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On the Relative Strength of Different Generalizations of Split Cuts

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Abstract

Split cuts are among the most important and well-understood cuts for general mixed-integer programs. In this paper we consider some recent generalizations of split cuts and compare their relative strength. More precisely, we compare the elementary closures of split, cross, crooked cross and general multi-branch split cuts as well as cuts obtained from multi-row and basic relaxations.

We present a complete containment relationship between the closures of split, rank 2 split, cross, crooked cross and general multi-branch split cuts. More specifically, we show that 3-branch split cuts strictly dominate crooked cross cuts, which in turn strictly dominate cross cuts. We also show that multi-branch split cuts are incomparable to rank 2 split cuts. In addition, we also show that cross cuts, and hence crooked cross cuts, cannot always be obtained from 2-row relaxations or from basic relaxations. Together, these results settle some open questions raised in earlier papers.

1 Introduction

Cutting planes are crucial for solving mixed-integer programs (MIPs), and the Gomory mixed-integer (GMI) cut is currently among the most effective cutting planes for general MIPs. Cook, Kannan and Schrijver [CKS90] studied a class of disjunctive cuts called *split cuts* (formal definitions are presented in Section 2). Despite their simplicity, many known families of cutting planes (or *cuts* for short) such as the GMI, lift-and-project, and flow cover cuts can be viewed as split cuts from very simple disjunctions. Due to their importance, these cuts have been extensively studied, both theoretically [NW90, MW01, OM01, CL01, BP03, ACL05, BCM12] and computationally [BCCN96, BS08, DGL10, DG10, Bon12, FS11].

In the following, we refer to a *mixed-integer set* as the set of mixed-integer solutions of a given set of linear equations or inequalities (a fixed subset of variables are restricted to be integral), and refer to the polyhedron defined by these linear constraints as the *the linear relaxation* of the mixed-integer set. Cuts for a mixed-integer set are linear inequalities valid for its convex hull. Split cuts (and other structured families of cuts) for a mixed-integer set are assumed to be derived using both the linear constraints and integrality restrictions defining the set. The *elementary closure* of a family of cuts for a mixed-integer set is the set of (real) points in its linear relaxation satisfying all cuts in the family.

When the linear constraints defining a mixed-integer set are given in inequality form, Andersen, Cornuéjols and Li [ACL05] proved that any split cut can be obtained as a split cut from a *basic* relaxation; also see [DGR11] for a simpler proof. A basic relaxation is the mixed-integer set defined by a maximal subset of linearly independent constraints of the linear relaxation and the original integrality restrictions (the remaining linear constraints are dropped). These relaxations generalize the corner relaxation introduced by Gomory [Gom69], as they also consider infeasible bases of the linear relaxation. When a mixed-integer set is defined by linear equations and nonnegativity constraints on some variables, then any split cut can be obtained as a *mixed-integer rounding* (MIR) inequality, as described by Nemhauser and Wolsey; see [NW88, NW90]. MIR inequalities are obtained by using nonnegativity constraints together with a single equation obtained as a linear combination of (a linearly independent subset of) the constraints of the linear relaxation. Therefore, depending on how the linear relaxation of the set is defined, it is possible to view split cuts as valid inequalities obtained from basic relaxations, or, as cuts obtained from 1-row relaxations.

Recently, split cuts have been generalized in different ways to obtain more effective cutting planes. One such generalization is to use two or more split disjunctions simultaneously to obtain valid inequalities. This gives rise to multi-branch split cuts, or *t-branch split cuts* when t split disjunctions are used. These cuts were first studied by Li and Richard [LR08] and recently Dash and Günlük extended some of their results [DG11]. Dash, Dey and Günlük [DDG, DDG11a] study 2-branch split cuts (and call them *cross cuts*) and *crooked cross cuts* (which are derived using three linearly dependent split disjunctions). Crooked cross cuts subsume cross cuts and are implied by 3-branch split cuts.

A different generalization of split cuts is obtained by considering multi-row relaxations of the mixed-integer set instead of one-row relaxations. This approach was introduced by Andersen, Louveaux, Weismantel and Wolsey [ALWW07] who study the so-called two-row continuous group relaxation and show that the convex hull of solutions of this relaxation is given by (two-dimensional) lattice-free cuts. This topic has received significant attention lately; see [CCZ] for a recent survey. More generally, a k-row relaxation of a mixed-integer set is constructed by aggregating the equations defining the linear relaxation into k equations for some small integer k.

In this paper we compare cuts obtained from different generalizations of split cuts. In particular, we compare the strength of split, cross, crooked cross and general t-branch split cuts as well as cuts obtained from multi-row and basic relaxations, by comparing their elementary closures. As mentioned earlier, some results comparing the strength of these closures are already present in the literature; we next review some of these results and highlight our contributions in this paper. We say that a family of cuts *dominates* another if for every mixed-integer set, the elementary closure of the first family of cuts for the set is contained in the elementary closure of the second family of cuts for the same set. We say that the dominance is *strict* if there are examples where the elementary closure of the first family is strictly contained in the elemetary closure of the second family. Henceforth, we refer to the elementary closure of a family $\mathcal{F}$ of cuts for a mixed-integer set as its $\mathcal{F}$-closure. Further, we define the *second* $\mathcal{F}$-closure as the elementary closure of the family $\mathcal{F}$ of cuts for the mixed-integer set defined by the constraints in the $\mathcal{F}$-closure of the original mixed-integer set along with its integrality restrictions.

Multi-branch split cuts. Recall that split cuts are the same as to 1-branch split cuts and cross cuts are the same as 2-branch split cuts. Cook, Kannan and Schrijver [CKS90] presented a simple mixed-integer set (with two integer variables and one continuous variable, see Example 2) such that its convex hull is strictly contained in its second split closure (actually in its nth split closure for any finite n).In stark contrast, Andersen, Louveaux, Weismantel and Wolsey [ALWW07] show that this convex hull can be obtained by adding a single two-dimensional lattice-free cut. Dash, Dey and Günlük [DDG] showed that this lattice-free cut is also a cross cut and therefore for this simple set, the cross closure is strictly contained in the split closure. This result was extended by Li and Richard [LR08] who showed that for $t > 2$, t-branch split cuts strictly dominate 2-branch split cuts. Subsequently, it was shown in [DG11] that for $t > k > 0$, t-branch split cuts strictly dominate k-branch split cuts.

In addition, it is known that 3-branch split cuts dominate crooked cross cuts which, in turn, dominate cross cuts [DDG11a, DDG]. However, these two dominance relationships were not known to be strict prior to our work. In [DDG11a] the authors show that there are crooked cross cuts that cannot be obtained by a *single* cross cut; however, this result does not rule out the possibility that the cross closure (which contains potentially infinitely many cuts) is *always equal* to the crooked cross closure.

In this paper we establish that 3-branch split cuts strictly dominate crooked cross cuts which, in turn, strictly dominate 2-branch split cuts.

Theorem 1.1. *There is a mixed-integer set such that its crooked cross closure is strictly contained in its cross closure.*

Theorem 1.2. *There is a mixed-integer set such that its* 3*-branch closure is strictly contained in its crooked cross closure.*

Dash et al. remark (Section 4.1 of [DGV11]) that although there are cross cuts (Example 2 of [CKS90]) that cannot be obtained via rank-2 split cuts, it is not known if in fact the cross closure strictly dominates the second split closure. This question is relevant in their computational procedure for generating cross cuts. We also answer this question and show that cross cuts and rank-2 split cuts are not comparable.

Theorem 1.3. *For every finite integer $t > 0$, there is a mixed-integer set whose second split closure is strictly contained in its t-branch split closure.*

In Figure 1 we summarize the dominance relationships between these closures, with a plain arrow from one closure to another if the first closure dominates the second closure, and a crossed arrow from a closure to another, if the first does not dominate the second one (in the sense that for some mixed-integer set, the first closure is not contained in the second closure). When both types of arrows are present between a pair of closures, then one closure strictly dominates the other. Dashed arrows indicate results known prior to this paper and solid arrows indicate results obtained in this paper. In the figure, we denote the closure of t-branch split cuts with tBC for $t = 1, 2, 3$ and we use 4^+BC for all $t > 3$. We denote the crooked cross cut closure by CCC and use SC^2 to denote the second split closure. Note that the displayed arrows can be used to infer the relationship between any pair of the closures considered.

Cuts from relaxations. Structured relaxations of mixed-integer sets have been widely used to generate cutting planes for the original sets. The literature on the theoretical aspects of different relaxations is extensive; for a small set of representative publications, see [MW99, CCZ11, ALWW07]. In this paper we focus on two relaxations: the *basic* and the *k-row* relaxations (see Section 2.2 for formal definitions).

When the linear relaxation of the mixed-integer set is given in equality form, Dash, Dey and Günlük [DDG] showed that every cross cut (resp. crooked cross cut) can be obtained as a cross cut

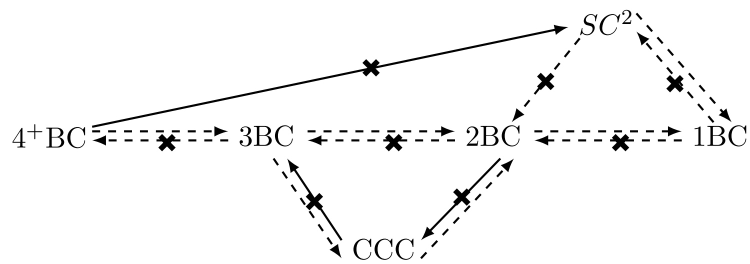

Figure 1: Comparing multi-branch split cuts with crooked cross cuts and rank 2 split cuts

(resp. crooked cross cut) from a 3-row relaxation. However, they left as an open question whether these cuts can also be obtained from 2-row relaxations. They also note that if crooked cross cuts can be obtained as crooked cross cuts from 2-row relaxations, then crooked cross cuts would be equivalent to cuts from all 2-row continuous group relaxations of the set. In this paper we answer this question.

Theorem 1.4. *There is a mixed-integer set such that its cross cut closure cannot be obtained by all cuts from its* 2*-row relaxations.*

Since the crooked cross closure is contained in the cross closure, the above theorem directly implies the following.

Corollary 1.5. *There is a mixed-integer set such that its crooked cross cut closure cannot be obtained by all cuts from its* 2*-row relaxations.*

Finally, we also show that unlike split cuts, t-branch split cuts in general cannot always be obtained from basic relaxations.

Theorem 1.6. *There is a mixed-integer set such that its cross cut closure cannot be obtained by all cuts from its basic relaxations.*

In Figure 2 we show some of the dominance relationships between these closures. We denote the closure of cuts from k-row relaxations by kR for $k = 1, 2, 3$ and we use CCC to denote crooked cross cuts, CC to denote cross cuts (2-branch split cuts) and SC to denote split cuts (1-branch split cuts). We use BR to denote cuts from basic relaxations. The fact that $2R$ does not dominate $3R$ follows from the fact that $2R$ does not dominate CC. The fact that $1R$ does not dominate $2R$ can be proved using the example of Cook, Kannan and Schrijver [CKS90] where the integer hull has infinite split rank. It is shown in [ALWW07] that the integer hull in this example can be obtained from a 2-row relaxation. On the other hand, it is possible (and nontrivial) to show that all cuts from 1-row relaxations of this example are split cuts, and thus cannot yield the integer hull. We believe that CCC does not dominate $1R$ but we cannot prove this.

Organization of the paper. The outline of the paper is as follows. In the next section we formally define the families of cuts studied in the paper and their closures. Unlike the case of the split closure, the polyhedrality of t-branch split closures for $t \geq 2$ is not known and consequently we need tools to tackle the interaction of potentially infinitely many cuts; Section 3 presents the main

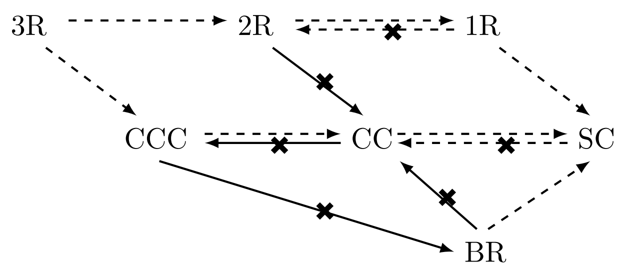

Figure 2: Comparing cuts from multi-row and basic relaxations with multi-branch split cuts

technical tool for this purpose, dubbed the "Height Lemma". In Sections 4, 5, and 6 we compare the closures of multi-branch split cuts and crooked cross cuts. In the last two sections we compare the strength of cross cuts with cuts obtained from multi-row and basic relaxations.

2 Preliminaries

In this paper we study mixed-integer sets of the following form: given rational matrices A, G, b with dimensions $r \times m$, $r \times n$ and $r \times 1$, respectively, and the mixed-integer lattice $I = \mathbb{Z}^m \times \mathbb{R}^n$, the polyhedron P is given by

$$P = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \ : \ Ax + Gy = b, \ y \geq 0\}, \tag{1}$$

and the associated mixed-integer set is

$$P^I = \{(x, y) \in \mathbb{Z}^m \times \mathbb{R}^n \ : \ Ax + Gy = b, \ y \geq 0\}. \tag{2}$$

P is called the *linear relaxation* of P^I. Clearly, different polyhedra, when intersected with I, can yield the same mixed-integer set. Throughout this paper, however, we associate the mixed-integer set P^I with a unique linear relaxation P. This association is necessary when considering elementary closures with respect to families of cuts as these cuts are derived using the linear relaxation of the mixed-integer set in hand.

2.1 Split Cuts and More General Disjunctive Cuts

A *split disjunction* for a mixed-integer lattice $\mathbb{Z}^m \times \mathbb{R}^n$ is a set of the form

$$D(\pi, \gamma) = \big\{(x, y) \in \mathbb{R}^{m+n} \ : \ \pi x \leq \gamma\big\} \cup \big\{(x, y) \in \mathbb{R}^{m+n} \ : \ \pi x \geq \gamma + 1\big\}$$

for some $\pi \in \mathbb{Z}^m, \gamma \in \mathbb{Z}$. Define the *split set* associated with the disjunction $D(\pi, \gamma)$ as

$$S(\pi, \gamma) = \{(x, y) \in \mathbb{R}^{m+n} \ : \ \gamma < \pi x < \gamma + 1\} = \mathbb{R}^{m+n} \setminus D(\pi, \gamma).$$

Split cuts. We call a linear inequality a *split cut* for P with respect to the disjunction $D(\pi, \gamma)$ if it is valid for $P \cap D(\pi, \gamma)$ [CKS90]. Notice that multiple split cuts can be generated from the same split disjunction. The *split closure* of P with respect to the lattice I is denoted by $SC(P)$

(the lattice I will be clear from the context). $SC(P)$ is defined as the points in P satisfying all split cuts for P derived from split disjunctions for I:

$$SC(P) = \bigcap_{(\pi,\gamma)\in\mathbb{Z}^{m+1}} \operatorname{conv}(P \cap D(\pi,\gamma)) = \bigcap_{(\pi,\gamma)\in\mathbb{Z}^{m+1}} \operatorname{conv}(P \setminus S(\pi,\gamma)),$$

where conv(.) denotes the convex hull operator. As I is contained in $D(\pi,\gamma)$ for all $(\pi,\gamma) \in \mathbb{Z}^{m+1}$, it follows that $P^I \subseteq SC(P)$, namely split cuts do not cut off any point in the set P^I. One can iterate the closure operator and define (for an integer k) the kth split closure $SC^k(P)$ recursively by setting $SC^k(P) = SC(SC^{k-1}(P))$ and $SC^1(P) = SC(P)$.

t-branch split cuts. Consider an integer t together with $\pi^i \in \mathbb{Z}^m$ and $\gamma_i \in \mathbb{Z}$ for $i = 1,\ldots,t$. The set $D(\pi^1,\ldots,\pi^t,\gamma_1,\ldots,\gamma_t)$ given by

$$D(\pi^1,\ldots,\pi^t,\gamma_1,\ldots,\gamma_t) = \bigcap_{i=1}^{t} D(\pi^i,\gamma_i) = \mathbb{R}^{m+n} \setminus \bigcup_{i=1}^{t} S(\pi^i,\gamma_i) \tag{3}$$

is called a *t-branch split disjunction* for I [LR08]. The fact that $I \subseteq D(\pi^i,\gamma_i)$ implies that $P^I \subseteq D(\pi^1,\ldots,\pi^t,\gamma_1,\ldots,\gamma_t)$. A linear inequality is a *t-branch split cut* for P with respect to a t-branch split disjunction D if it is valid for $P \cap D$. The *t-branch split closure* of P with respect to I, denoted by $tBC(P)$, is defined as the set of points in P which satisfy all t-branch split cuts:

$$tBC(P) = \bigcap_{(\pi^1,\gamma_1),\ldots,(\pi^t,\gamma_t)\in\mathbb{Z}^{m+1}} \operatorname{conv}(P \cap D(\pi^1,\ldots,\pi^t,\gamma_1,\ldots,\gamma_t)).$$

Similar to the split closure, $tBC(P)$ depends on the mixed-integer lattice I and throughout the paper I will be clear from the context. Notice that in the case $t = 1$ we have $1BC(P) = SC(P)$. Again, also notice that $P^I \subseteq tBC(P)$.

In [DDG], 2-branch split disjunctions are called *cross disjunctions*, and 2-branch split cuts are called *cross cuts*. In this case, we have the equivalent definition of the *cross closure* as

$$CC(P) = \bigcap_{(\pi^1,\gamma_1),(\pi^2,\gamma_2)\in\mathbb{Z}^{m+1}} \operatorname{conv}(P \setminus (S(\pi^1,\gamma_1) \cup S(\pi^2,\gamma_2))).$$

Crooked cross cuts. Given $\pi^1,\pi^2 \in \mathbb{Z}^m$ and $\gamma_1,\gamma_2 \in \mathbb{Z}$ we define the sets

$$\begin{aligned}
D_1^c(\pi^1,\pi^2,\gamma_1,\gamma_2) &= \{(x,y) \in \mathbb{R}^{m+n} : \pi^1 x \le \gamma_1, (\pi^2-\pi^1)x \le \gamma_2-\gamma_1\},\\
D_2^c(\pi^1,\pi^2,\gamma_1,\gamma_2) &= \{(x,y) \in \mathbb{R}^{m+n} : \pi^1 x \le \gamma_1, (\pi^2-\pi^1)x \ge \gamma_2-\gamma_1+1\},\\
D_3^c(\pi^1,\pi^2,\gamma_1,\gamma_2) &= \{(x,y) \in \mathbb{R}^{m+n} : \pi^1 x \ge \gamma_1+1, \pi^2 x \le \gamma_2\},\\
D_4^c(\pi^1,\pi^2,\gamma_1,\gamma_2) &= \{(x,y) \in \mathbb{R}^{m+n} : \pi^1 x \ge \gamma_1+1, \pi^2 x \ge \gamma_2+1\}.
\end{aligned}$$

We call the set $D^c(\pi^1,\pi^2,\gamma_1,\gamma_2) = \bigcup_{i=1}^4 D_i^c(\pi^1,\pi^2,\gamma_1,\gamma_2)$ a *crooked cross disjunction* for I. A linear inequality is a *crooked cross cut* for P if it is valid for $P \cap D^c$ for some crooked cross disjunction D^c. The *crooked cross closure* of P, which we denote by $CCC(P)$, is again defined as $\bigcap_{(\pi^1,\gamma_1),(\pi^2,\gamma_2)\in\mathbb{Z}^{m+1}} \operatorname{conv}(P \cap D^c(\pi^1,\pi^2,\gamma_1,\gamma_2))$. Notice that $P^I \subseteq CCC(P)$.

2.2 Relaxations of Mixed-integer Sets

k-row relaxation. Consider a polyhedral set P as in (1). A *k-row relaxation* of P is obtained by combining the r equality constraints defining the set into $k \le r$ equalities. More precisely, it is $P(M) \triangleq \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n : MAx + MGy = Mb, y \ge 0\}$, where M is a $k \times r$ matrix. Any inequality valid for $P^I(M) = P(M) \cap I$ is called a *cut from a k-row relaxation.*

Basic relaxation. We now consider a polyhedron defined in inequality form. Let $P = \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n : Ax + Gy \le b\}$ where A, G and b have $r \ge m+n$ rows. For a subset $J \subseteq \{1,\ldots,r\}$ of row indices, we use A_J to denote the submatrix of A consisting of the rows of A corresponding to the indices in J. We define G_J and b_J similarly. Then a *basic relaxation* of P is obtained by keeping in the linear relaxation only linearly independent constraints, namely it is a set of the form $P_{[J]} = \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n : A_J x + G_J y \le b_J\}$ for some $J \subseteq \{1,2,\ldots,r\}$ such that the matrix $[A_J \; G_J]$ has full-row rank. A basic relaxation of the mixed-integer set P^I is obtained as follows: $P^I_{[J]} = P_{[J]} \cap I$ and any inequality valid for $P^I_{[J]}$ is called *a cut from a basic relaxation.*

2.3 Notation

We use $\|\cdot\|$ to denote the ℓ_2 norm. Given a point $x \in \mathbb{R}^n$ and a positive real $r > 0$, we use $B(x,r) = \{y \in \mathbb{R}^n : \|x - y\| < r\}$ to denote the ball centered at x with radius r. For a set $S \subseteq \mathbb{R}^n$ we use $\operatorname{conv}(S)$ to denote the convex hull of S, and $\operatorname{aff}(S)$ to denote the affine hull of S. Given a set of vectors $V \subseteq \mathbb{R}^n$ we use $\operatorname{span}(V)$ to denote the subspace spanned by V. Given a matrix $M \in \mathbb{R}^{n \times m}$, we use $\operatorname{rowspan}(M)$ to denote the subspace spanned by the rows of M.

3 Height Lemma

In preparation for the proof of our results we present the main technical tool used, called *Height Lemma* (this generalizes a similar result in [DG11]). Intuitively this lemma states the following: consider a collection (of arbitrary cardinality) of full dimensional pyramids, all sharing the same base. If we have a uniform lower bound on the height of the pyramids, plus the property that their apexes are not arbitrarily far from each other, then the intersection of all these pyramids contains a point outside of the common base. The motivation is that these pyramids will later represent what is 'left over' of P when we employ a subset of the cuts of interest, so this result allow us to talk about the left over of P when we add all these cuts together. In the formal statement below, the points $s^1, s^2, \ldots, s^n$ form the base of the pyramids and the points in Q are the apexes.

Lemma 3.1 (Height Lemma)**.** *Let $(a,b) \in \mathbb{R}^n \times \mathbb{R}$ with $a \neq 0$ and let $s^1, s^2, \ldots, s^n$ be affinely independent points in the hyperplane $\{x \in \mathbb{R}^n : ax = b\}$. Take $b' > b$ and $U \ge 0$ and define $Q = \{x \in \mathbb{R}^n : ax \ge b', \|x\| \le U\}$. Then there exists a point x in $\bigcap_{q \in Q} \operatorname{conv}(s^1, s^2, \ldots, s^n, q)$ satisfying the strict inequality $ax > b$.*

Proof. Let $H = \{x \in \mathbb{R}^n : ax = b\}$ and $S = \operatorname{conv}(s^1, s^2, \ldots, s^n)$. We say that a point x that satisfies $ax > b$ has positive *height*; so our goal is to find a point in $\bigcap_{q \in Q} \operatorname{conv}(s^1, s^2, \ldots, s^n, q)$ with positive height. To simplify the notation, we assume without loss of generality that $\|a\| = 1$.

Clearly S is an $(n-1)$-dimensional simplex contained in H and, by comparing dimensions, the affine hull of S equals H. Consider a point x^* in the relative interior of S, and let $r > 0$ be such that the ball $B(x^*, r) \cap H$ is contained in S. Let U' be an upper bound on the norm of the points in S (this exists as S is bounded). We show that the point $x^* + (b' - b)\frac{r}{U+U'}a$ belongs to $\bigcap_{q \in Q} \mathrm{conv}(s^1, s^2, \ldots, s^n, q)$, which gives the desired result.

Consider $q \in Q$ and let q^* denote its orthogonal projection into H, namely $q^* = q - b''a$ for $b'' = b' - b$. The idea is to show that x^* can be written as a convex combination $\alpha q^* + (1-\alpha)y^*$ for some point y^* in S (see Figure 3). Then replacing q^* by q in this expression, we get by convexity that $\alpha q + (1-\alpha)y^* = x^* + \alpha b'' a$ belongs to $\mathrm{conv}(s^1, s^2, \ldots, s^n, q)$ and has positive height. Importantly, our construction will guarantee that we can lower bound α independently of the choice of q.

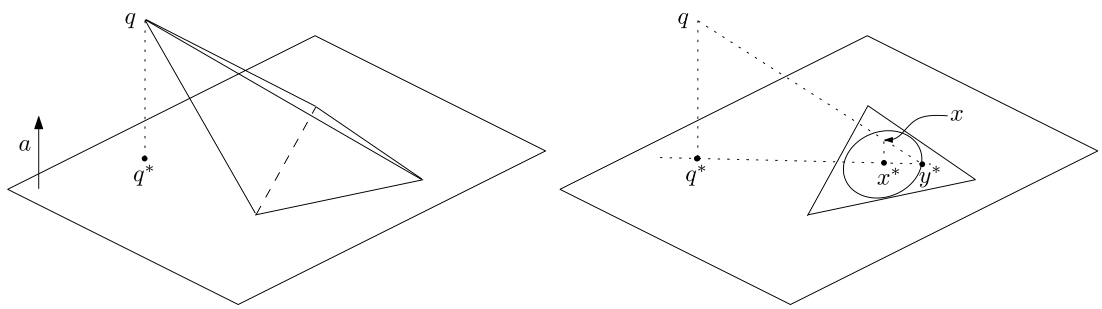

Figure 3: The left picture shows $\mathrm{conv}(s_1, s_2, \ldots, s_n, q)$ and the hyperplane $H = \{x \in \mathbb{R}^n : ax = b\}$. The right picture shows the construction of y^* and the point x which belongs to $\mathrm{conv}(s_1, s_2, \ldots, s_n, q)$ and has positive height, namely it satisfies $ax > b$.

To make this construction, consider the line $\{q^* + \lambda(x^* - q^*) : \lambda \in \mathbb{R}\}$ passing though the points q^* and x^*, and notice that it lies in the hyperplane H. This line intersects the boundary of the closed ball $\bar{B}(x^*, r) \cap H$ in two points, so let y^* denote such point which is farthest from q^* (notice that this point belongs to S); specifically, we have $y^* = q^* + \lambda^*(x^* - q^*)$ for $\lambda^* = 1 + \frac{r}{\|x^* - q^*\|}$, and notice that $\|y^* - q^*\| = \lambda^* \|x^* - q^*\| = r + \|x^* - q^*\|$. Rearranging, we can write explicitly x^* as a convex combination of q^* and y^*: $x^* = \alpha q^* + (1-\alpha)y^*$ for $\alpha = \frac{r}{\|y^* - q^*\|} \in [0, 1]$. As mentioned previously, we get that the point $\alpha q + (1-\alpha)y^* = x^* + \frac{r}{\|y^* - q^*\|} b'' a$ belongs to $\mathrm{conv}(s^1, s^2, \ldots, s^n, q)$. Using the triangle inequality, we get that

$$\frac{r}{\|y^* - q^*\|} b'' \geq \frac{r}{\|y^*\| + \|q^*\|} b'' \geq \frac{r}{U + U'} b''.$$

Using convexity we conclude that the point $x^* + b'' \frac{r}{U+U'} a$ belongs to $\mathrm{conv}(s^1, s^2, \ldots, s^n, q)$. Since the point is independent of q, it belongs to $\bigcap_{q \in Q} \mathrm{conv}(s^1, s^2, \ldots, s^n, q)$ and the result follows.

$\square$

Also note that in the proof we do not use the property that the norms of the points in Q are bounded, we only use the fact that their projection on H has bounded norm. It is therefore possible to generalize this result slightly to unbounded Q that has bounded projection on H.

By employing an affine transformation, this lemma also carries over to affine subspaces of $\mathbb{R}^n$.

Corollary 3.2. *Let $A \in \mathbb{R}^n$ be an affine subspace of dimension k. Fix $(a, b) \in \mathbb{R}^n \times \mathbb{R}$ such that $a \neq 0$ and let $s^1, s^2, \ldots, s^k \in \mathbb{R}^n$ be affinely independent points in $A \cap \{x \in \mathbb{R}^n : ax = b\}$. Take $b' > b$ and $U \geq 0$ and define $Q = \{x \in A : ax \geq b', \|x\| \leq U\}$. Then there exists a point x in $\bigcap_{q \in Q} \mathrm{conv}(s^1, s^2, \ldots, s^n, q)$ satisfying the strict inequality $ax > b$.*

For a vector $v \in \mathbb{R}^n$, a $n \times n$ matrix M and a set $S \subseteq \mathbb{R}^n$, let $S - v = \{s - v : s \in S\}$ and $MS = \{Ms : s \in S\}$. To see that the corollary follows from Lemma 3.1, let M be an $n \times n$ matrix with determinant one such that $M(A - s^1) = \mathbb{R}^k \times \{0\}^{n-k}$. Applying this affine transformation and subsequently removing the last k coordinates, the corollary reduces to the previous lemma applied to objects in $\mathbb{R}^k$ (points in Q are mapped to points in $\mathbb{R}^k \times \{0\}^{n-k}$ with bounded norm).

4 Crooked cross Closure Versus Cross Closure

In this section we prove Theorem 1.1 by constructing a polyhedral set P whose cross closure $CC(P)$ is strictly contained in its crooked cross closure $CCC(P)$. One important component of the construction is a triangle that cannot be covered by a cross set.

Theorem 4.1 ([DDG+11b]). *There exists a rational triangle $T^* \subseteq \mathbb{R}^2$ satisfying the following: (i) T^* does not contain integer points in its interior; (ii) T^* contains the points $(0,0)$, $(1,0)$, $(0,1)$ in its boundary; (iii) there is $\delta > 0$ such that for any pair of split sets S_1, S_2 for $\mathbb{Z}^2$, the set $T^* \setminus (S_1 \cup S_2)$ has area at least δ.*

Let T^* be such a triangle and let x^* be a point in the interior of T^*, say its centroid (which has rational coordinates). In this section we work with the polyhedron P defined as

$$P = \left\{(x, y) \in \mathbb{R}^2 \times \mathbb{R} : (x, y) \in \operatorname{conv}\left(T^* \times \{0\}\right) \cup \left(x^* \times \{1\}\right)\right\},$$

and the mixed integer lattice $I = \mathbb{Z}^2 \times \mathbb{R}$. The associated mixed-integer set $P^I = P \cap (\mathbb{Z}^2 \times \mathbb{R})$. We also define $T_\epsilon \triangleq P \cap \{x \in \mathbb{R}^3 : x_3 = \epsilon\}$ for $\epsilon \geq 0$ and define T^*_ϵ to be the projection of T_ϵ onto the first 2 coordinates. We next obtain the following result.

Lemma 4.2. *The inequality $x_3 \leq 0$ is valid for $CCC(P)$.*

Proof. Notice that $T^*_0 = T^*$ and $T^*_1 = x^*$, and as the latter belongs to the interior of T^*, we conclude that T_ϵ is contained in the interior of T^* for all $\epsilon > 0$. As T^* does not contain any integer points in its interior, $T^*_\epsilon \cap \mathbb{Z}^2 = \emptyset$ for all $\epsilon > 0$ and therefore $\operatorname{conv}(P) = T^* \times \{0\}$. Consequently, the inequality $x_3 \leq 0$ is valid for $\operatorname{conv}(P)$.

To conclude the proof, we recall the fact that the convex hull of any polyhedral mixed-integer set in $\mathbb{Z}^2 \times \mathbb{R}$ is given by crooked cross cuts. In particular, $\operatorname{conv}(P^I) = CCC(P)$ and consequently $x_3 \leq 0$ is valid for $CCC(P)$, concluding the proof. □

We next show that the inequality $x_3 \leq 0$ is not valid for $CC(P)$; since $CC(P)$ always contains $CCC(P)$ – which equals $\operatorname{conv}(P^I)$ in this section – such a result would imply that $CC(P)$ strictly contains $CCC(P)$, as desired. We start by showing that a single cross disjunction cannot imply the cut $x_3 \geq 0$.

Lemma 4.3. *There exists $\epsilon^* > 0$ such that for any pair of split sets S_1, S_2 for P, the set $T_{\epsilon^*} \setminus (S_1 \cup S_2)$ is non-empty.*

Proof. Notice that $\operatorname{area}(T^*_0) > \operatorname{area}(T^*_1) = 0$ and $\operatorname{area}(T^*_\epsilon)$ is continuous as a function of ϵ. Let $\delta > 0$ be given by Theorem 4.1 and take $\epsilon^* > 0$ such that $\operatorname{area}(T^*_{\epsilon^*}) \geq \operatorname{area}(T^*_0) - \delta/2$; the existence of ϵ^* is guaranteed by the Intermediate Value Theorem.

Let S^*_1 denote the projection of the split set S_1 onto the first 2 (integer) coordinates, and notice that $S_1 = S^*_1 \times \mathbb{R}$ and that S^*_1 is a split set for $(T^*, \mathbb{Z}^2)$. Define S^*_2 similarly. It then follows that

$T_{\epsilon^*} \setminus (S_1 \cup S_2)$ is non-empty if and only if $T^*_{\epsilon^*} \setminus (S^*_1 \cup S^*_2)$ is non-empty; we prove the latter. Theorem 4.1 guarantees that the set $T^* \setminus (S^*_1 \cup S^*_2)$ has area at least δ, and so $T^*_{\epsilon^*} \setminus (S^*_1 \cup S^*_1)$ has area at least $\delta/2$. Therefore $T_{\epsilon^*} \setminus (S_1 \cup S_2)$ is non-empty. $\square$

Together with the previous lemma, the Height Lemma directly implies that the cut $x_3 \leq 0$ is not valid for the cross closure of P; the proof is exactly the same as in Lemma 6.4 and is omitted.

Lemma 4.4. *The inequality $x_3 \leq 0$ is not valid for $CC(P)$.*

Employing Lemmas 4.2 and 4.4 we obtain Theorem 1.1:

Theorem 1.1 (restated). *$CCC(P) \subsetneq CC(P)$.*

5 Crooked Cross Cuts Versus 3-branch Split Cuts

In this section we prove Theorem 1.2 by constructing an integer set $P^I = P \cap I$ where $I = \mathbb{Z}^3$ such that $3BC(P) = \text{conv}(P^I) = \emptyset$ but $CCC(P) \neq \emptyset$. We define the polyhedron P to be the intersection of a specific octahedron with the unit cube, i.e.,

$$P = \left\{ x \in [0,1]^3 : \sum_{i \in I} x_i - \sum_{i \notin I} x_i \leq |I| - \frac{1}{2}, \forall I \subseteq \{1,2,3\} \right\}.$$

Notice that P^I is the empty set.

We first claim that $3BC(P) = \emptyset$. To see this, consider the 3-branch split disjunction $D = D(e^1, e^2, e^3, 0, 0, 0)$, where e^i is the ith unit vector in $\mathbb{R}^3$. Notice that x belongs to D iff $x_i \notin (0,1)$ for all $i = 1,2,3$, and therefore $x \in P \cap D$ if and only if x is a 0-1 vector. Therefore $P \cap D = \emptyset$. Since $3BC(P) \subseteq P \cap D$, the claim follows.

Now we need to show that $CCC(P) \neq \emptyset$; in particular, we show that $(1/2, 1/2, 1/2)$ belongs to $CCC(P)$. For that, we need the following characterization of the crooked cross closure.

Theorem 5.1. *([DDG11a, Theorem 3.1]) For any polyhedron $\tilde{P} \subseteq \mathbb{R}^n \times \mathbb{R}^l$ and mixed-integer lattice $I = \mathbb{Z}^n \times \mathbb{R}^l$,*

$$CCC(\tilde{P}) = \bigcap_{\pi^1, \pi^2 \in \mathbb{Z}^n} \text{conv}\left(\tilde{P} \cap \{(x,y) : \pi^1 x \in \mathbb{Z}, \pi^2 x \in \mathbb{Z}\} \right).$$

Lemma 5.2. *The point $(1/2, 1/2, 1/2)$ belongs to $CCC(P)$.*

Proof. Consider an arbitrary pair of vectors $\pi^1, \pi^2 \in \mathbb{Z}^3$ and define $P_{\pi^1,\pi^2} = \text{conv}(P \cap \{x \in \mathbb{R}^3 : \pi^1 x \in \mathbb{Z}, \pi^2 x \in \mathbb{Z}\})$. Given Theorem 5.1, it suffices to show that

$$(1/2, 1/2, 1/2) \in P_{\pi^1,\pi^2}. \tag{4}$$

For that, let $v \in \mathbb{R}^3$ be a non-zero vector orthogonal to π^1 and π^2. We will prove (4) for the case $v_1 \neq 0$; the proof for the cases $v_2 \neq 0$ or $v_3 \neq 0$ is identical. The idea in the analysis is that the set $\{x \in \mathbb{R}^3 : \pi^1 x \in \mathbb{Z}, \pi^2 x \in \mathbb{Z}\}$ contains all lines in the direction of v that pass through an integer point. We are interested in the lines that cross the intersection of P with the plane $x_1 = 1/2$; therefore, it suffices to project $\mathbb{Z}^3$ onto this plane along v and analyze the obtained set of points Λ, and show that $\text{conv}(P \cap \Lambda)$ contains $(1/2, 1/2, 1/2)$.

Define the integer points $w^1 = (0, -\lfloor \frac{v_2}{2} \rfloor, -\lfloor \frac{v_3}{2} \rfloor)$ and $w^2 = (1, 1 + \lfloor \frac{v_2}{2} \rfloor, 1 + \lfloor \frac{v_3}{2} \rfloor)$; clearly $w^j \pi^i \in \mathbb{Z}$ for $i, j \in \{1, 2\}$. Now consider the points $u^1 = w^1 + v/2$ and $u^2 = w^2 - v/2$, which lie in the plane $x_1 = 1/2$. We can use the fact that v is orthogonal to π^1, π^2 to deduce that $u^j \pi^i \in \mathbb{Z}$ for $i, j \in \{1, 2\}$. Also, notice that u_2^j and u_3^j belong to the interval $[0, 1]$ for $j \in \{1, 2\}$. Now any point in $[0, 1]^3$ with one component equal to $1/2$ is contained in P, and therefore so are u^1, u^2. Therefore, these points belong to P_{π^1,π^2}. By convexity of P_{π^1,π^2}, the point $(u^1 + u^2)/2 = (1/2, 1/2, 1/2)$ also belongs to it, which concludes the proof of the lemma. □

The fact that $3BC(P) = \emptyset \neq CCC(P)$ then concludes the proof of Theorem 1.2:

Theorem 1.2 (restated). $3BC(P) \subsetneq CCC(P)$.

6 t-branch Split Closure Versus Second Split Closure

In this section we prove Theorem 1.3, which states that there is a polyhedral set P whose t-branch split closure is not contained in its second split closure. More specifically, we will work with the integer lattice $I = \mathbb{Z}^{n+1}$ and the distorted simplex P defined by

$$P = \left\{ x \in \mathbb{R}^{n+1} \ : \ \sum_{i=1}^{n} x_i + 2x_{n+1} \leq n + 2 - \epsilon, \quad x_i \geq \epsilon, \ i = 1, \ldots, n \right\},$$

where $\epsilon > 0$ is a small scalar defined in the proof of Lemma 6.3 below (see Figure 4).

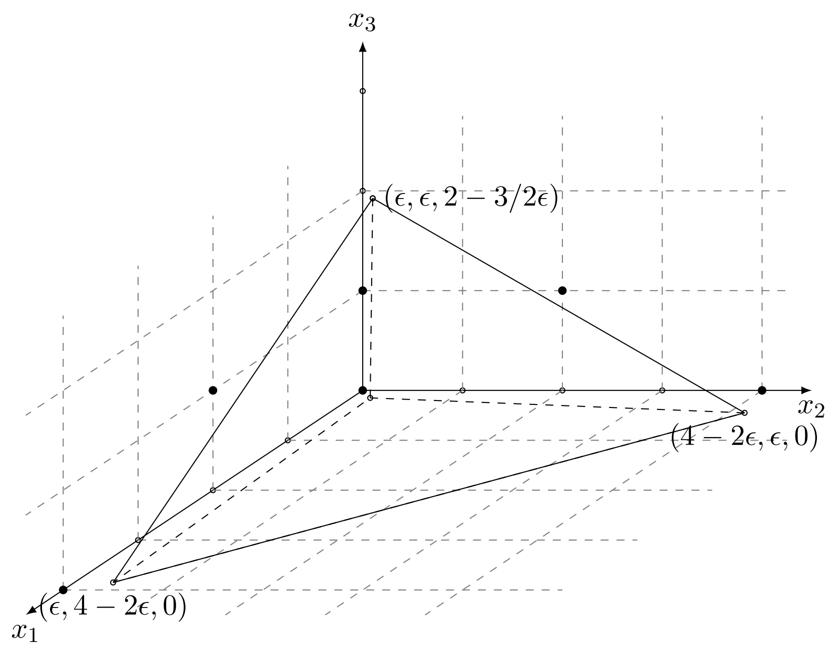

Figure 4: The set P when $n = 2$

We claim that the cut $x_{n+1} \leq 0$ is valid for $SC^2(P)$. First notice that Chvátal-Gomory cuts [NW88] for P can be obtained by rounding the right-hand sides of the constraints above. Since every Chvátal-Gomory cut is also a split cut, we observe that

$$
\begin{aligned}
SC(P) &\subseteq \left\{ x \in \mathbb{R}^{n+1} \;:\; \sum_{i=1}^{n} x_i + 2x_{n+1} \leq n+1, \quad x_i \geq 1,\; i = 1, \ldots, n \right\} \\
&\subseteq \left\{ x \in \mathbb{R}^{n+1} \;:\; x_{n+1} \leq \frac{1}{2} \right\}.
\end{aligned}
$$

Again by using Chvátal-Gomory cuts, we get that

$$SC^2(P) \subseteq \{x \in \mathbb{R}^{n+1} \;:\; x_{n+1} \leq 0\},$$

which proves the claim. Since $P \subseteq SC^2(P)$, we get the following.

Lemma 6.1. *The inequality $x_{n+1} \leq 0$ is valid for $SC^2(P)$. Furthermore, it is facet defining as it contains the following $n+1$ affinely independent points in P^I:*

$$s_1 = (2, 1, \ldots, 1, 0), s_2 = (1, 2, \ldots, 1, 0), \ldots, s_n = (1, 1, \ldots, 2, 0), s_{n+1} = (1, 1, \ldots, 1, 0). \tag{5}$$

We next argue that the inequality $x_{n+1} \leq 0$ is not valid for the t-branch split closure of P when $t < n$. First we show that a single t-branch split cut cannot imply the cut $x_{n+1} \leq 0$. The main tool used is the fact that simplices cannot be covered by a small collection of split sets. More precisely, define the simplex

$$\Delta_n = \left\{ x \in \mathbb{R}^n \;:\; \sum_{i=1}^{n} x_i \leq n, \quad x_i \geq 0,\; i = 1, \ldots, n \right\}.$$

Theorem 6.2 ([DG11])**.** *For every integer $n > 0$, there exists a constant $\delta > 0$ such that the volume of the n-dimensional simplex Δ_n not covered by any collection of $n-1$ split sets is at least δ.*

Lemma 6.3. *Let $S_1, \ldots, S_t$ be a collection of split sets for P with $t < n$ and let $S = \bigcup_{i=1}^{t} S_i$ be their union. Then the set $P \setminus S$ contains a point x such that $x_{n+1} = 1$.*

Proof. Consider the slice of P with $x_{n+1} = 1$, namely

$$
\begin{aligned}
T \triangleq\; & P \cap \{x \in \mathbb{R}^{n+1} \;:\; x_{n+1} = 1\} \\
=\; & \left\{ x \in \mathbb{R}^{n+1} \;:\; \sum_{i=1}^{n} x_i \leq n - \epsilon, \quad x_{n+1} = 1, \quad x_i \geq \epsilon,\; i = 1, \ldots, n \right\}.
\end{aligned}
$$

We will show that if $t < n$ then $T \setminus S \neq \emptyset$, which proves the lemma. Let $\pi^i \in \mathbb{Z}^{n+1}$ and $\gamma_i \in \mathbb{Z}$ be such that $S_i = \{x \in \mathbb{R}^{n+1} \;:\; \gamma_i < \pi^i x < \gamma_i + 1\}$. Notice that

$$T \cap S_i = T \cap \left\{ x \in \mathbb{R}^{n+1} \;:\; \gamma_i - \pi^i_{n+1} < \sum_{i=1}^{n} \pi^i x_i < \gamma_i - \pi^i_{n+1} + 1 \right\}$$

and therefore, $T \cap S_i = T \cap (S_i^* \times \mathbb{R})$, where S_i^* is the split set $S(\pi^i, \gamma_i - \pi^i_{n+1})$ contained in $\mathbb{R}^n$. Let $S^* = \bigcup_{i=1}^{n} S_i^*$ and observe that $T \cap S = T \cap (S^* \times \mathbb{R})$. Let T^* denote the projection of T onto the first n coordinates, namely

$$T^* = \left\{ x \in \mathbb{R}^n \;:\; \sum_{i=1}^{n} x_i \leq n - \epsilon, \quad x_i \geq \epsilon\; i = 1, \ldots, n \right\},$$

and notice that $T \setminus S \neq \emptyset$ if and only if $T^* \setminus S^* \neq \emptyset$, so it suffices to prove the latter.

Now notice that T^* is a perturbation of the simplex Δ_n, depending on ϵ. Choosing $\epsilon > 0$ small enough, we get the volume of $T^* \setminus S^*$ arbitrarily close to the volume of $\Delta_n \setminus S^*$, which is strictly positive by Theorem 6.2. This implies that $T^* \setminus S^*$ is non-empty, which concludes the proof of the lemma. □

Applying the height lemma, we can make a statement about the simultaneous effect of every possible collection of $t < n$ split sets $S_1, S_2, \dots, S_t$ on P.

Lemma 6.4. *For $t < n$, the inequality $x_{n+1} \leq 0$ is not valid for the t-branch split closure of P.*

Proof. Let $\mathcal{S}_t$ denote the family of t-branch split sets for $I = \mathbb{Z}^{n+1}$, namely sets of the form $\bigcup_{i=1}^t S_i$ where each S_i is a split set for I. To prove the lemma, we show that $\bigcap_{S \in \mathcal{S}_t} \operatorname{conv}(P \setminus S)$ contains a point x with $x_{n+1} > 0$.

For each $S \in \mathcal{S}_t$, let x^S be the point given by Lemma 6.3. As $x^S \in T$, we have $||x^S|| \leq n+1$, we can apply the Height Lemma with parameters $a = (0, 0, \dots, 0, 1)$, $b = 0$, $b' = 1$, $U = n+1$, and $s_1, s_2, \dots, s_{n+1}$ defined in (5) to get that $\bigcap_{S \in \mathcal{S}_t} \operatorname{conv}(s_1, s_2, \dots, s_{n+1}, x^S)$ contains a point x with $x_{n+1} > 0$. Notice that for each $S \in \mathcal{S}_t$ we have $\operatorname{conv}(s_1, s_2, \dots, s_{n+1}, x^S) \subseteq \operatorname{conv}(P \setminus S)$ (since the integer points $s_1, s_2, \dots, s_{n+1}$ belong to $P^I \subseteq \operatorname{conv}(P \setminus S)$), which implies that $\bigcap_{S \in \mathcal{S}_t} \operatorname{conv}(P \setminus S)$ contains a point x with $x_{n+1} > 0$. This concludes the proof. □

Using Lemmas 6.1 and 6.4 we now prove Theorem 1.3.

Theorem 1.3 (restated). *For any positive integer $t < n$, $SC^2(P)$ is strictly contained in $tBC(P)$.*

Proof. We have already showed that

$$SC^2(P) \subseteq \left\{ x \in \mathbb{R}^{n+1} \ : \ \sum_{i=1}^{n} x_i + 2x_{n+1} \leq n+1, \quad x_{n+1} \leq 0, \quad x_i \geq 1, \ i = 1, \dots, n \ \right\}. \tag{6}$$

Note that, the polyhedron on the right is defined by $n+2$ inequalities in $\mathbb{R}^{n+1}$ and therefore, it has at most $n+2$ extreme points that are obtained by intersecting all but one of the defining hyperplanes. It can be checked that the only fractional point that can be obtained by intersecting $n+1$ of these hyperplanes is obtained by excluding the inequality $x_{n+1} \leq 0$. The corresponding point, however violates $x_{n+1} \leq 0$ and therefore is not an extreme point of the polyhedron. Consequently, inequality (6) in fact holds as equality and $SC^2(P)$ is integral. This implies that $\operatorname{conv}(P^I) = SC^2(P)$ and by Lemma 6.4 we conclude that $SC^2(P)$ is strictly contained in $tBC(P)$. □

Further, as P is defined by $n+1$ linearly independent linear inequalities in $n+1$ variables, P is a basic relaxation of itself, and therefore $\operatorname{conv}(P^I)$ can be obtained by cuts from basic relaxations. This yields the following corollary.

Corollary 6.5. *For any positive integer $t < n$, the set of points satisfying all cuts from basic relaxations of P is strictly contained in $tBC(P)$.*

7 Cross Cuts from Basic Relaxations

In this section we prove Theorem 1.6 by constructing a polyhedral set P with the property that the intersection of all cuts from its basic relaxations does not dominate its cross closure. We will work with the polyhedron (see Figure 5)

$$P = \Big\{(x,w) \in \mathbb{R}^2 \times \mathbb{R} : \; -x_1 - x_2 + w \le 0, \quad x_1 + x_2 + w \le 2, \quad -x_1 + x_2 + w \le 1, \quad x_1 - x_2 + w \le 1 \Big\}, \tag{7}$$

and the mixed-integer lattice $I = \mathbb{Z}^2 \times \mathbb{R}$. For $j = 1,2,3,4$, let P_j denote the relaxation of P obtained by dropping the jth constraint in (7); also let $P_j^I = P_j \cap I$.

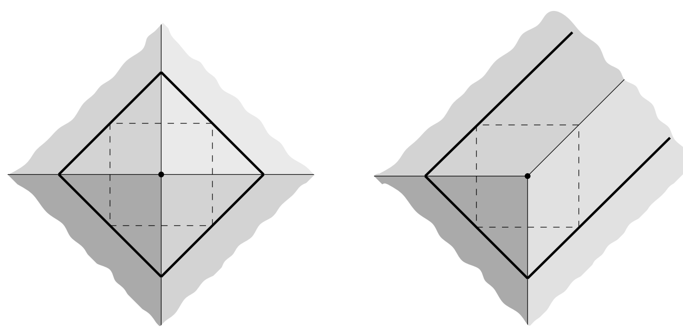

Figure 5: The left picture shows P along the x_1, x_2 axis, with the unit square $[0,1]^2$ in dashed lines and the intersection of P with the plane $t = 0$ in bold. The right picture shows the basic relaxation P_1, which gives rise to the set $P_1^I = P_1 \cap I$.

As $P \subseteq \mathbb{R}^3$ is defined by 4 constraints, the sets P_j for $j = 1,2,3,4$ give all the basic relaxations of P. Thus we want to show that $\bigcap_{j=1}^4 \operatorname{conv}(P_j^I) \not\subseteq CC(P)$. For that, we show that $w \le 0$ is a cross cut for P but it is not valid for $\bigcap_{j=1}^4 \operatorname{conv}(P_j^I)$.

Lemma 7.1. *The inequality $w \le 0$ is a valid cross cut for P.*

Proof. We will show that $w \le 0$ is a cross cut for P derived from the cross disjunction $D(e^1, e^2, 0, 0) = \mathbb{R}^3 \backslash (S_1 \cup S_2)$ where e^i is the ith unit vector in $\mathbb{R}^3$ and S_1 is the split set $\{(x,w) \in \mathbb{R}^2 \times \mathbb{R} : 0 < x_1 < 1\}$ and S_2 is the split set $\{(x,w) \in \mathbb{R}^3 : 0 < x_2 < 1\}$.

This statement would be false only if there exists some point (x,w) belonging to both P and $D(e^1, e^2, 0, 0)$ with $w > 0$. But if (x,w) belongs to P and $w > 0$, the inequalities in (7) immediately imply that $0 < x_1 + x_2 < 2$ and $-1 < x_1 - x_2 < 1$. Therefore (see Figure 6)

$$\Big\{(x,w) \in P : w > 0\Big\} \subseteq S_1 \cup S_2 = \mathbb{R}^3 \setminus D(e^1, e^2, 0, 0),$$

and hence (x,w) does not belong to $D(e^1, e^2, 0, 0)$. The result then follows. $\square$

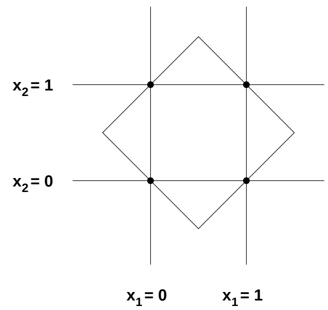

Figure 6: The set $\{x \in \mathbb{R}^2 : 0 < x_1 + x_2 < 2, -1 < x_1 - x_2 < 1\}$ is the interior of the depicted quadrilateral.

Next we show that this cut cannot be obtained from basic relaxations.

Lemma 7.2. *The inequality $w \leq 0$ is not valid for $\bigcap_{j=1}^4 \operatorname{conv}(P_j^I)$.*

Proof. Observe that the points

$$p_1 = (0,0,0),\ p_2 = (1,1,0),\ p_3 = (0,1,0),\ p_4 = (1,0,0)$$

all belong to P^I and therefore to $\operatorname{conv}(P_j^I)$ for $j = 1, \dots, 4$. Also, the points

$$q_1 = (0,0,1),\ q_2 = (1,1,1),\ q_3 = (0,1,1),\ q_4 = (1,0,1)$$

belong to, respectively, $P_1^I, \dots, P_4^I$ (q_j violates only the jth constraint defining P). But $(p_j + q_j)/2 = (1/2, 1/2, 1/2)$ for $j = 1, \dots, 4$, and therefore the point $(1/2, 1/2, 1/2)$ belongs to $\bigcap_{j=1}^4 \operatorname{conv}(P_j^I)$ but violates $w \leq 0$. □

Theorem 1.6 follows from the previous two lemmas:

Theorem 1.6 (restated). *Let P_j for $j \in J$ denote the set of basic relaxations of P, then*

$$P \cap \left(\bigcap_{j \in J} \operatorname{conv}(P_j^I) \right) \not\subseteq CC(P).$$

8 Cross Cuts that Cannot be Obtained from 2-row Relaxations

In this section we prove Theorem 1.4, namely we exhibit a polyhedral set such that the intersection of all cuts from its 2-row relaxations does not dominate its cross closure. The polyhedron we work with in this section is

$$\begin{aligned} P = \Big\{ (x,y) \in \mathbb{R}^2 \times \mathbb{R}^4 \ : x_1 &= \frac{1}{2} + \frac{1}{2} y_1 - \frac{1}{2} y_4, \\ x_2 &= \frac{1}{2} + \frac{1}{2} y_1 - \frac{1}{2} y_3, \\ -y_1 - y_2 + y_3 + y_4 &= 0, \qquad y \geq 0 \ \Big\}, \end{aligned} \tag{8}$$

and the associated mixed-integer set is $P^I = P \cap I$ where the mixed-integer lattice $I = \mathbb{Z}^2 \times \mathbb{R}^4$.

Observation 8.1. *The set P^I contains the points $p^k = (x^k, y^k)$ for $k = 1, \dots, 4$ given by*

$$\begin{aligned} x^1 &= (0,0), y^1 = (0,2,1,1) \\ x^2 &= (1,1), y^2 = (2,0,1,1) \\ x^3 &= (0,1), y^3 = (1,1,0,2) \\ x^4 &= (1,0), y^4 = (1,1,2,0). \end{aligned} \tag{9}$$

Moreover, the points p^1, p^2, and p^3 are affinely independent.

For convenience, we define

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1/2 & 0 & 0 & -1/2 \\ 1/2 & 0 & -1/2 & 0 \\ -1 & -1 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

so that $P = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^4 : Dx - Ay = b, y \geq 0\}$. We use A_i to denote the ith row of A. Note that $\text{rank}(A) = 3$, and so $\dim(P) \leq 3$. On the other hand, P contains the affinely independent points p^1, p^2, p^3 and $(1/2, 1/2, 0, 0, 0, 0)$, and so $\dim(P) = 3$.

P can be obtained from the polyhedron in (7) by: (i) introducing slack variable y_i to convert the ith ($i = 1, \dots, 4$) inequality to an equation, e.g., $-x_1 - x_2 + w + y_1 = 0$; (ii) Replacing w in the second to the fourth equations by $x_1 + x_2 - y_1$ (obtained from the first equation) and, (iii) subtracting the third and fourth equations from the second equation, and then dividing the third and fourth equations by 2. It follows from the above operations that there is a one-to-one correspondence between the solutions of (7) and (8). For any solution (x_1, x_2, w) of (7), one gets a solution $(x_1, x_2, y_1, \dots, y_4)$ of (8) by keeping x_1, x_2 unchanged and letting $y_1, \dots, y_4$ stand for the slacks of the inequalities in (8). Conversely, for any solution $(x_1, x_2, y_1, \dots, y_4)$ of (8), $(x_1, x_2, x_1 + x_2 - y_1)$ or $(x_1, x_2, 1 - (y_1 + \cdots + y_4)/4)$ is a solution of (7). The latter claim follows from the fact that adding up the four constraints in (7) (after introducing the slack variables) yields $4w + y_1 + y_2 + y_3 + y_4 = 4$.

Any 2-row relaxation of P is of the form

$$P(M) = \left\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^4 : Mx - MAy = Mb, y \geq 0\right\}$$

for a 2×3 matrix M. To prove Theorem 1.4, we will show that

$$P \cap \Big(\bigcap_{M \in \mathbb{R}^{2\times 3}} \text{conv}(P^I(M)) \Big) \not\subseteq CC(P).$$

Before starting, we observe that it is sufficient to consider matrices M that have full row rank.

Lemma 8.2. *For any $M \in \mathbb{R}^{2\times 3}$, there is a rank 2 matrix $M' \in \mathbb{R}^{2\times 3}$ such that* $\text{conv}(P^I(M')) \subseteq \text{conv}(P^I(M))$.

Proof. Clearly there exists a rank 2 matrix $M' \in \mathbb{R}^{2\times 3}$ such that $\text{rowspan}(M') \supseteq \text{rowspan}(M)$. It is easy to verify that such M' satisfies $P(M') \subseteq P(M)$, and hence $\text{conv}(P^I(M')) \subseteq \text{conv}(P^I(M))$. □

We start by showing that the inequality $cy \geq 4$, where $c = (1, 1, 1, 1)$, is a cross cut for P. Notice that the inequality $cy \geq 4$ translates to the inequality $w \leq 0$ for the polyhedron (7).

Lemma 8.3. *The inequality $cy \geq 4$ is a cross cut for P.*

Proof. We will show that $cy \geq 4$ is a cross cut for P derived from the cross disjunction $D(e^1, e^2, 0, 0) = \mathbb{R}^6 \setminus (S_1 \cup S_2)$, where e^i is the ith unit vector in $\mathbb{R}^6$, $S_1 = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^4 : 0 < x_1 < 1\}$ and $S_2 = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^4 : 0 < x_2 < 1\}$.

This statement would be false only if there exists some point $(\bar{x}, \bar{y})$ belonging to both P and $D(e^1, e^2, 0, 0)$ with $c\bar{y} < 4$. But if $(\bar{x}, \bar{y})$ belongs to P and $c\bar{y} < 4$, then $(\bar{x}, \bar{w})$ with $\bar{w} = 1 - cy/4$ is a solution of (7) with $\bar{w} > 0$. As in the proof of Lemma 7.1, we then infer that $0 < \bar{x}_1 + \bar{x}_2 < 2$ and $-1 < \bar{x}_1 - \bar{x}_2 < 1$, hence $(\bar{x}, \bar{y}) \in S_1 \cup S_2 = \mathbb{R}^6 \setminus D(e^1, e^2, 0, 0)$ and thus $(\bar{x}, \bar{y})$ does not belong to $D(e^1, e^2, 0, 0)$. The result then follows. □

We will next show that there exists a point $(\bar{x}, \bar{y}) \in P \cap \left(\bigcap_{M \in \mathbb{R}^{2\times 3}} \operatorname{conv}(P^I(M))\right)$ such that $c\bar{y} < 4$, and hence the cross cut $cy \geq 4$ is not valid for this set. To this end, we will show that for any $M \in \mathbb{R}^{2\times 3}$ we can construct a point $(x(M), y(M)) \in P \cap \operatorname{conv}(P^I(M))$ such that $cy(M) \leq 3$. We will then apply the Height Lemma using these points and a common base formed by points p^1, p^2 and p^3 presented in Observation 8.1. The following lemma, whose proof is deferred to Section 8.1, shows the existence of the points mentioned above.

Lemma 8.4. *Consider a matrix $M \in \mathbb{R}^{2\times 3}$ of rank 2. Then, there is a point (x, y) with the following properties: (i) $(x, y) \in P \cap \operatorname{conv}(P^I(M))$; (ii) $cy \leq 3$; (iii) $\|(x, y)\| \leq 6$.*

Using Lemma 8.4 we next prove Theorem 1.4:

Theorem 1.4 (restated). *The crooked cross cut closure of P cannot be obtained by all cuts from its 2-row relaxations. More precisely,*

$$P \cap \Big(\bigcap_{M \in \mathbb{R}^{2\times 3}} \operatorname{conv}\left(P^I(M)\right) \Big) \not\subseteq CC(P).$$

Proof. Consider a matrix $M \in \mathbb{R}^{2\times 3}$. Using Lemma 8.4 (and Lemma 8.2 if necessary), find a point $(x(M), y(M))$ in $P \cap \operatorname{conv}(P^I(M))$ such that $cy(M) \leq 3$ and $\|(x(M), y(M))\| \leq 6$. Also, for $i = 1, 2, 3$, the affinely independent points p^i in Observation 8.1 belong to $P \cap \operatorname{conv}(P^I(M))$ and satisfy $cy^i = 4$. Then applying Corollary 3.2 (with $A = \operatorname{aff}(P)$, $a = (0, 0, -c)$, $b = -4$ and $b' = -3$), we conclude that the set

$$Q = \bigcap_{M \in \mathbb{R}^{2\times 3}} \operatorname{conv}\left(p^1, p^2, p^3, (x(M), y(M))\right)$$

contains a point (x^*, y^*) satisfying $cy^* < 4$. Note that it is possible to apply Corollary 3.2 because the dimension of $\operatorname{aff}(P)$ is 3.

Since $P \cap (\bigcap_{M \in \mathbb{R}^{2\times 3}} \operatorname{conv}(P^I(M)))$ contains Q, it also contains (x^*, y^*). This shows that the cut $cy \geq 4$ is not valid for this set; together with Lemma 8.3, this concludes the proof of Theorem 1.4. □

8.1 Proof of Lemma 8.4

Let $M \in \mathbb{R}^{2\times 3}$ be a rank 2 matrix. Since $\operatorname{rank}(A) = 3$, this implies that $\operatorname{rank}(MA) = 2$. We will construct the points $(x(M), y(M))$ satisfying the properties of the lemma in three steps. In the first step, we will construct points in $\operatorname{aff}(P)$, which violate $cy \geq 4$, but do not belong to $P^I(M)$;

informally, these points almost belong to $P \cap P^I(M)$, except that they do not satisfy the required non-negativity condition. In the second step, we create two directions d^1 and d^2 in order to 'correct' the points constructed in the first step. In the final step, we use these directions to correct the points created in the first step, obtaining the desired point $(x(M), y(M))$ in $P \cap P^I(M)$ but still violating $cy \geq 4$.

Step 1. Consider the points $(x^i, y^i) \in P^I$ for $i = 1, \ldots, 4$ from Observation 8.1, and recall that they all satisfy $cy^i = 4$. Since they belong to P^I, we have $Dx^i - Ay^i = b$ for $i = 1, \ldots, 4$. Moreover, since $Ac = 0$, we have $Dx^i - A(y^i - c/2) = b$ for all i, which then implies $MDx^i - MA(y^i - c/2) = Mb$ for all i. In other words, the points $(x^i, \bar{y}^i) = (x^i, y^i - c/2)$ $(i = 1, \ldots, 4)$ satisfy the equations defining both P and $P^I(M)$ but violate one non-negativity inequality each, as

$$\begin{aligned}
\bar{y}^1 &= (0,2,1,1) - c/2 = (-1,3,1,1)/2 \\
\bar{y}^2 &= (2,0,1,1) - c/2 = (3,-1,1,1)/2 \\
\bar{y}^3 &= (1,1,0,2) - c/2 = (1,1,-1,3)/2 \\
\bar{y}^4 &= (1,1,2,0) - c/2 = (1,1,3,-1)/2.
\end{aligned} \tag{10}$$

Note that each point above has exactly one negative coefficient which equals $-1/2$, and the remaining coefficients are strictly positive and at least $1/2$. These four points also violate the inequality $cy \geq 4$, as $c \cdot c = 4$ and therefore, $(x^i, y^i - c/2)$ satisfies $c(y^i - c/2) = 2$.

Step 2. We now define the 'correcting' directions $d^1, d^2 \in \mathbb{R}^4$. To do so, recall that rowspan(A) has dimension 3 and by assumption rowspan(MA) is a 2-dimensional subspace of rowspan(A). If $A_3 \notin$ rowspan(MA), let $i^* = 3$, and if $A_3 \in$ rowspan(MA), let $i^* \in \{1, 2\}$ be the index such that A_{i^*} does not belong to rowspan(MA). Notice that the rows of MA together with A_{i^*} span exactly rowspan(A).

Now define $d^1, d^2 \in \mathbb{R}^4$ to be solutions of the following two systems of four equations each (the coefficient η is specified later):

$$\begin{bmatrix} MA \\ A_{i^*} \\ c \end{bmatrix} d^1 = \begin{bmatrix} \mathbf{0} \\ 1 \\ \eta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} MA \\ A_{i^*} \\ c \end{bmatrix} d^2 = \begin{bmatrix} \mathbf{0} \\ -1 \\ \eta \end{bmatrix}. \tag{11}$$

As the rows of MA together with A_{i^*} span exactly rowspan(A) and the vector c is orthogonal to the rows of A and hence to rowspan(A), the matrix in the left-hand side of equations in (11) (which is the same) is invertible. Therefore, these systems have unique nonzero solutions.

We will show that for some η, there are scalars $\lambda^1, \lambda^2 > 0$ such that $\lambda^1 d^1$ and $\lambda^2 d^2$ are nonzero vectors satisfying the following properties:

1. $MA(\lambda^1 d^1) = MA(\lambda^2 d^2) = 0$.
2. There exists an $\alpha \in (0,1)$ such that $A(\alpha \lambda^1 d^1 + (1-\alpha)\lambda^2 d^2) = 0$.
3. $\max_i \lambda^1 d^1_i = 1/2 = \max_i |\lambda^1 d^1_i|$ and $\max_i \lambda^2 d^2_i = 1/2 = \max_i |\lambda^2 d^2_i|$.
4. $c\lambda^1 d^1 \leq 1$ and $c\lambda^2 d^2 \leq 1$.

The motivation for these properties is the following: (i) the first and second properties will ensure that the 'corrected' vectors $(x^i, \bar{y}^i + (\alpha d^1 + (1-\alpha)d^2))$ still satisfy all the constraints of P and $P^I(M)$, except for the non-negativity conditions; (ii) we will use the third property to argue that there is an index i such that the corresponding corrected vector satisfies the non-negativity conditions, and hence belongs to $P \cap P^I(M)$; (iii) the fourth property will ensure that the corrected vector does not satisfy the inequality $cy \geq 4$.

Note that Properties 1 and 2 hold for all λ independent of the choice of η. Property 1 follows directly from the first two equations in both systems in (11). For Property 2, since $A_{i^*}d^1 = 1$ and $A_{i^*}d^2 = -1$, we have that $A_{i^*}(d^1 + d^2) = 0$. Therefore, $d^1 + d^2$ is orthogonal to the rows of MA and to A_{i^*}, and hence to the rows of A (as rows of MA and A_{i^*} span rowspan(A)).

In order to obtain Properties 3 and 4 we need to rescale the vectors d^1 and d^2, but notice that this operation preserves Properties 1 and 2. We consider two cases depending on whether A_3 belongs to rowspan(MA) or not, and set the coefficient η appropriately.

Case 1: $A_3 \in$ rowspan(MA)**.** Set $\eta = 0$. In this case, the last constraint in both systems in (11) (which are identical) guarantee that $\lambda^1 d^1$ and $\lambda^2 d^2$ satisfy Property 4 for all λ^1, λ^2.

We now consider Property 3 for a rescaling of d^1; the proof for d^2 is identical. Since A_3 belongs to rowspan(MA), the first two constraints in the first system in (11) guarantee that $A_3 d^1 = 0$, and therefore $d^1_1 + d^1_2 = d^1_3 + d^1_4$. The last constraint implies that $d^1_1 + d^1_2 + d^1_3 + d^1_4 = 0$. In addition, $d^1 \neq 0$ as $A_{i^*}d^1 \neq 0$. Therefore, $d^1_1 + d^1_2 = d^1_3 + d^1_4 = 0$ and hence $\max_i d^1_i = \max_i |d^1_i|$, so we can multiply d^1 by an appropriate positive scalar λ^1 so that $\max_i \lambda^1 d^1_i = 1/2$. The vector $\lambda^1 d^1$ then satisfies Properties 1,2,3, and 4.

Case 2: $A_3 \notin$ rowspan(MA)**.** Set $\eta = 1$. In this case $i^* = 3$, namely both systems in (11) contain a constraint of the form $A_3 d = \pm 1$ (instead of the implied constraint $A_3 d = 0$ in the previous case). Adding the third and fourth constraints in the first system in (11), we get $d^1_3 + d^1_4 = 1$. Subtracting the third constraint from the fourth constraint, we get $d^1_1 + d^1_2 = 0$. Therefore $\max_i d^1_i = \max_i |d^1_i| \geq 1/2$. We can then rescale d^1 by $\lambda^1 \in (0,1]$ so that $\lambda^1 d^1$ satisfies Property 3. Further, $\lambda^1 d^1$ satisfies Property 4, since $cd^1 \leq 1$. Therefore, $\lambda^1 d^1$ satisfies Properties 1,2,3 and 4.

As for d^2, adding and subtracting constraints as in the case of d^1, we see that $d^2_3 + d^2_4 = 0$ and $d^2_1 + d^2_2 = 1$. Once again we can scale d^2 so that it satisfies all properties.

Step 3. Consider the vectors $\lambda^1 d^1$ and $\lambda^2 d^2$ from the previous step. Let $i = \text{argmax}_k\, d^1_k$ and $j = \text{argmax}_k\, d^2_k$. As $\lambda^1 d^1$ is nonzero, and because of Property 3, we have $\lambda^1 d^1_i = 1/2$ and $\bar{y}^i + \lambda^1 d^1 \geq 0$. Property 1 implies that $MDx - MA(\bar{y}^i + \lambda^1 d^1) = MDx - MA\bar{y}^i = Mb$, and therefore $(x^i, \bar{y}^i + \lambda^1 d^1)$ belongs to $P^I(M)$ (but not to P, since we can still have $Dx^i - A(\bar{y}^i + \lambda^1 d^1) \neq b$). Also, Property 4 implies that $c(\bar{y}^i + \lambda^1 d^1) \leq 3$, and hence the point does not satisfy the inequality $cy \geq 4$. Similarly, Properties 1 and 3 imply that $(x^j, \bar{y}^j + \lambda^2 d^2) \in P^I(M)$, and $c(\bar{y}^j + \lambda^2 d^2) \leq 3$.

Finally, by Property 2 there is an $\alpha \in (0,1)$ such that the point

$$(x(M), y(M)) \triangleq \alpha(x^i, \bar{y}^i + \lambda^1 d^1) + (1-\alpha)(x^j, \bar{y}^j + \lambda^2 d^2)$$

satisfies $Dx(M) - Ay(M) = Dx^i - A\bar{y}^i = b$. Therefore, this point $(x(M), y(M))$ belongs to $P \cap \text{conv}(P^I(M))$. In addition, we clearly have $cy(M) \leq 3$, and it is easy to verify that $\|(x(M), y(M))\| \leq 6$. This concludes the proof of Lemma 8.4.

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