P_L and P_R. As before P_L (and P_R – though again we do not exploit this latter fact) is a polygon of the type being considered in the statement of this theorem, and, moreover, in this case, has two consecutive non-reflex vertices so falls into Case (ii) of this proof, and thus necessarily has a monochromatic vertex. If P_L is to have a monochromatic vertex that is not monochromatic in P, that vertex must be x. If m together with the one or two other edges of P meeting at x are X-colored, then if x is not monochromatic in P, there must be a Y-colored line h passing through x and into the interior of P_R. This line h partitions P_R into two pieces P_{R_R} and P_{R_L} with common edge $[x, y]$ as depicted in Figure 19. P_{R_R} is again a

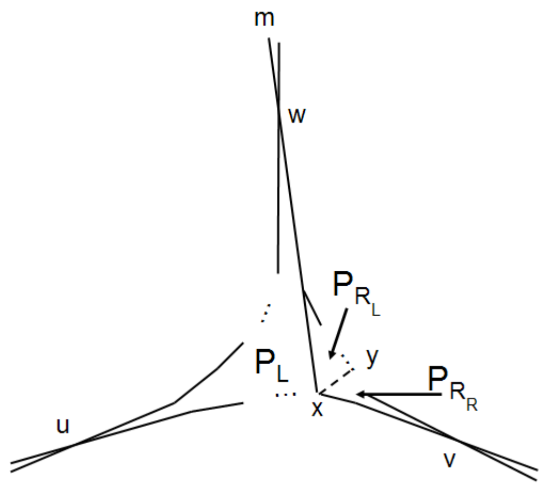

Figure 19. Case (iii) of the proof of Theorem 13 where the three non-reflex vertices, labeled u, v, and w are separated by one or more reflex vertices. The line m partitions P into polygons P_L and P_R. P_L necessarily has a monochromatic vertex and if P_L is to have a monochromatic vertex that is not monochromatic in P, that vertex must be x. If m together with the one or two other edges meeting at x are X-colored, then if x is not monochromatic in P, there must be a Y-colored line h passing through x and into the interior of P_R. This (dashed) line h partitions P_R into two pieces P_{R_R} and P_{R_L} with common edge $[x,y]$ as shown. P_{R_R} again must have a monochromatic vertex. Since x is bichromatic in P_{R_R}, the only possible monochromatic vertex not monochromatic in P is the vertex y. Since y is monochromatic in P_{R_R} there is a Y-colored line, call it ℓ, through y which forms an edge of P_{R_R}. We consider the following four lines: m, the line containing the edge of P_{R_R} containing x that is not $[x,y]$, h and ℓ. Then the first two of these lines are X-colored (from the fact that x was monochromatic in P_L), while the second two are Y-colored (from the fact that y was monochromatic in P_{R_R}), and the lines satisfy the conditions of Lemma 12 – thereby guaranteeing a monochromatic vertex in P.

polygon of the type considered in case (ii) and the only possible monochromatic vertex in P_{R_R} which would not necessarily be monochromatic in P is the vertex y – i.e. the vertex at which h intersects the set of edges between non-reflex vertices w and v. Since y is monochromatic in P_{R_R} there is a Y-colored line, call it ℓ, through y which forms an edge of P_{R_R}. Now consider the following four lines: m, the line containing the edge of P_{R_R} containing x that is not $[x,y]$, h and ℓ. Then the first two of these lines are X-colored, while the second two are Y-colored, and the lines satisfy the conditions of Lemma 12. We conclude that P necessarily contains a monochromatic vertex. Case (iii) is therefore settled and the theorem is proved. □

6 Concluding Remarks

Many interesting facts about ordinary points and bichromatic arrangements in polygons surely remain to be discovered. We do not know whether there are non-convex hexagons with just a single ordinary point. A small variation of the construction in Figure 9 shows that there is a decagon (10-sided polygon) with two reflex vertices without ordinary points. Is it true that any polygon with no more than $5+2R$ sides and R reflex vertices necessarily has an ordinary point? Is it possible to expand the table in Figure 11 to provide tight lower bounds on the number of ordinary points as a function of the number of sides *and* the number of reflex vertices?

In all the extremal examples we have given, the lines intersecting the polygons completely partition the polygon into triangles. Is this partitioning coincidence or a consequence of the associated Melchior expression

$$Y_P = \sum_{j\geq 2}(3-j)t_j + \sum_{k\geq 3}(3-k)p_k \tag{26}$$

for the given polygon? In our analysis we have made use of excess crossings, which show up as part of the first summation term, but we have never made particular use of possible excess in the second term, i.e. $\sum_{k\geq 4}(k-3)p_k$. We may call such excess, "polygonal excess" – since any time we have polygonal excess, just like when we have excess crossings, it makes way for additional ordinary points.

Can one formulate interesting theorems regarding the number of ordinary points in polygons as a function of the number of lines intersecting the interior of the polygon? What if we add the constraint that the lines intersecting in the polygon should have no intersection points outside of the polygon, as would be the case for the bounding polygon (i.e. the union of all finite cells) of a usual line arrangement? Are there any interesting constraints on the ways polygons with few ordinary points (and possibly many intersecting lines) can be pieced together? In other words, can we simultaneously bound the number of ordinary points in a polygon P and a neighboring polygon P' where P and P' share a defining edge or defining vertex if we assume that all lines pass through the interior of both polygons?

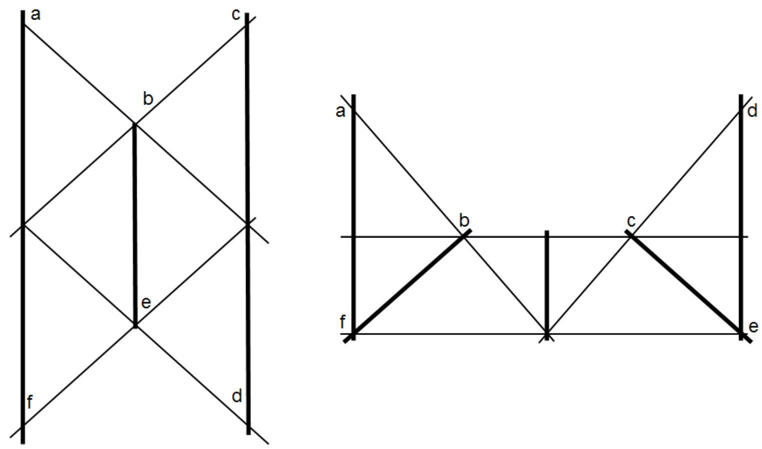

Figure 20. Bichromatic arrangements in hexagons without any monochromatic vertices, each with two reflex vertices (at b and e, and b and c respectively. In the right-hand example b and c are consecutive reflex vertices.

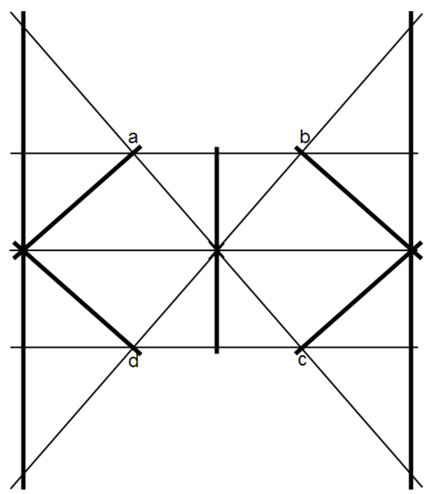

Figure 21. A bichromatic arrangement in an octagon, with four reflex vertices a, b, c and d, and no monochromatic vertices.

We may have just scratched the surface of the possible theorems involving bichromatic arrangements in polygons. Can one find examples, for arbitrarily large n, of bichromatic arrangements of n-gons with $n-4$ reflex vertices and no monochromatic vertex? Figure 20 gives examples of bichromatic arrangements in hexagons with two reflex vertices and no monochromatic vertex. Doubling the right hand example in Figure 20 gives the example in Figure 21, which is a bichromatic octagon with 4 reflex vertices and no monochromatic vertex. However, we don't know if there are additional examples. Finally, are there interesting constraints that one can place on polygons which would guarantee multiple monochromatic vertices?

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