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Simple Extended Formulation for the Dominating Set Polytope via Facility Location

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SIMPLE EXTENDED FORMULATION FOR THE DOMINATING SET POLYTOPE VIA FACILITY LOCATION

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Abstract. In this paper we present an extended formulation for the dominating set polytope via facility location. We show that with this formulation we can describe the dominating set polytope for cacti graphs, though its description in the natural node variables dimension has been only partially obtained. Moreover, the inequalities describing this polytope have coefficients in $\{-1, 0, 1\}$. This is not the case for the dominating set polytope in the node-variables dimension, because it is known from [9] that for any integer p, there exists a facet defining inequality having coefficients in $\{1, \dots, p\}$. We also give a linear time algorithm to solve the minimum weight dominating set problem in cacti graphs.

Then we study the p-dominating set problem, where the cardinality of the set is required to be exactly p. We show that the natural linear programming formulation gives an integral polytope when the graph is a cycle. We also give a polynomial combinatorial algorithm for cacti.

1. Introduction

let $G = (V, E)$ be an undirected graph. A set $D \subseteq V$ is called a *dominating set* if every node of $V \setminus D$ is adjacent to a node of D. The *minimum weight dominating set problem* (MWDSP) consists of finding a dominating set D that minimizes $\sum_{v \in D} w(v)$, where $w(v)$ is a weight associated with each node $v \in V$. The natural linear relaxation of the MWDSP is defined by the linear program below

$$\min \sum_{v \in V} w(v)x(v) \tag{1}$$

$$\sum_{u \in N[v]} x(u) \geq 1 \quad \forall v \in V, \tag{2}$$

$$x(v) \geq 0 \quad \forall v \in V, \tag{3}$$

$$x(v) \leq 1 \quad \forall v \in V, \tag{4}$$

where $N[v]$ denotes the set of neighbors of v including it. Define $DSP(G)$ to be the convex hull of the integer vectors satisfying (2)-(4).

The MWDSP is a special case of the set covering problem. It is NP-hard even when all the weights are equal to 1, this may be shown using a simple reduction from the vertex cover problem. A large literature is devoted to this case and many of its variants, for a deep understanding of the subject we refer to [20, 19]. It has been shown that when the weights are all equal to 1, the MWDSP is solvable in many classes of graphs, a non-exhaustive list is cactus graphs [24], trees [24], series-parallel graphs [21], permutation graphs [11, 12, 13, 17], cocomparability graphs [22], (see chapter 2 in [19] for more

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classes). For the weighted case of the MDWDSP we only know three classes of graphs where this problem may be solved in polynomial time, namely for threshold graphs [23], for cycles [8] and for strongly chordal graphs [16]. Little is known from the point of view of polyhedral approach, and complete characterizations of the polytope are known only for the three classes of graphs mentioned above. For the case of strongly chordal graphs Farber [16] gives a primal-dual algorithm to solve the MWDSP this shows that $DSP(G)$ is defined by (2)-(4).

The polytope $DSP(G)$ has been first characterized for cycle graphs in [8], and later published in [9]. This result has also been established in [26] using a different approach. Namely if $C = (V, E)$ is a a cycle, $V = \{1, \ldots, n\}$, they proved that two types of inequalities have to be added to (2)-(4) to define $DSP(C)$. These are

$$\sum_{v \in V} x(v) \geq \left\lceil \frac{|V|}{3} \right\rceil, \tag{5}$$

when $|V|$ is not a multiple of 3. And

$$2 \sum_{v \in W} x(v) + \sum_{v \in V \setminus W} x(v) \geq \sum k_i + \left\lceil \frac{p}{2} \right\rceil, \tag{6}$$

where $W = \{v_1, \ldots, v_p\} \subset V$, $v_1 < v_2 < \ldots < v_p$, $p \geq 3$, p is odd, and

$$|C(v_i + 1, v_{i+1} - 1)| = 3k_i,$$

$k_i \geq 1$, for $i = 1, \ldots, p \pmod p$. Here $C(u, v)$ denotes the path $u, u+1, \ldots, u+t$ between u and v, where t is such that $u + t = v$, (the integers are taken modulo n). Notice that for each set W satisfying the definition above, one inequality (6) is required.

For a family $\mathcal{F}$ of inequalities, the *separation problem* consists of given a vector $\bar{x}$, finding an inequality in $\mathcal{F}$ violated by $\bar{x}$, or show that none exists. In [9] they gave a polynomial algorithm for the separation problem for inequalities (6). This combined with the Ellipsoid Method [18] shows that the MWDSP is polynomially solvable for cycles. A combinatorial algorithm for the MWDSP in cycles was not given in [9].

One may also use the results related to the set covering polytope [15, 5, 6, 14, 25], to cite a few, to establish new results for the MWDSP. The set covering polytope is the convex hull of $\{x \in \mathbb{R}^n : \ Ax \geq 1, \ x \in \{0,1\}^n\}$, where A is an $m \times n$ matrix with 0,1 entries. For example, the polytope $DSP(G)$ when G is a cycle with n nodes coincide with the set covering polytope when A is the C_n^3 circulant matrix. Recently in [7] a complete description of the set covering polytope is established when A is the circulant matrix C_{2k}^k or C_{3k}^k, $k \geq 3$.

Let $G = (V, E)$ be an undirected connected graph, the graph G is a *cactus* if each edge of G is contained in at most one cycle of G. We give an extended formulation via facility location to completely characterize the $DSP(G)$ when G is a cactus. This description has been studied in the original dimension that is $\mathbb{R}^{|V|}$ in [8, 9]. They developed several facet defining inequalities for this case, and showed that this polytope has a more complicated structure than the case when G is a cycle. Even with the 1-sum composition developed in [10], the complete characterization of $DSP(G)$ in cactus graphs has not been found. The main difficulty reported in [8, 9] is the description of the polytope when restricted to the auxiliary graphs obtained after the decomposition. In our work we show that with the extended formulation this task is easy and allows us to completely describe this polytope in a higher dimension. Moreover in [8, 9], it has been shown that for any fixed integer p, there exist a cactus G such that $DSP(G)$ has a facet defining inequality with coefficients $1, \ldots, p$. In our description all the facets defining inequalities have coefficients in $\{0, -1, +1\}$.

We also use results about the p-median problem to study the p-dominating set problem, i.e., when the dominating set is required to have a fixed cardinality. For cycles we show that the natural formulation gives a polytope with all integer extreme points. We also give a polynomial combinatorial algorithm for cacti.

We complete this introduction with some definitions. An undirected graph $G = (V, E)$ decomposes by means of a 1-sum, if G may be decomposed into two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with $V = V_1 \cup V_2$, $V_1 \cap V_2 = \{u\}$ and $E_1 \cup E_2 = E$, $E_1 \cap E_1 = \emptyset$. For directed graphs a 1-sum is defined similarly. If a graph is a cactus, it can be obtained by means of 1-sums of cycles and paths.

To an undirected graph $G = (V, E)$ we associate a directed graph $\overset{\leftrightarrow}{G} = (V, A)$, where for each edge $uv \in E$ we include the arcs (u, v) and (v, u) in A.

For a directed graph $G = (V, A)$ and a set $W \subset V$, we denote by $\delta^+(W)$ the set of arcs $(u, v) \in A$, with $u \in W$ and $v \in V \setminus W$. Also we denote by $\delta^-(W)$ the set of arcs (u, v), with $v \in W$ and $u \in V \setminus W$. We write $\delta^+(v)$ and $\delta^-(v)$ instead of $\delta^+(\{v\})$ and $\delta^-(\{v\})$, respectively. If there is a risk of confusion we use δ^+_G and δ^-_G.

This paper is organized as follows. In Section 2 we review polyhedral results on Facility location. In Section 3 we give an extended formulation of $DSP(G)$ for cacti. In Section 4 we study algorithmic aspects. In Section 5 we study the p-dominating set problem.

2. Facility location

Here we review results on the facility location polytope that will be used in the next section. If $G = (V, A)$ is a directed graph, not necessarily connected, where each arc and each node has a cost (or a profit) associated with it. Consider the following version of the *uncapacitated facility location problem* (UFLP), where each location $v \in V$ has a weight $w(v)$ that corresponds to the revenue obtained by opening a facility at that location, minus the cost of building this facility. Each arc $(u, v) \in A$ has a weight $w(u, v)$ that represents the revenue obtained by assigning the customer u to the opened facility at location v, minus the cost originated by this assignment. The goal is to select some nodes where facilities are opened, and the non selected nodes might be assigned in such a way that the overall profit is maximized. This version of the UFLP is called the *prize-collecting uncapacitated facility location problem* (pc-UFLP). The following is a linear programming relaxation of the pc-UFLP.

$$\max \sum_{(u,v)\in A} w(u, v)y(u, v) + \sum_{v\in V} w(v)x(v) \tag{7}$$

$$\sum_{(u,v)\in A} y(u, v) + x(u) \le 1 \quad \forall u \in V, \tag{8}$$

$$y(u, v) \le x(v) \quad \forall (u, v) \in A, \tag{9}$$

$$y(u, v) \ge 0 \quad \forall (u, v) \in A, \tag{10}$$

$$x(v) \ge 0 \quad \forall v \in V, \tag{11}$$

For each node u, the variable $x(u)$ takes the value 1 if the node u is selected and 0 otherwise. For each arc (u, v) the variable $y(u, v)$ takes the value 1 if u is assigned to v

and 0 otherwise. Inequalities (8) express the fact that either node u can be selected or it can be assigned to another node. Inequalities (9) indicate that if a node u is assigned to a node v then this last node should be selected.

Let $P(G)$ be the polytope defined by (8)-(10), and let $LP(G)$ be the convex hull of $P(G) \cap \{0,1\}^{|V|+|A|}$. Clearly

$$LP(G) \subseteq P(G),$$

and in most cases new inequalities should be added to (8)-(11) to obtain $LP(G)$. Below we study this for cycles and cacti.

2.1. **Facility location polytope of a bidirected cycle.** A directed graph $G = (V, A)$ is called a bidirected cycle (BIC_r), if $V = \{1, \ldots, r\}$, and for each $i = 1, \ldots, r$, the arcs $(i, i+1)$ and $(i+1, i)$ are in A, the indices are taken modulo r. The set of arcs is $A(BIC_r)$. A bidirected path is defined in a similar way. Two types of inequalities are needed for BIC_r, they are shown below.

2.1.1. *Bidirected cycle inequalities.* It may be easily seen that the inequality

$$\sum_{a \in A(BIC_r)} y(a) \leq \left\lfloor \frac{2|r|}{3} \right\rfloor, \tag{12}$$

is valid for $LP(G)$. This inequality is called the *bidirected cycle inequality.* It has been introduced in [1].

2.1.2. *Lifted g-odd cycle inequalities.* A simple cycle C is an ordered sequence

$$v_0, a_0, v_1, a_1, \ldots, a_{p-1}, v_p,$$

where

- v_i, $0 \leq i \leq p-1$, are distinct nodes,
- a_i, $0 \leq i \leq p-1$, are distinct arcs,
- either v_i is the tail of a_i and v_{i+1} is the head of a_i, or v_i is the head of a_i and v_{i+1} is the tail of a_i, for $0 \leq i \leq p-1$, and
- $v_0 = v_p$.

By setting $a_p = a_0$, we associate with C three more sets as below.

- We denote by $\hat{C}$ the set of nodes v_i, such that v_i is the head of a_{i-1} and also the head of a_i, $1 \leq i \leq p$.
- We denote by $\dot{C}$ the set of nodes v_i, such that v_i is the tail of a_{i-1} and also the tail of a_i, $1 \leq i \leq p$.
- We denote by $\tilde{C}$ the set of nodes v_i, such that either v_i is the head of a_{i-1} and also the tail of a_i, or v_i is the tail of a_{i-1} and also the head of a_i, $1 \leq i \leq p$.

Notice that $|\hat{C}| = |\dot{C}|$. A cycle is called *g-odd* (generalized odd) if $p + |\dot{C}|$ (or $|\dot{C}| + |\tilde{C}|$) is odd, otherwise it is called *g-even.* A cycle C with $\dot{C} = \hat{C} = \emptyset$ is a *directed* cycle. The set of arcs in C is denoted by $A(C)$.

Let C be a g-odd cycle. Now we define the *lifting set* $\tilde{A}(C)$ as follows. For each node $i \in \dot{C}$ we have two cases:

- If $i-1$ and $i+1$ are in $\tilde{C}$, we pick arbitrarily one arc from $\{(i-1, i), (i+1, i)\}$ and add it to $\tilde{A}(C)$.
- If only one of the neighbors of i is in $\tilde{C}$, say the node j. We add (j, i) to $\tilde{A}(C)$.

Once the set $\tilde{A}(C)$ has been defined, a *lifted g-odd cycle inequality* has the form

$$\sum_{a\in A(C)} y(a) + \sum_{a\in \tilde{A}(C)} y(a) - \sum_{v\in \hat{C}} x(v) \leq \frac{|\tilde{C}| + |\hat{C}| - 1}{2}. \tag{13}$$

Notice that given a g-odd cycle C, we might have several lifting sets $\tilde{A}(C)$, therefore we might have several lifted g-odd cycle inequalities.

The following characterization of $LP(BIC_n)$ was proved in [4].

Theorem 1. *$LP(BIC_n)$ is described by the constraints (8)-(11), the bidirected cycle inequality (12) with respect to BIC_n, if $n = 3k+1$, and the lifted g-odd cycle inequalities (13). Moreover, these inequalities describe a minimal system for $LP(BIC_n)$.*

2.2. **Polytope description for a cactus.** Let $G = (V, E)$ be an undirected graph that is a cactus. We plan to build the directed graph $\overset{\leftrightarrow}{G}$ and study $LP(\overset{\leftrightarrow}{G})$. Since G is obtained by means of 1-sums of cycles and paths, we need the theorem below, proved in [2]. We are going to use z to denote the vector (x, y). Also we use $z(S)$ to denote $\sum_{e\in S} z(e)$.

Theorem 2. *Let D be a directed graph that is a 1-sum of $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$, with $V_1 \cap V_2 = \{u\}$. Let D_1' be the graph obtained from D_1 by replacing u with u', and D_2' is obtained from D_2 by replacing u with u''. Suppose that the system*

$$Az' \leq b \tag{14}$$

$$z'\left(\delta^+_{D_1'}(u')\right) + z'(u') \leq 1 \tag{15}$$

describes $LP(D_1')$. Suppose that (14) *contains the inequalities* (8)-(11) *except for* (15)*. Similarly suppose that*

$$Cz'' \leq d \tag{16}$$

$$z''\left(\delta^+_{D_2'}(u'')\right) + z''(u'') \leq 1 \tag{17}$$

describes $LP(D_2')$. Also (16) *contains the inequalities* (8)-(11) *except for* (17)*. Then the system below describes an integral polyhedron.*

$$Az' \leq b \tag{18}$$

$$Cz'' \leq d \tag{19}$$

$$z'\left(\delta^+_{D_1'}(u')\right) + z''\left(\delta^+_{D_2'}(u'')\right) + z'(u') \leq 1 \tag{20}$$

$$z'(u') = z''(u''). \tag{21}$$

Thus the theorem below follows from Theorem 1 and Theorem 2.

Theorem 3. *If G is a cactus, then $LP(\overset{\leftrightarrow}{G})$ is described by the constraints (8)-(11), the bidirected cycle inequalities (12), and the lifted g-odd cycle inequalities (13).*

2.3. **Algorithmic decomposition.** The polyhedral decomposition shown in the last subsection, has the following algorithmic counterpart.

Consider the pc-UFLP in D. To decompose it we first treat in D_1 the following three cases.

- Let λ_0 be the value of an optimal solution in D_1 with the restriction $x(u') + y(\delta^+_{D_1}(u')) = 0$.

- Let λ_1 be the value of an optimal solution in D_1 with the restriction $x(u') = 1$.
- Let λ_2 be the value of an optimal solution in D_1 with the restriction $y(\delta^+_{D_1}(u')) = 1$.

Then we add a new node t to D_2 and the arc (u'', t). Let D_2' be this new graph. Then in D_2' we give the weight $\lambda_1 - \lambda_0$ to u'', and the weight $\lambda_2 - \lambda_0$ to (u'', t''). Let W be the weight of an optimal solution in D_2' with these weights, then the weight of an optimal solution of D is $W + \lambda_0$.

3. Extended formulation of $DSP(G)$

Here we use the results from the preceding section to give an extended formulation for $DSP(G)$, when G is a cactus. Starting from an undirected graph $G = (V, E)$ we build $\overleftrightarrow{G} = (V, A)$, and associate the following system.

$$\sum_{(u,v)\in A} y(u,v) + x(u) = 1 \quad \forall u \in V, \tag{22}$$

$$y(u,v) \le x(v) \quad \forall (u,v) \in A, \tag{23}$$

$$y(u,v) \ge 0 \quad \forall (u,v) \in A, \tag{24}$$

$$x(v) \ge 0 \quad \forall v \in V, \tag{25}$$

Let $P^=(\overleftrightarrow{G})$ be the polytope defined by this system. This is a face of the polytope defined by (8)-(10), because inequalities (8) have been transformed into equations. Let $LP^=(\overleftrightarrow{G})$ be the convex hull of $P^=(\overleftrightarrow{G}) \cap \{0,1\}^{|V|+|A|}$, $LP^=(\overleftrightarrow{G})$ is a face of $LP(\overleftrightarrow{G})$. First we need the following lemma.

Lemma 4. *For any undirected graph G, the projection of $LP^=(\overleftrightarrow{G})$ onto the x's variables is exactly $DSP(G)$.*

Proof. We have to prove

$$DSP(G) = \{x \mid \text{ there is a vector } y \text{ such that } (x,y) \in LP^=(\overleftrightarrow{G})\}$$

The proof consists of two parts.

(i) First consider $\bar{x} \in DSP(G)$. We have

$$\bar{x} = \sum \alpha_i x^i, \quad \sum \alpha_i = 1, \quad \alpha \ge 0,$$

where $\{x^i\}$ are extreme points of $DSP(G)$.

Consider now a particular vector x^k. Let $D^k = \{u \mid x^k(u) = 1\}$. For each $v \in V \setminus D^k$, there is at least one of its neighbors in D^k, w_v say. We set $y^k(v, w_v) = 1$. We set $y^k(i,j) = 0$ for all other arcs (i,j) in $\overleftrightarrow{G}$.

Each vector (x^k, y^k) is an extreme point of $LP^=(\overleftrightarrow{G})$. So

$$(\bar{x}, \bar{y}) = \sum \alpha_i (x^i, y^i)$$

is a vector in $LP^=(\overleftrightarrow{G})$.

(ii) Consider now $(\bar{x}, \bar{y}) \in LP^=(\overleftrightarrow{G})$. We have

$$(\bar{x}, \bar{y}) = \sum \alpha_i (x^i, y^i), \quad \sum \alpha_i = 1, \quad \alpha \ge 0,$$

where each vector (x^i, y^i) is an extreme point of $LP^=(\overleftrightarrow{G})$. Then each vector x^i is the incidence vector of a dominating set D^i, therefore it is an extreme point of $DSP(G)$. Then

$$\bar{x} = \sum \alpha_i x^i$$

is a vector in $DSP(G)$. □

From this lemma and Theorem 3 we obtain the following.

Corollary 5. *If G is a cactus, then $LP^=(\overleftrightarrow{G})$ is described by the constraints (22)-(25), the bidirected cycle inequalities (12), and the lifted g-odd cycle inequalities (13). Since $DSP(G)$ is a projection of $LP^=(\overleftrightarrow{G})$, we have an extended formulation for $DSP(G)$.*

4. Algorithmic consequences

In [9] the authors they give the first polynomial algorithm to solve the minimum weighted dominating set problem (MWDSP) in a cycle. They showed that the separation of the inequalities defining the dominating set polytope in a cycle can be done in polynomial time, then their algorithm is based on the ellipsoid method [18]. In this section we show that using facility location techniques one can derive a simple combinatorial algorithm. In the next subsection, we give a simple linear time combinatorial algorithm to solve the uncapacitated facility location problem when the underlying graph is a bidirected cycle. As a consequence, we obtain a linear time algorithm to solve the MWDSP in cycles. In subsection 4.2 we give the first polynomial time algorithms to solve the UFLP in $\overleftrightarrow{G}$ when G is a cactus graph. As a consequence we obtain the first polynomial time algorithm to solve the MWDSP in cacti.

4.1. **Linear time algorithm for bidirected cycles.** In this subsection we give an algorithm to solve the prize-collecting uncapacitated facility location (pc-UFLP), when $G = BIC_n$. That is we want to solve (7)-(11) with the additional constraint that (x, y) must be a 0-1 vector.

For any index i we can decompose in the following three cases:

- Neither of $(i, i+1)$ nor $(i+1, i)$ is in the solution.
- $(i, i+1)$ is in the solution.
- $(i+1, i)$ is in the solution.

Each of the three preceding cases reduces to a pc-UFLP problem in a bidirected path. Now let us solve pc-UFLP in a bidirected path.

Suppose that we deal with a bidirected path with nodes $1, \ldots, n$, and $n \geq 4$. The algorithm consists of the following two parts.

- First consider the bidirected path induced by $n-2, n-1, n$. We denote it by P_0. We keep the original weights, but we set $w(n-2) = 0$. Let λ_0 be the weight of an optimal solution in P_0 without the arcs $(n-2, n-1)$ and $(n-1, n-2)$. Let λ_1 be the weight of an optimal solution in P_0 with $(n-2, n-1)$ in the solution. Let λ_2 be the weight of an optimal solution in P_0 with $(n-1, n-2)$ in the solution.
- Then denote by P_1 the bidirected path induced by $1, \ldots, n-1$. We give the weight $\lambda_1 - \lambda_0$ to $(n-2, n-1)$ and the weight $\lambda_2 - \lambda_0$ to $(n-1, n-2)$. All other nodes and arcs keep their original weights. Let W be the weight of an optimal

solution in P_1, then the weight of an optimal solution in the original path is $W + \lambda_0$.

The same procedure is applied recursively to P_1. Since dealing with P_0 takes constant time, we have a linear time algorithm. Also, since treating a bidirected cycle reduces to treating three bidirected paths, we have a linear time algorithm to pc-UFLP when the underlying graph is a bidirected cycle.

Notice that the same algorithm is applied to solve the uncapacitated facility location problem (UFLP). In this problem, all inequalities (8) are replaced by equations. The MWDSP in a cycle reduces to the UFLP in a bidirected cycle. As a consequence we have the following result

Theorem 6. *The MWDSP in a cycle (and the UFLP in a bidirected cycle) can be solved in linear time.*

4.2. **Polynomial time algorithm for cacti.** First we will a give a cutting-plane polynomial time algorithm to solve pc-UFLP in the graph $\overset{\leftrightarrow}{G}$ when G is a cactus. From Theorem 3 it suffices to develop a polynomial time algorithm to solve the separation problem associated with inequalities (12) and (13). Recall that $\overset{\leftrightarrow}{G}$ may be decomposed by means of 1-sum into bidirected cycles and bidirected paths. The number of bidirected cycles is at most the number of nodes of G and hence one can easily introduce the bidirected cycle inequalities (12) in any linear program. Thus we only need to solve the separation problem for the lifted g-odd inequalities (13) for each component of $\overset{\leftrightarrow}{G}$ that is a bidirected cycle.

4.2.1. *Separating lifted g-odd inequalities in a bidirected cycle.* Given a vector $(\bar{x}, \bar{y})$ we want to find a lifted g-odd cycle inequality (13) violated by $(\bar{x}, \bar{y})$, if there is any.

Theorem 7. *The g-odd lifted cycle inequalities* (13) *may be separated in linear time for bidirected cycles.*

Proof. A lifted g-odd cycle inequality (13) has the form

$$\sum_{a \in A(C)} y(a) + \sum_{a \in \tilde{A}(C)} y(a) - \sum_{v \in \hat{C}} x(v) \le \frac{|\tilde{C}| + |\hat{C}| - 1}{2},$$

with $|A(C)| + |\hat{C}|$ odd. It can also be written as

$$\sum_{a \in A(C)} 2y(a) + \sum_{a \in \tilde{A}(C)} 2y(a) + \sum_{v \in \hat{C}} (1 - 2x(v)) \le |A(C)| - 1,$$

or

$$\sum_{a \in A(C)} (1 - 2y(a)) - \sum_{a \in \tilde{A}(C)} 2y(a) + \sum_{v \in \hat{C}} (2x(v) - 1) \ge 1. \tag{26}$$

Thus we look for a cycle that violates (26). For that we create a directed graph $D' = (V', A')$ as follows. For every arc $(i, i+1)$ and $(i+1, i)$ we create a node in D'. The arcs in A' are as below. See Figure 1.

- From $(i, i+1)$ to $(i+1, i+2)$ we create an arc with weight $1 - 2\bar{y}(i+1, i+2)$ and label "odd."

- From $(i, i+1)$ to $(i+2, i+1)$ we create an arc with weight $2\bar{x}(i+1) - 2\bar{y}(i+2, i+1)$ and label "even."
- From $(i+1, i)$ to $(i+1, i+2)$ we create an arc with weight $1 - 2\bar{y}(i+1, i+2)$ and label "odd."
- From $(i+1, i)$ to $(i+2, i+1)$ we create an arc with weight $1 - 2\bar{y}(i+2, i+1)$ and label "odd."
- From $(i, i-1)$ to $(i+1, i+2)$ we create an arc with weight
$$2 - 2\bar{y}(i, i+1) - 2\bar{y}(i+1, i) - 2\bar{y}(i+1, i+2)$$
and label "even." This arc corresponds to the case when either $(i, i+1)$ or $(i+1, i)$ is in the lifting set $\tilde{A}(C)$.

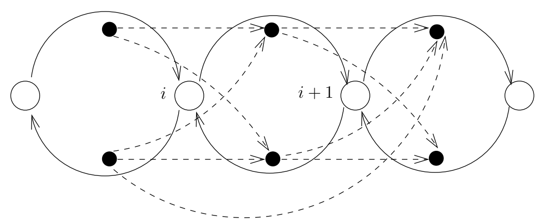

FIGURE 1

Then we look for a minimum weight directed cycle with an odd number of odd arcs in D'. If the weight of such a cycle is less than one, we have found a violated inequality.

Now we give the details of how to find a minimum weight directed cycle with an odd number of odd arcs. We pick and index i, and remove the arcs entering $(i, i+1)$ and $(i+1, i)$. We add an extra node s and connect it to $(i, i+1)$ and $(i+1, i)$ with even arcs of weight zero. For each node v in D' let $f_o(v)$ (resp. $f_e(v)$) be the weight of a shortest path from s to v having an odd (resp. even) number of odd arcs. We set $f_e(s) = 0$, $f_o(s) = f_o(v) = f_e(v) = \infty$ for every other node v in D'. We call the labels of s permanent and all others temporary. For each arc (u, v) we denote by $w(u, v)$ its weight. Then for a node v such that all its predecessors have permanent labels we update its labels as below.

$$f_o(v) = \min\Big\{ \min_u\{f_o(u) + w(u,v) : (u,v) \text{ is even}\}, \min_u\{f_e(u) + w(u,v) : (u,v) \text{ is odd}\}\Big\} \tag{27}$$

$$f_e(v) = \min\Big\{ \min_u\{f_o(u) + w(u,v) : (u,v) \text{ is odd}\}, \min_u\{f_e(u) + w(u,v) : (u,v) \text{ is even}\}\Big\} \tag{28}$$

Then the labels of v are called permanent, and we continue.

Once all labels are permanent, we use the arcs entering $(i, i+1)$ and $(i+1, i)$ to find a shortest directed cycle with an odd number of odd arcs and including either $(i, i+1)$ or $(i+1, i)$. Next we have to consider the case when neither $(i, i+1)$ nor $(i+1, i)$ is in the shortest cycle. This is when the arc from $(i, i-1)$ to $(i+1, i+2)$ is part of the shortest cycle. For that we repeat the same procedure with $i' = i+1$.

Since the indegree of each node in D' is at most three, the labels in (27) and (28) are computed in constant time for each node. Therefore this is a linear time algorithm. □

The above discussion combined with the ellipsoid method implies the following.

Theorem 8. *If G is a cactus, then the MWDSP can be solved in polynomial time. Also the pc-UFLP in $\overset{\leftrightarrow}{G}$ can be solved in polynomial time.*

4.2.2. *Linear time combinatorial algorithm.* We conclude this section by noticing that the algorithmic decomposition given in subsection 2.3 together with the algorithm of subsection 4.1 imply the following.

Theorem 9. *If G is a cactus, then the MWDSP can be solved in linear time. Also the pc-UFLP in $\overset{\leftrightarrow}{G}$ can be solved in linear time.*

5. The p-dominating set problem

In this section we extend results from the p-median problem to the dominating set problem. For a graph $G = (V, E)$ and a positive integer p, we consider now dominating sets $D \subseteq V$ with $|D| = p$. In this section we show that for a cycle it is enough to add an equation to system (2)-(4) to have an integral polytope. Surprisingly inequalities (12) and (13) are not needed. This is stated in Theorem 10 below.

Theorem 10. *If $G = (V, E)$ is a cycle, then system below defines an integral polytope.*

$$x(N[v]) \geq 1 \quad \forall v \in V, \tag{29}$$

$$x(v) \geq 0 \quad \forall v \in V, \tag{30}$$

$$x(v) \leq 1 \quad \forall v \in V, \tag{31}$$

$$\sum_{v \in V} x(v) = p. \tag{32}$$

Proof. Consider $\overset{\leftrightarrow}{G} = (V, A)$, and the system below,

$$\sum_{(u,v) \in A} y(u, v) + x(u) = 1 \quad \forall u \in V, \tag{33}$$

$$y(u, v) \leq x(v) \quad \forall (u, v) \in A, \tag{34}$$

$$y(u, v) \geq 0 \quad \forall (u, v) \in A, \tag{35}$$

$$x(v) \geq 0 \quad \forall v \in V, \tag{36}$$

$$\sum_{v \in V} x(v) = p. \tag{37}$$

This is a linear relaxation of the p-median problem in $\overset{\leftrightarrow}{G}$, where the number of open facilities is required to be exactly p. It was shown in [3] that this system defines an integral polytope if and only if G is a path or a cycle.

To complete the proof we have to show that when we project the variables y, we obtain the system (29)-(32). In order to apply Fourier-Motzkin elimination, we re-write the system (33)-(37) as below.

$$\sum_{(u,v)\in A} y(u,v) + x(u) \leq 1 \quad \forall u \in V, \tag{38}$$

$$-\sum_{(u,v)\in A} y(u,v) - x(u) \leq -1 \quad \forall u \in V, \tag{39}$$

$$y(u,v) - x(v) \leq 0 \quad \forall (u,v) \in A, \tag{40}$$

$$-y(u,v) \leq 0 \quad \forall (u,v) \in A, \tag{41}$$

$$-x(v) \leq 0 \quad \forall v \in V, \tag{42}$$

$$\sum_{v\in V} x(v) \leq p, \tag{43}$$

$$-\sum_{v\in V} x(v) \leq -p. \tag{44}$$

Then to eliminate a variable $x(u,v)$, we sum an inequality where this variable has the coefficient 1 with an inequality where the variable has the coefficient -1. This is done for all pairs of inequalities with these characteristics. The details are as follows.

- The combination of (39) and (40) gives inequalities (29).
- The combination of (38) and (41) gives inequalities (31).
- The combination of (40) and (41) gives inequalities (30).
- Inequalities (42), (43) and (44) remain unchanged.

The proof is complete. □

We obtain the result below.

Corollary 11. *The p-dominating set problem in cycles is polynomially solvable.*

Remark 12. *In general, for cacti the polytope defined by (29)-(32) is not integral. To see this, consider the graph $G = (V, E)$, where $V = \{1, 2, 3, 4, 5, 6, 7\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 5\}, \{4, 6\}, \{5, 6\}, \{6, 7\}\}$. Consider the vector $x(1) = 1$, $x(2) = x(3) = 0$, $x(4) = x(5) = x(6) = x(7) = 1/2$. This is an extreme point, to see this, notice that it is the unique solution of the following system of equations.*

$$\begin{array}{rcl}
x(1) & = & 1 \\
x(2) & = & 0 \\
x(3) & = & 0 \\
x(2) + x(4) + x(6) & = & 1 \\
x(3) + x(5) + x(6) & = & 1 \\
x(6) + x(7) & = & 1 \\
x(1) + x(2) + x(3) + x(4) + x(5) + x(6) + x(7) & = & 3
\end{array}$$

5.1. **Extension to Cacti.** Here we give an algorithm for the p-median problem in cacti, that is used to solve the p-dominating set problem.

5.1.1. *Decomposition algorithm.* Suppose that D is a directed graph that is a 1-sum of $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$, and $V_1 \cap V_2 = \{u\}$. We need the definitions below.

- Let $\lambda_0(k)$ be the optimal value of the k-median problem in $D_2 \setminus u$.

- Let $\lambda_1(k)$ be the optimal value of the k-median problem in D_2 with the constraint that u is included in the solution.
- Let $\lambda_2(k)$ be the optimal value of the k-median problem in D_2 with the constraint that u is not included in the solution. In this case an arc (u, t) should be in the solution, with $t \in V_2 \setminus u$.

Let D_1' be the graph obtained from D_1 by adding the node u', and the arc (u, u'). Then in D_1' we give the weight $\lambda_1(k+1) - \lambda_0(k)$ to u, and the weight $\lambda_2(k) - \lambda_0(k)$ to the arc (u, u'). If we solve the $(p-k+1)$-median problem in D_1', we obtain a solution of value $\alpha(k)$. It contains $p-k$ nodes in V_1, and can be combined with a solution in D_2 with k nodes in $V_2 \setminus u$. This gives a solution in D whose value is $\beta(k) = \alpha(k) + \lambda_0(k)$. From this we obtain the optimal value of the p-median problem as

$$\min_{0 \leq k \leq p} \beta(k).$$

5.1.2. *Combinatorial algorithm for a cycle.* Here we assume that $D = (V, A)$ is a bidirected cycle, and D' has been obtained by adding a node u' for each $u \in V$, and also the arc (u, u'). These new arcs are called *artificial*. This is the general case that has to be used in the decomposition algorithm above.

For any index i we decompose in the following three cases:

- Neither of $(i, i+1)$ nor $(i+1, i)$ is in the solution.
- $(i, i+1)$ is in the solution.
- $(i+1, i)$ is in the solution.

In each case we have a bidirected path with the artificial arcs defined above. This can be decomposed by means of 1-sums. So the decomposition algorithm above is applied, where one piece consists of a bidirected path with two original nodes and two artificial nodes. This piece is treated in constant time. Since the algorithm uses p values of the parameter k, we have an algorithm that requires quadratic time. We state this below.

Theorem 13. *The p-dominating set problem in a cycle $C = (V, E)$ can be solved in $O(|V|^2)$ time.*

When we apply this decomposition algorithm to a cactus, each piece is a cycle. Treating one piece once takes quadratic time, and since at most p values of the parameter k are needed, the algorithm requires cubic time. We have the following.

Theorem 14. *The p-dominating set problem in a cactus $G = (V, A)$ can be solved in $O(|V|^3)$ time.*

6. Concluding remarks

We have used results for facility location to study the dominating set problem. Instead of having only the node variables, having the arc variables allowed us to not only to derive polyhedral characterizations, but to also obtain linear time algorithms.

We also used results from the p-median problem to study the p-dominating set problem, and prove Theorem 10. We have not been able to find a direct proof of this apparently simple result.

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